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A PROBABILISTIC APPROACH TO THE TWO-DIMENSIONAL NAVIER–STOKES EQUATIONS

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We turn the Navier-Stokes equations for a 2-dimensional viscous incompressible fluid into a system of functional integrals in the trajectory space of a suitable diffusion process. Using probabilistic techniques as Girsanov's transformation and Bismut-Elworthy formula, we prove the existence of a unique global solution of this system in a constructive way.

1. Introduction. We consider here the Navier–Stokes (NS) equations for velocity u and pressure p in a viscous, incompressible, planar fluid with initial velocity u_0 in the absence of external forces,

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= \nu \Delta u \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,\\
\nabla \cdot u &= 0 \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2
\end{aligned}
\tag{1.1}
$$

with initial and boundary conditions

$$
\begin{aligned}
u(0,x) &= u_0(x) \qquad \forall\ x \in \mathbb{R}^2,\\
\lim_{|x| \to +\infty} u(t,x) &= 0 \qquad \forall\ t \in \mathbb{R}^+
\end{aligned}
\tag{1.2}
$$

We will call stream function a function ψ such that

$$
\nabla^{\perp} \psi(t,x) = \begin{pmatrix} \dfrac{\partial \psi}{\partial x_2}(t,x) \\ -\dfrac{\partial \psi}{\partial x_1}(t,x) \end{pmatrix} = u(t,x).
$$

We recall the formulation of system (1.1), (1.2) in terms of the vorticity $\xi = \mathrm{rot}\, u = \partial_1 u^{(2)} - \partial_2 u^{(1)}$ (and the corresponding stream function vanishing

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at infinity ψ) and the velocity

$$
\begin{aligned}
&\frac{\partial \xi}{\partial t} + (\nabla^{\perp}\psi \cdot \nabla)\xi = \nu \Delta \xi, && (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
&\xi(0,x) = \operatorname{rot} u_0(x) = \xi_0(x), && x \in \mathbb{R}^2, \\
&-\Delta \psi_t = \xi_t, && (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2 \\
&\lim_{|x| \to +\infty} \psi(t,x) = 0, && t \in \mathbb{R}^+ \\
&u = \nabla^{\perp}\psi.
\end{aligned} \tag{1.3}
$$

(See [3], page 44.)

The aim of this work is to analyze system (1.3) with probabilistic methods. This was suggested to the author by Mark Freidlin. Using probabilistic representations we transform (1.3) in the problem

$$
\begin{aligned}
&\xi(t,x) = E\left[\xi_0\left(X_t^{t,x}\right)\right], \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
&\text{where} \quad \begin{aligned} &dX_s^{t,x} = -u\left(t-s, X_s^{t,x}\right) ds + \sqrt{2\nu}\, dW_s, \\ &X_0^{t,x} = x, \end{aligned} \\
&u(t,x) = -\frac{1}{2}\int_0^{\infty} \frac{1}{s} E\left[\xi(t, x + W_s) W_s^{\perp}\right] ds, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,
\end{aligned} \tag{1.4}
$$

where $W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$ is a two-dimensional Brownian motion (BM) and $W_t^{\perp} = \begin{pmatrix} W_t^2 \\ -W_t^1 \end{pmatrix}$.

We denote by C_0 the space of continuous functions on $\mathbb{R}^2$ vanishing at infinity, equipped with the $L^{\infty}(\mathbb{R}^2)$, by C_K the set of continuous function on $\mathbb{R}^2$ whose support is compact, and for all Banach space X, we denote by $BC([0,\infty[, X)$ the space $C([0,\infty[, X) \cap L^{\infty}([0,\infty[, X)$. Furthermore, for sake of simplicity, we use the notation L^r for $L^r(\mathbb{R}^2)$ and $\| \ \|_r$ for $\| \ \|_{L^r(\mathbb{R}^2)}$.

Ben-Artzi in [1] showed that, for initial vorticity ξ_0 in L^1, system (1.3) has a long-time solution. He proved that this solution is unique and regular and, for ξ_0 in $C_0^{\infty} \cap L^1$, he proposed an iterative method to obtain the solution.

Initially, our plan was to recover the same results for system (1.4) by probabilistic techniques. We realize it just partially: we prove existence and uniqueness under the assumption that the initial vorticity belongs to $L^p \cap L^q$ with $1 \le p < 2 < q$, and we do not prove regularity results for the solution. However, for ξ_0 in $L^p \cap L^q$, we provide a method constructing the solution which extends that proposed in [1] for initial vorticities in $L^1 \cap C_0^{\infty}$.

The work is organized as follows: In Section 2 we turn the NS system into the system of functional integrals (1.4). The third and fourth sections contain some preliminary results which we need in our proofs of the existence and the uniqueness of the solution. In Section 3 we provide some useful estimates for the $L^p(\Omega)$ norms of the Girsanov densities corresponding to stochastic differential equations (SDE) with additive noise, and in Section 4 we prove that the operators corresponding to stochastic differential equations (SDE) with additive noise and divergence-free drift do not increase the L^p norms. In Section 5, using these results, we show that, if $\xi_0 \in L^p \cap L^q$, system (1.4) has a unique global solution. The last section deals with SDEs where the noise is additive and the drift satisfies a "quasi-Lipschitz" estimate which the fluid velocity satisfies, provided that the initial vorticity belongs to $L^p \cap L^\infty$ for some $p \in [1, 2[$ (we prove that Lebesgue measure is invariant with respect to those SDEs).

Subjects partially related to those of the present paper are studied in a different way by Marchioro and Pulvirenti in [8].

2. The transformation of NS equations into a system of functional integrals. We consider here system (1.3) and we convert it into a system of functional integrals. Some difficulties arise when we try to transform the elliptic equations since we need a probabilistic representation of the solution of Poisson's equation. Section 2.1 contains a failed attempt to represent, when $f \in C_K^\infty$, the unique solution in C_0 of $\Delta\psi = f$ using the probabilistic expression of the heat semigroup. In Section 2.2 we provide probabilistic representations of the derivatives of ψ. Using these representations and the representation of the solution of parabolic equations with initial condition via SDE, in Section 2.3 we rewrite (1.3) as a system of functional integrals.

2.1. *Probabilistic representation of the solution of Poisson's problem in* $\mathbb{R}^2$. Consider the Poisson equation

$$\Delta\psi = f. \tag{2.1}$$

The operator $A = \frac{1}{2}\Delta$ generates a strongly continuous semigroup of contractions on C_b (the set of all bounded and uniformly continuous functions) and, for all $\lambda > 0$, the resolvent $(A - \lambda I)^{-1}$ can be expressed as follows:

$$\begin{aligned}(A - \lambda I)^{-1}(f)x &= \int_0^\infty e^{-\lambda s} P_s(f)(x)\,ds \\ &= \int_0^\infty e^{-\lambda s} E[f(x + W_s)]\,ds \qquad \forall\, f \in C_b.\end{aligned}$$

So one could hope that, for $f \in C_b$, the integral

$$\psi(x) = \frac{1}{2}\int_0^\infty E[f(x + W_s)]\,ds \tag{2.2}$$

converges providing the solution of (2.1). Unfortunately, that integral does not converge in general, as the next proposition proves. This result is known (see [9], Exercise 2.29), but we include a short proof.

PROPOSITION 2.1.1. *Let* $f: \mathbb{R}^2 \to \mathbb{R}^+$ *be a continuous function*, $f \neq 0$. *Then, for every* $x \in \mathbb{R}^2$, *we have*

$$\int_0^{+\infty} P_t(f)(x)\,dt = +\infty.$$

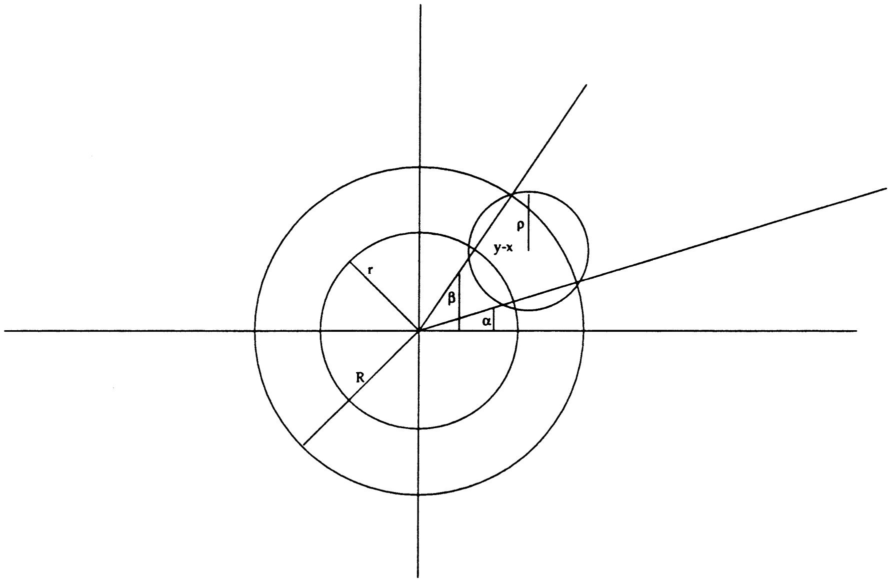

FIG. 1.

PROOF. There exists $a > 0$ such that $\{f > a\} \neq \varnothing$. Since f is continuous, there exist y in $\mathbb{R}^2$ and $\rho > 0$ such that $B(y, \rho) \subset \{f > a\}$. Fix $x \in \mathbb{R}^2$. There exist $\beta, \alpha \in [0, 2\pi[$ and $r, R \in \mathbb{R}$ such that $0 \leq \alpha < \beta < 2\pi$, $0 < r < R$ and $\{z \mid r < |z| < R,\ \alpha < \arg(z) < \beta\} \subset B(y - x, \rho)$ (see Figure 1). It follows that

$$\begin{aligned} P(W_t + x \in B(y, \rho)) &= P(W_t \in B(y - x, \rho)) \\ &\geq \frac{(\beta - \alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right) \end{aligned}$$

and

$$\int_0^{+\infty} E[f(W_t + x)]\,dt \geq a \int_0^{+\infty} \frac{(\beta - \alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right) dt.$$

Note that

$$\frac{(\beta - \alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right) \sim \frac{1}{t}$$

and, in consequence,

$$\int_0^{+\infty} E[f(W_t + x)]\,dt \geq a \int^{+\infty} \frac{c}{t}\,dt = \infty. \qquad \square$$

Hence, if $f \in C_K^+$ and $f \neq 0$, we cannot write the solution of (2.1) in the form (2.2).

REMARK 2.1.1. From the preceding proposition we deduce that, if f is a continuous function which changes sign, then, for all $x \in \mathbb{R}^2$, the integrals $\int_0^{+\infty} E[f^+(x + W_t)]\,dt$ and $\int_0^{+\infty} E[f^-(x + W_t)]\,dt$ diverge. Nevertheless, it would be an error to deduce that the integral $\int_0^{+\infty} E[f(x + W_t)]\,dt$ is undetermined, because the equivalence $E[f(x + W_t)]^+ = E[f^+(x + W_t)]$ does not hold in general [for instance, if $f(x) = x$, then $\int_0^{+\infty} E[f(0 + W_t)]\,dt = \int_0^{+\infty} E[W_t]\,dt = 0$].

2.2. *Probabilistic representations of the derivatives of the solution of Poisson's problem in* $\mathbb{R}^2$. Consider the function $x \mapsto E[f(x + W_t)]$. Under suitable assumptions on f, it is differentiable, and the partial derivatives can be expressed by the Bismut-Elworthy formula

$$\frac{\partial}{\partial x_i} E\left[f(x + W_t)\right] = \frac{1}{t} E\left[f(x + W_t) W_t^i\right]$$

(see [4]). Note that no derivatives of f appear in this formula. Applying formally the Bismut-Elworthy formula to differentiate the expression

$$\psi(x) = \tfrac{1}{2} \int_0^{+\infty} E\left[f(x + W_t)\right] dt,$$

we obtain

$$\frac{\partial \psi}{\partial x_i}(x) = \frac{1}{2} \int_0^{+\infty} \frac{1}{t} E\left[f(x + W_t) W_t^i\right] dt.$$

We now want to investigate under which conditions these integrals converge. The next proposition provides some sufficient conditions.

PROPOSITION 2.2.1. *Let* $f \in L^p \cap L^q$ *with* $1 \le p < 2 < q \le \infty$. *Then the integrals*

$$\psi_i(x) = \int_0^{+\infty} \frac{1}{t} E\left[f(x + W_t) W_t^i\right] dt, \qquad i = 1, 2$$

converge and there exists a constant $c_{p,q}$ *such that*

$$|\psi_i(x)| \le c_{p,q} \|f\|_{p,q} \qquad \forall\ x \in \mathbb{R}^2.$$

Here and in the following, the notation $\| \ \|_{p,q}$ denotes the norm $\| \ \|_p + \| \ \|_q$.

The proof relies on the following two remarks.

REMARK 2.2.1. For all $p \in [1, +\infty[$ there exists a constant c_p such that

$$E\left[|W_t^{(i)}|^p\right]^{1/p} \le c_p \sqrt{t} \qquad \forall\ t \in [0, +\infty[.$$

This property is a consequence of a well-known fact about Gaussian random variables.

REMARK 2.2.2. For all $r \in]1, \infty]$ there exists a constant c such that

$$E\left[|f(x + W_t) W_t^{(i)}|\right] \le c_r \|f\|_r t^{-1/r + 1/2} \qquad \forall\ f \in L^r\ \forall\ t > 0.$$

PROOF. Using Hölder's inequality and the preceding remark, one can write the following chain of inequalities:

$$\begin{aligned}E\big[|f(x+W_t)W_t^{(i)}|\big] &\le \|f(x+W_t)\|_{L^r(\Omega)}\,\|W_t^{(i)}\|_{L^{r'}(\Omega)}\\ &\le \left(\frac{1}{2\pi t}\int_{\mathbb{R}^2} f^r(x+y)\exp\left(-\frac{|y|^2}{2t}\right)dy\right)^{1/r} c_{r'}\sqrt{t}\\ &\le \|f\|_p c_{r'} t^{-1/r+1/2}.\end{aligned}$$

□

PROOF OF PROPOSITION 2.2.1. It is enough to prove the theorem for $1 < p < 2 < q < \infty$. By the preceding remark, for all $t > 0$, we have

$$\begin{aligned}\frac{1}{t}E\big[|f(x+W_t)W_t^{(i)}|\big] \le c_{p'}\|f\|_p t^{-1/p-1/2} I_{]1,\infty]}(t)\\ + c_{q'}\|f\|_q t^{-1/q-1/2} I_{]0,1]}(t).\end{aligned}$$

Since $-1/p - 1/2 < -1$ and $-1/q - 1/2 > -1$, the function on the right is integrable. Therefore the integrals $\psi_i(x)$ converge and

$$\begin{aligned}|\psi_i(x)| &\le c_{p'}\|f\|_p \int_1^\infty t^{-1/p-1/2}\,dt + c_{q'}\|f\|_q\int_0^1 t^{-1/q-1/2}\,dt\\ &= c_{p'}\|f\|_p\left(\frac{1}{p}+\frac{1}{2}-1\right) + c_{q'}\|f\|_q\left(1-\frac{1}{q}-\frac{1}{2}\right)\\ &= c_{p'}\|f\|_p\left(\frac{1}{p}-\frac{1}{2}\right) + c_{q'}\|f\|_q\left(\frac{1}{2}-\frac{1}{q}\right).\end{aligned}$$

□

REMARK 2.2.3. If $f \in L^1 \cap C_b^1$, then

$$\frac{\partial}{\partial x_i}E[f(x+W_t)] = E\left[\frac{\partial f}{\partial x_i}(x+W_t)\right]$$

and the integrals

$$\int_0^\infty E\left[\frac{\partial f}{\partial x_i}(x+W_t)\right]dt \tag{2.3}$$

converge. This seems to be in contradiction with Proposition 2.2.1. It is not: the functions $\partial f/\partial x_i$ have to change the sign (while $f \in C_b \cap L^1$) and, as we have remarked, from Proposition 2.2.1 we cannot argue that the integrals (2.3) are undetermined.

Now we prove some regularity results for the functions ψ_i.

PROPOSITION 2.2.2. *If $f \in L^p \cap L^q$ with $1 < p < 2 < q$, then the functions $\psi_i(x)$ are uniformly continuous.*

PROOF. The inequality

$$\int_0^\infty \left| \frac{1}{t} E\left[f(x + W_t) W_t^{(i)} \right] - \frac{1}{t} E\left[f(\tilde{x} + W_t) W_t^{(i)} \right] \right| dt$$
$$\leq c_1 \| \tau_{x-\tilde{x}} f - f \|_p + c_2 \| \tau_{x-\tilde{x}} f - f \|_q ,$$

together with the uniform continuity of the shift in L^r for $r \in]1, \infty[$, leads us to conclude that the functions $\psi_i(x)$ are uniformly continuous. □

PROPOSITION 2.2.3. *If $f \in L^p \cap L^q$ and $1 \leq p < 2 < q$, then the functions $\psi_i(x)$ are in C_0.*

PROOF. For all $R > 0$, we have

$$\frac{1}{t} E\left[\left| f(x + W_t) W_t^{(i)} \right| \right]$$
$$= \frac{1}{t} E\left[\left| f(x + W_t) W_t^{(i)} \right| I_{\{|W_t| \leq R\}} \right] + \frac{1}{t} E\left[\left| f(x + W_t) W_t^{(i)} \right| I_{\{|W_t| > R\}} \right].$$

Concerning the first addendum, we have

$$\frac{1}{t} E\left[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| \leq R\}} \right] \leq c_p \| f I_{\{|y-x| \leq R\}} \|_p t^{-1/p - 1/2} I_{]1, \infty[}(t)$$
$$+ c_q \| f I_{\{|y-x| \leq R\}} \|_q t^{-1/q - 1/2} I_{]0,1[}(t).$$

Hence, for all fixed R,

$$\int_0^\infty \frac{1}{t} E\left[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| \leq R\}} \right] dt$$
$$\leq C_1 \| f I_{\{|y-x| \leq R\}} \|_p + C_2 \| f I_{\{|y-x| \leq R\}} \|_q \to 0 \qquad \text{as } |x| \to \infty,$$

where the convergence follows from the fact that, for all $|x| > R$, the inclusion $\{|y - x| \leq R\} \subset \{|y| > |x| - R\}$ holds. Consider now the second addendum. For all $t > 0$, we get

$$\sup_x \left(\frac{1}{t} E\left[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| > R\}} \right] \right)$$
$$\leq \frac{1}{t} t^{-1/q} \| f \|_q t^{1/2} I_{]0,1]} + \frac{1}{t} t^{-1/p} \| f \|_p t^{1/2} I_{]1, \infty[} .$$

Since the function of t on the left-hand side is dominated (as R varies) and converges to 0 as $R \to \infty$, the dominated convergence theorem entails that

$$\sup_x \left(\int_0^\infty \frac{1}{t} E\left[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| > R\}} \right] dt \right) \to 0 \qquad \text{as } R \to \infty.$$

Fix an arbitrary $\varepsilon > 0$. Choosing R so that, for each x,

$$\int_0^\infty \frac{1}{t} E\Big[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| > |x|/2\}} \Big] \, dt \le \varepsilon,$$

we obtain

$$\begin{aligned}
&\lim_{|x| \to \infty} \int_0^\infty \frac{1}{t} E\big[|f(x + W_t) W_t^{(i)}| \big] \, dt \\
&\quad \le \lim_{|x| \to \infty} \int_0^\infty \frac{1}{t} E\Big[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| \le R\}} \Big] \, dt \\
&\qquad + \lim_{|x| \to \infty} \int_0^\infty \frac{1}{t} E\Big[|f(x + W_t) W_t^{(i)}| I_{\{|W_t| \le R\}} \Big] \, dt \\
&\quad \le \varepsilon.
\end{aligned}$$

The result follows from the arbitrariness of ε. □

We can summarize Propositions (2.2.1), (2.2.2) and (2.2.3) with the following assertion: the map

$$\begin{aligned}
s \colon L^p \cap L^q(\mathbb{R}^2) &\mapsto C_0(\mathbb{R}^2), \\
f &\mapsto \int_0^{+\infty} \frac{1}{t} E\big[f(x + W_t) W_t^i \big] \, dt
\end{aligned}$$

is well defined and continuous.

How are this integrals $\psi_i(x)$ related to the Poisson equation? It is known that, if $f \in C_K^\infty$, the Poisson equation

$$\Delta \psi = f \tag{2.4}$$

has a unique solution in C_0^∞, given by

$$\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| f(y) \, dy$$

and its partial derivatives may be expressed by

$$\begin{aligned}
\frac{\partial \psi}{\partial x_i}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|x - y|^2} (x_i - y_i) f(y) \, dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|y|^2} y_i f(y + x) \, dy
\end{aligned}$$

(see [6], Chapter 4). Since $f \in C_K^\infty \subset L^1 \cap L^\infty$, the integrals

$$\psi_i(x) = \int_0^{+\infty} \frac{1}{t} E\big[f(x + W_t) W_t^i \big] \, dt, \qquad i = 1, 2$$

converge. Applying the Fubini–Tonelli theorem, we get

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{t} E\left[f(x + W_t) W_t^i \right] dt \\
&\quad = \int_0^{+\infty} \frac{1}{t} \int_{\mathbb{R}^2} f(x+y) y^i \exp\left(-\frac{1}{2t} |y|^2 \right) \frac{1}{2\pi t}\, dy\, dt \\
&\quad = \frac{1}{\pi} \int_{\mathbb{R}^2} f(x+y) y^i \int_0^{+\infty} \frac{1}{2t^2} \exp\left(-\frac{1}{2t} |y|^2 \right) dt\, dy \\
&\quad = \frac{1}{\pi} \int_{\mathbb{R}^2} f(x+y) y^i \left[\frac{1}{|y|^2} \exp\left(-\frac{1}{2t} |y|^2 \right) \right]_0^{+\infty} dy \\
&\quad = \frac{1}{\pi} \int_{\mathbb{R}^2} f(x+y) y^i \frac{1}{|y|^2}\, dy.
\end{aligned}$$

In conclusion, if $f \in C_K^\infty$, the unique solution in C_0^∞ of (2.4) has derivatives

$$\frac{\partial \psi}{\partial x_i}(x) = \frac{1}{2} \int_0^{+\infty} \frac{1}{s} E\left[f(x + W_s) W_s^i \right] ds.$$

So, if the functions $\xi(t, \cdot)$ belong to C_K^∞, we obtain for the velocity field $u(t, x)$ the representation formula

$$u(t, x) = -\frac{1}{2} \int_0^{+\infty} \frac{1}{s} E\left[\xi(t, x + W_s) W_s^{\perp} \right] ds.$$

2.3. *A system of functional integrals for vorticity and velocity.* We can now summarize our efforts. We are not able to give a probabilistic expression for the stream function ψ. On the contrary we have a probabilistic representation of velocity u in term of vorticity ξ,

$$u(t, x) = \frac{1}{2} \int_0^{+\infty} \frac{1}{s} E\left[\xi(t, x + W_s) W_s^{\perp} \right] ds.$$

On the other hand, it is known that, if $\xi_0 \in C_b$, the solution of the problem

$$\frac{\partial \xi}{\partial t} + (u \cdot \nabla) \xi = \nu \Delta \xi, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

$$\xi(0, x) = \xi_0(x), \qquad x \in \mathbb{R}^2$$

may be written in the form

$$\xi(t, x) = E\left[\xi_0\left(X_t^{t,x} \right) \right],$$

where $X_s^{t,x}$ solves

$$\begin{aligned}
dX_s^{t,x} &= -u\left(t - s, X_s^{t,x}\right) ds + \sqrt{2\nu}\, dW_s, \\
X_0^{t,x} &= x
\end{aligned}$$

(see [5]). Therefore we are interested in studying the system of functional integrals,

$$\xi(t,x) = E\big[\xi_0\big(X_t^{t,x}\big)\big], \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$
$$\text{where } dX_s^{t,x} = -u\big(t-s, X_s^{t,x}\big)\,ds + \sqrt{2\nu}\,dW_s,\ X_0^{t,x} = x,$$
$$u(t,x) = -\frac{1}{2}\int_0^\infty \frac{1}{s} E\big[\xi_t(x+W_s)W_s^{\perp}\big]\,ds, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

3. The Girsanov densities. The reader will see that our fundamental tool in the sequel is the Girsanov formula which plays the role that the integral formula for evolutionary equations plays in [1]. The system (1.4) contains a SDE with an additive noise multiplied by a positive constant; that is, a SDE of the following type:

$$\begin{aligned} dX_t^x &= u(t, X_t^x)\,dt + \mu dW_t, \\ X_0^x &= x. \end{aligned} \tag{3.1}$$

We will find that the drift u is bounded and, more precisely, belongs to $BC([0,\infty[, C_0)$. So u is not necessarily Lipschitzian in x, and (3.9) has generally just weak solutions. That is, there exists a Brownian motion depending on x such that (3.1) admits a solution, but this does not happen for all Brownian motions. Moreover, for system (3.1) the pathwise uniqueness (which entails the existence with respect to each Brownian motion) does not hold, but two solutions (corresponding eventually to two different BM) must have the same law (see [9], Chapter IX).

Fix $f \in L^\infty$ and consider the function

$$\mathbb{R}^2 \ni x \mapsto E^{P^x}\big[f\big(X_t^{x,u}\big)\big] \in \mathbb{R}^2,$$

where $(\Omega^x, F^x, F_t^x, W_t^x, X_t^{x,u}, P^x)$ is a solution of (3.1). As x varies in $\mathbb{R}^2$, in order to get solutions of (3.1), we have to change the underlying BM. For that reason, to obtain continuity results for $x \mapsto E^{P^x}[f(X_t^{x,u})]$ and to approximate that function by maps $x \mapsto E^{P^x}[f(X_t^{x,u_n})]$ corresponding to more regular drifts u_n, it will be convenient to use the Girsanov formula.

Take $u \in BC([0,\infty[, C_b)$ and fix a BM (Ω, F, F_t, W_t, P). Let $(\Omega^x, F^x, F_t^x, W_t^x, X_t^{x,u}, P^x)$ be a solution of (3.1). The Girsanov theorem and the uniqueness in law of the solution of (3.1) allow us to conclude that

$$E^{P^x}\big[f(X_t^x)\big] = E^P\big[f(x+\mu W_t)Z_t^x\big] \qquad \forall\, f \in L^\infty,$$

where Z_t^x is the process

$$Z_t^x = \exp\left\{\int_0^t \frac{1}{\mu} u(s, x+\mu W_s)\,dW_s - \frac{1}{2}\int_0^t \frac{1}{\mu^2}|u(s, x+\mu W_s)|^2\,ds\right\} \tag{3.2}$$

and solves the SDE

$$\begin{aligned} dZ_s^x &= \frac{1}{\mu} u(s, x+\mu W_s) Z_s^x dW_s, \\ Z_0^x &= 1. \end{aligned} \tag{3.3}$$

Due to the boundedness of u, for all $T > 0$, the Girsanov density process $(Z_s^x)_{0 \le s \le T}$ is a martingale bounded in L^p for all $p \in [1, \infty[$ (see [9], Chapter VIII, paragraph 1).

We will find some useful upper bounds for the norm in $L^p(\Omega)$ of these Girsanov densities and of the difference of two such densities using the Burkholder–Davis–Gundy (BDG) inequalities, which are the content of next theorem.

THEOREM 3.0.1. *Let $p \in]0, \infty[$. There exist two positive constants c_p and C_p such that, for all continuous local martingales M vanishing at zero,*

$$E\left[\langle M, M\rangle_\infty^{p/2}\right] \le c_p E[M_\infty^*] \le C_p E\left[\langle M, M\rangle_\infty^{p/2}\right]$$

(see [9], page 151).

We indicate by $|||\cdot|||_\infty$ the norm

$$|||u|||_\infty = \sup_{t\in[0,\infty[} \|u(t,\cdot)\|_\infty \quad \forall\ u \in BC([0,\infty[, C_b),$$

and, for each $u \in BC([0,\infty[, C_b)$ and $x \in \mathbb{R}^2$, we use the notation $Z_t^{x,u}$ to denote the corresponding process defined by (3.2).

The next lemma provides an upper bound for the norm $E[|Z_t^{x,u}|^q]$.

LEMMA 3.0.1. *For each $q \in]2, \infty[$, there exists a constant $c > 0$ such that, for all $u \in BC([0,\infty[, C_b)$,*

$$E\left[|Z_t^{x,u}|^q\right] \le c \exp\left(c\left(\frac{|||u|||_\infty}{\mu}\right)^q t^{q/2}\right) \qquad \forall\ t \in [0,\infty[,\ \forall\ x \in \mathbb{R}^2.$$

PROOF. Since $Z_t^{x,u}$ solves (3.3), we have

$$E\left[|Z_t^{x,u}|^q\right] \le c_q\left(1 + E\left[\left|\int_0^t \frac{1}{\mu} u(s, x+\mu W_s) Z_s^{x,u}\, dW_s\right|^q\right]\right) \qquad \forall\ t \ge 0.$$

Therefore, by BDG inequalities,

$$E\left[|Z_t^{x,u}|^q\right] \le c_q\left(1 + c\frac{1}{\mu^q} E\left[\left(\int_0^t |u(s, x+\mu W_s)|^2 |Z_s^{x,u}|^2\, ds\right)^{q/2}\right]\right) \qquad \forall\ t \ge 0,$$

and from Hölder's inequality it follows

$$\begin{aligned} E\left[|Z_t^{x,u}|^q\right] &\le c_q + c'_q \frac{1}{\mu^q} E\left[t^{(q/2)-1} \int_0^t |u(s, x+\mu W_s)|^q |Z_s^{x,u}|^q\, ds\right] \\ &\le c_q + c'_q\left(\frac{|||u|||_\infty}{\mu}\right)^q t^{(q/2)-1} \int_0^t E\left[|Z_s^{x,u}|^q\right] ds. \end{aligned}$$

Finally, applying Gronwall's lemma, we get

$$E\left[|Z_t^{x,u}|^q\right] \le c_q \exp\left(c'_q\left(\frac{|||u|||_\infty}{\mu}\right)^q t^{q/2}\right) \qquad \forall\ t \ge 0. \qquad \square$$

REMARK. In the same way, one can prove that, for each $q \in]2, \infty[$, there exists a constant $c > 0$ such that

$$E\big[|Z_t^{x,u}|^q\big] \le c \exp\left(c\left(\frac{|||\, u \,|||_{\infty,t}}{\mu}\right)^q t^{q/2}\right) \qquad \forall\ t \in [0, \infty[, \forall\ x \in \mathbb{R}^2,$$

where $|||\, u \,|||_{\infty,t} = \sup_{s \in [0,t]} \|u(s,\cdot)\|_\infty\ \forall\ u \in BC([0,\infty[\ ,\ C_b)$.

COROLLARY 3.0.1. *For each $p \in [1,2]$ there exists a constant $c > 0$ such that*

$$E\big[|Z_t^{x,u}|^p\big] \le c \exp\left(c\left(\frac{|||\, u \,|||_{\infty}}{\mu}\right)^2 t\right) \qquad \forall\ t \in [0, \infty[, \forall\ x \in \mathbb{R}^2.$$

PROOF. For $p = 2$ the proof is very similar to that of the preceding lemma and slightly simpler (it does not avoid the use of Hölder's inequality). If $p < 2$, then $E[|Z_t^{x,u}|^p] \le E[|Z_t^{x,u}|^2]^{p/2}$. □

Now we estimate the $L^q(\Omega)$ norm of the difference of two densities.

LEMMA 3.0.2. *For each $q \in]2, \infty[$, there exists a constant c such that, for all $u, v \in BC([0,\infty[, C_b)$,*

$$E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] \le c\left(\frac{|||\, u - v \,|||_{\infty}}{\mu}\right)^q t^{q/2} \exp\left(c\frac{|||\, u \,|||_{\infty}^q + |||\, v \,|||_{\infty}^q}{\mu^q} t^{q/2}\right)$$
$$\forall\ t \in [0, \infty[,\ \forall\ x \in \mathbb{R}^2.$$

PROOF. Since $Z_t^{x,u}$ solves

$$dZ_s^x = \frac{1}{\mu} u(s, x + \mu W_s) Z_s^x\, dW_s,$$
$$Z_0^x = 1$$

and $Z_t^{x,v}$ solves

$$dZ_s^x = \frac{1}{\mu} v(s, x + \mu W_s) Z_s^x\, dW_s,$$
$$Z_0^x = 1,$$

we have

$$\begin{aligned} E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] &\le c_q E\left[\left|\int_0^t \frac{1}{\mu}(u - v)(s, x + \mu W_s) Z_s^{x,u}\, dW_s\right|^q\right] \\ &\quad + c_q E\left[\left|\int_0^t \frac{1}{\mu} v(s, x + \mu W_s)(Z_s^{x,u} - Z_s^{x,v})\, dW_s\right|^q\right] \end{aligned}$$

and, applying the BDG inequality, we get

$$\begin{aligned}
&E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] \\
&\quad\le c'_q \frac{1}{\mu^q} E\left[\left(\int_0^t |(u-v)(s, x+\mu W_s)|^2 |Z_s^{x,u}|^2\, ds\right)^{q/2}\right] \\
&\qquad + c'_q \frac{1}{\mu^q} E\left[\left(\int_0^t |v(s, x+\mu W_s)|^2 |Z_s^{x,u} - Z_s^{x,v}|^2\, dr\right)^{q/2}\right].
\end{aligned}$$

According to Hölder's inequality, we get

$$\begin{aligned}
&E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] \\
&\quad\le c'_q \frac{1}{\mu^q} E\left[t^{q/2-1} \int_0^t |(u-v)(s, x+\mu W_s)|^q |Z_s^{x,u}|^q\, ds\right] \\
&\qquad + c'_q \frac{1}{\mu^q} E\left[t^{(q/2)-1} \int_0^t |v(s, x+\mu W_s)|^q |Z_s^{x,u} - Z_s^{x,v}|^q\, ds\right] \\
&\quad\le c'_q \frac{1}{\mu^q} t^{(q/2)-1} |||\, u - v\, |||_\infty^q \int_0^t E\big[|Z_s^{x,u}|^q\big]\, ds \\
&\qquad + c'_q \frac{1}{\mu^q} t^{(q/2)-1} |||\, v\, |||_\infty^q \int_0^t E\big[|Z_s^{x,u} - Z_s^{x,v}|^q\big]\, ds \\
&\quad\le c'_q \frac{1}{\mu^q} t^{(q/2)-1} |||\, u - v\, |||_\infty^q tc \exp\left(c\left(\frac{|||\, u\, |||_\infty}{\mu}\right)^q t^{q/2}\right) \\
&\qquad + c'_q \frac{1}{\mu^q} t^{(q/2)-1} |||\, v\, |||_\infty^q \int_0^t E\big[|Z_s^{x,u} - Z_s^{x,v}|^q\big]\, ds.
\end{aligned}$$

Finally, applying Gronwall's lemma, we obtain

$$E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] \le c\left(\frac{|||\, u - v\, |||_\infty}{\mu}\right)^q t^{q/2} \exp\left(c\frac{|||\, u\, |||_\infty^q + |||\, v\, |||_\infty^q}{\mu^q} t^{q/2}\right). \quad \square$$

REMARK. In the same way, one can prove that

$$E\big[|Z_t^{x,u} - Z_t^{x,v}|^q\big] \le c\left(\frac{|||\, u - v\, |||_{\infty,t}}{\mu}\right)^q t^{q/2} \exp\left(c\frac{|||\, u\, |||_{\infty,t}^q + |||\, v\, |||_{\infty,t}^q}{\mu^q} t^{q/2}\right)$$
$$\forall\ t \in [0,\infty[,\ \forall\ x \in \mathbb{R}^2.$$

COROLLARY 3.0.2. *Let $p \in [1,2]$. For all $q > 2$ there exists c such that*

$$E\big[|Z_t^{x,u} - Z_t^{x,v}|^p\big] \le c\left(\frac{|||\, u - v\, |||_\infty}{\mu}\right)^p t^{p/2} \exp\left(c^{p/q}\left(\frac{|||\, u\, |||_\infty^q + |||\, v\, |||_\infty^q}{\mu^q}\right) t^{q/2}\right)$$
$$\forall\ t \in [0,\infty[,\ \forall\ x \in \mathbb{R}^2.$$

PROOF. It is enough to note that for all random variable X and for all $p < q$, the following inequality holds:

$$E[|X|^p]^{1/p} \le E[|X|^q]^{1/q}. \qquad \square$$

4. The operators corresponding to certain SDEs on $\mathbb{R}^2$ do not increase the L^1 norm.

Fix μ in $\mathbb{R}^+$ and let u: $[0, +\infty[\times\mathbb{R}^2 \to \mathbb{R}^2$ be a continuous and bounded function. Consider the SDE

$$\begin{aligned} dX_t^x &= u(t, X_t^x)\,dt + \mu\,dW_t \qquad \forall\ t \ge 0,\\ X_0^x &= x. \end{aligned} \tag{4.1}$$

As we remarked in the previous section, for each $x \in \mathbb{R}^2$, there exist a Brownian motion $(\Omega^x, F^x, F_t^x, W_t^x, P^x)$ and a process X_t^x which solve the equation. Furthermore, two different solutions have the same law.

For all $t \ge 0$, we denote by P_t the map

$$P_t\colon L^\infty \to L^\infty$$

defined by

$$P_t f(x) = E^{P^x}[f(X_t^x)] \qquad \forall\ x \in \mathbb{R}^2,$$

where $(\Omega^x, F^x, F_t^x, W_t^x, X_t^{x,u}, P^x)$ is a solution of (4.1). [Here, in order to be more precise, we have to define P_t: $L^\infty(\mathbb{R}^2, B(\mathbb{R}^2), \mathscr{L}^2) \to L^\infty(\mathbb{R}^2, B(\mathbb{R}^2), \mathscr{L}^2)$. We have to take the σ-algebra of Borel to overcome measurability problems.] We will show that, if $u \in BC([0,\infty[, C_b)$ and $\operatorname{div}_x u = 0$ in the distributional sense, the operators P_t do not increase the L^1 norm; that is,

$$\int_{\mathbb{R}^2} \left| E^{P^x}[f(X_t^x)] \right| dx \le \int_{\mathbb{R}^2} |f(x)|\,dx \qquad \forall\ f \in L^1 \cap L^\infty(\mathbb{R}^2). \tag{4.2}$$

We first prove it for $u \in BC([0,\infty[, C_b^1)$ (in Section 4.1) and then we extend the result for $u \in BC([0,\infty[, C_b)$ (in Section 4.2).

We state in advance a useful remark.

REMARK ON MEASURABILITY. If f belongs to L^p for some $p \in [1,\infty[$ and X_t^x is a process such that, for all ω, the functions $(x,t) \mapsto X_t^x(\omega)$ are continuous then the functions $(x,\omega) \mapsto f(X_t^x(\omega))$ are measurable. Therefore the maps $x \mapsto P_t f(x)$ are Lebesgue measurable.

PROOF. Fix $t \ge 0$. Since the maps $x \mapsto X_t^x(\omega)$ are continuous, the function $(x,\omega) \mapsto X_t^x(\omega)$ is measurable with respect to the σ-algebras $(B(\mathbb{R}^2) \times F, B(\mathbb{R}^2))$. Hence, for all $g \in C_0$, the map $(x,\omega) \mapsto g(X_t^x(\omega))$ is measurable with respect to $(B(\mathbb{R}^2) \times F, B(\mathbb{R}^+))$. In view of the existence of a sequence in C_0 which converges almost surely to f, we can conclude that the map $(x,\omega) \mapsto f(X_t^x(\omega))$ is measurable with respect to $(\mathscr{M} \times F, B(\mathbb{R}^+))$ where $\mathscr{M}$ is the σ-algebra of the Lebesgue measurable sets of $\mathbb{R}^2$. Finally we note that, fixing a BM (Ω, F, F_t, W_t, P), we have $P_t f(x) = E[f(x + W_t)Z_t^x]\ \forall\ x \in \mathbb{R}^2$.

Using this expression for $P_t f$, it is not difficult to prove the last assertion. □

4.1. *When the drift is in* $BC([0,\infty[, C_b^1)$. Suppose that u satisfies the following hypotheses:

A1. u is uniformly continuous and bounded.
A2. u admits derivatives $\partial u/\partial x_1$ and $\partial u/\partial x_2$ uniformly continuous and bounded.
A3. $\partial u/\partial x_1 + \partial u/\partial x_2 = 0$.

Fix a standard planar Brownian motion $(\Omega, F, (F_t)_{t\geq 0}, (W_t)_{t\geq 0}, P)$ in (4.1). For each x, (4.1) has a solution unique up to indistinguishability. Moreover, there exists a process continuous in x and t which, for all x, is indistinguishable from a solution. We will say that such a process is a continuous solution of (4.1).

THEOREM 4.1.1. *Let* $(\Omega, F_\infty, F_t, W_t, X_t^x, P)$ *be a continuous solution of* (4.1) *and fix* $t \geq 0$. *Then, for P-almost all* $\omega \in \Omega$, *the function*

$$\mathbb{R}^2 \ni x \mapsto X_t^x(\omega) \in \mathbb{R}^2$$

is a diffeomorphism and its differential has determinant everywhere equal to 1.

Kunita (in [7], page 218) showed that, if $d \in \mathbb{N}$, $b \in BC([0,T], C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d))$ and $\sigma \in BC([0,T], C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^{d\times d}))$, the SDE

$$\begin{aligned} dX_t^x &= b(t, X_t^x)\,dt + \sigma(t, X_t^x)\,dW_t, \\ X_0^x &= x, \end{aligned} \tag{4.3}$$

where W_t is a fixed d-dimensional Brownian motion on $(\Omega, F, (F_t)_{t\geq 0}, P)$, has a solution $(X_t^x)_{t\in[0,T]}$ such that for all $\beta < \alpha$, for P-a.a. ω, the map $x \mapsto X_t^x(\omega)$ is in $C^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d)$. We note that Kunita requires more smoothness for the derivatives: our case is not contained in Kunita's theorem. We do not need any Hölder continuity for the derivatives, because our SDE has an additive noise.

PROOF OF THEOREM 4.1.1. Since the noise is additive, we can construct a continuous solution of (4.1) in the following way. Take a version W_t of the fixed BM such that all the paths are continuous. For all $\omega \in \Omega$ and $x \in \mathbb{R}^2$, the equation

$$Y_t^x(\omega) = x + \int_0^t u(s, Y_s^x(\omega) + \mu W_s(\omega))\,ds \qquad \forall\ t \in [0,T]$$

has a unique solution (u is uniformly Lipschitzian in x). We denote $(Y_t^x(\omega))_{t\geq 0}$ that solution and set

$$Y_t^x(\omega) = X_t^x(\omega) + \mu W_t(\omega) \qquad \forall\ x, \forall\ t, \forall\ \omega.$$

Clearly, for each x and ω, we have

$$X_t^x(\omega) = x + \int_0^t u(s, X_s^x(\omega)\, ds + \mu W_s(\omega)) \qquad \forall\ t \in [0, T]. \tag{4.4}$$

It is not difficult to prove that, for all fixed $T > 0$ and ω, the function $[0, T] \times \mathbb{R}^2 \ni (t, x) \mapsto X_t^x(\omega) \in \mathbb{R}^2$ is uniformly Lipschitz continuous in x. Since, by definition, for all x and ω, the map $t \mapsto X_t^x(\omega)$ is continuous, we conclude that for all ω the map $(t, x) \mapsto X_t^x(\omega)$ is continuous. We underline that for a fixed BM we have built a continuous solution which solves (4.4) for all fixed ω and x.

Fixing $t \in [0, T]$ and $\omega \in \Omega$, we see that $X_t^x(\omega) = Y_t^x(\omega) + \mathit{constant}$ [where the constant is $\mu W_t(\omega)$], so it is enough to check that for $\forall\ t \geq 0$ for P-almost all ω, the function

$$\mathbb{R}^2 \ni x \mapsto Y_t^x(\omega) \in \mathbb{R}^2$$

is a diffeomorphism and its Jacobian is equal to the constant 1.

Fix ω. For all $x \in \mathbb{R}^2$, the function $[0, +\infty[\ni t \mapsto Y_t^x(\omega) \in \mathbb{R}^2$ is the solution of the Cauchy problem

$$v'(t) = u\big(t, v(t) + \mu W_t(\omega)\big) \qquad \forall\ t \geq 0,$$
$$v(0) = x.$$

Since u is continuous, bounded and uniformly Lipschitz in x, for every given $s \geq 0$ and $x \in \mathbb{R}^2$, the problem

$$v'(t) = u\big(t, v(t) + \mu W_t(\omega)\big) \qquad \forall\ t \geq 0,$$
$$v(s) = x,$$

has a unique solution in $[0, +\infty[$. It follows that $\mathbb{R}^2 \ni x \mapsto Y_t^x(\omega) \in \mathbb{R}^2$ is bijective. As u has partial derivatives with respect to x in C_b, the function $\mathbb{R}^2 \ni x \mapsto Y_t^x(\omega) \in \mathbb{R}^2$ is differentiable and $\partial Y(t, x)(\omega)/\partial x_i$ solves the Cauchy problem

$$v'(t) = D_x u\big(t, Y_t^x(\omega) + \mu W_t(\omega)\big) v(t),$$
$$v(0) = e_i.$$

In view of the fact that $\operatorname{div}_x u = 0$, we have $\operatorname{Tr}[d_x u] = 0$ and

$$\det\left(\frac{\partial Y_t^x(\omega)}{\partial x_1}, \frac{\partial Y_t^x(\omega)}{\partial x_2}\right) = 1 \qquad \forall\ x, \forall\ t.$$

Therefore, for all fixed t and ω, the map $x \mapsto X_t^x(\omega)$ is a diffeomorphism with Jacobian everywhere equal to 1. We conclude by noting that, if Z_t^x is a continuous solution of (4.1), then $P(Z_t^x = X_t^x, \forall\ t, \forall\ x) = 1$. □

We are now able to prove the following theorem.

THEOREM 4.1.2. *Suppose that u satisfies hypotheses* (A1)–(A3). *Let X_t^x be a continuous solution of* (4.1) *and fix $t > 0$. Then, for P-almost all ω, we have*

$$\int_{\mathbb{R}^2} f(X_t^x(\omega))\, dx = \int_{\mathbb{R}^2} f(x)\, dx \qquad \forall\ f \in L^1.$$

Therefore, the operators P_t can be extended to contractions on L^1.

PROOF. Choosing ω so that $x \mapsto X_t^x(\omega)$ is a diffeomorphism with Jacobian everywhere equal to 1, by changing the variable in the integral, we obtain

$$\int_{\mathbb{R}^2} f(X_t^x(\omega))\,dx = \int_{\mathbb{R}^2} f(x)\,dx.$$

Since the preceding inequality holds for $P-$ a.a. ω, and the function $(x,\omega) \mapsto f(X_t^x(\omega))$ is measurable, by changing the integration order, we obtain

$$\int_{\mathbb{R}^2} E[|f(X_t^x)|]\,dx = E\left[\int_{\mathbb{R}^2} |f(X_t^x)|\,dx\right] = \int_{\mathbb{R}^2} |f(x)|\,dx. \tag{4.5}$$

Therefore, the integral $E[f(X_t^x)]$ converges for almost all x and the function $x \mapsto E[f(X_t^x)]$ belongs to L^1. Hence operators P_t can be extended to L^1, and, by (4.5) those extensions do not increase the L^1 norm. (We cannot conclude that they preserve the L^1 norm, since in general $|E[f(X_t^x)]|$ does not coincide with $E[|f(X_t^x)|]$.) $\square$

We can extend the result for $p > 1$.

THEOREM 4.1.3. *Suppose that u satisfies hypotheses* (A1)–(A3). *Let $f \in L^p$ for some $p \in [1,\infty[$ ($f \in L^\infty$ and is Borel measurable), and fix $t > 0$. Then the integral $E[f(X_t^x)]$ converges for almost all $x \in \mathbb{R}^2$ and the function $x \mapsto E[f(X_t^x)]$ belongs to L^p (belongs to L^∞ and is Borel measurable). Moreover,*

$$\|E[f(X_t^x)]\|_p \le \|f\|_p, \qquad (\|E[f(X_t^x)]\|_\infty \le \|f\|_\infty).$$

PROOF. The case $p = \infty$ is trivial. Let $p \in [1,\infty[$. By the Jensen inequality we have

$$\left(\int_{\mathbb{R}^2} |E[f(X_t^x)]|^p\,dx\right)^{1/p} \le \left(\int_{\mathbb{R}^2} E[|f(X_t^x)|^p]\,dx\right)^{1/p} = \|f\|_p \tag{4.6}$$

So $E[|f(X_t^x)|] < +\infty$ for almost all $x \in \mathbb{R}^2$, and the integral $E[f(X_t^x)]$ converges for almost all $x \in \mathbb{R}^2$. Finally, by (4.6), the map $x \mapsto E[f(X_t^x)]$ belongs to L^p, and

$$\|E[f(X_t^x)]\|_p \le \|f\|_p. \qquad \square$$

4.2. *When the drift is in $BC([0,\infty[, C_b)$.* We refer again to the SDE (4.1) and suppose that u satisfies weaker hypotheses.

A1*. u belongs to $BC([0,+\infty[, C_b)$.
A2*. $\mathrm{div}_x\, u = 0$ in the distributional sense.

We remark that, if $\xi \in BC([0,+\infty[, L^p \cap L^q)$ for some p and q such that $1 \le p < 2 < q$, the function

$$u(t,x) = -\frac{1}{2}\int_0^\infty \frac{1}{s} E\left[\xi(t, x + W_s)W_s^{\perp}\right] ds$$

satisfies hypotheses A1* and A2*.

Fix a two-dimensional BM (Ω, F, F_t, W_t, P). We will approach u with a suitable sequence $\{u_n\}$ in $C[0,\infty[, C_b^1)$ and obtain inequality (4.2) for a fixed function $f \in L^1$ by passing to the limit as $n \to \infty$ in the sequence $E[f(x + W_t)Z_t^{x,u_n}]$.

Let $\{\rho_n\}$ be a sequence of mollifiers in $\mathbb{R}^2$ and set $u_n(t,x) = (u(t,\cdot) * \rho_n(\cdot))(x)\ \forall\ n$. One can easily check that, for all n, $u_n \in BC([0,\infty[, C_b^1)$, $\|u_n\|_\infty \le \|u\|_\infty$ and $\operatorname{div}_x u_n = 0$. Moreover, for all $t > 0$, $u_n \to u$ uniformly in $[0,t] \times \mathbb{R}^2$.

Fix $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ such that $f \ge 0$. Then

$$E\big[f(x + \mu W_t)Z_t^{x,u_n}\big] \to E\big[f(x + \mu W_t)Z_t^{x,u}\big] \quad \text{uniformly in } x.$$

In fact, by Lemma 3.0.2,

$$\begin{aligned}
&\big|E\big[f(x + \mu W_t)(Z_t^{x,u_n} - Z_t^{x,u})\big]\big| \\
&\quad \le \|f\|_\infty E\big[|Z_t^{x,u_n} - Z_t^{x,u}|^q\big]^{1/q} \\
&\quad \le \|f\|_\infty \left(c\left(\frac{\|u - u_n\|_{\infty,t}}{\mu}\right)^q t^{q/2} \exp\left(c\frac{2\,|||\,u\,|||_\infty^q}{\mu^q} t^{q/2}\right)\right)^{1/q} \to 0.
\end{aligned}$$

Therefore, in view of Fatou's lemma, we have

$$\int_{\mathbb{R}^2} E\big[f(x + \mu W_t)Z_t^{x,u}\big]\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^2} E\big[f(x + \mu W_t)Z_t^{x,u_n}\big]\,dx = \|f\|_1,$$

which implies

$$\int_{\mathbb{R}^2} E\big[f(x + \mu W_t)Z_t^{x,u}\big]\,dx \le \|f\|_1 \qquad \forall\ f \in L^1 \cap L^\infty,\ f \ge 0.$$

Using once more Fatou's lemma we conclude that, for each positive $f \in L^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} E\big[f(x + \mu W_t)Z_t^{x,u}\big]\,dx \le \|f\|_1.$$

(We have just to apply Fatou's lemma to the sequence $E[\min(f(x + \mu W_t), n)Z_t^{x,u}]$.)

We extend this inequality to all $f \in L^1$ by noting that $|f| = f^+ + f^-$. Hence, for all $f \in L^1(\mathbb{R}^2)$, the integral $\int_{\mathbb{R}^2} E[|f(x + \mu W_t)|Z_t^{x,u}]\,dx$ converges, and therefore for almost all $x \in \mathbb{R}^2$, $E[|f(x + \mu W_t)|Z_t^{x,u}]$ converges. It turns out that we can extend the operators P_t to L^1 and the extensions are contractions on L^1.

We have thus proved the following result.

THEOREM 4.2.1. *Let u satisfy hypotheses* A1* *and* A2* *and let f belong to L^1. Then, for a.a. $x \in \mathbb{R}^2$, the integral $E[f(x + \mu W_t)Z_t^{x,u}]$ converges. Moreover, the function $x \mapsto E[f(x + \mu W_t)Z_t^{x,u}]$ belongs to $L^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} \big|E\big[f(x + \mu W_t)Z_t^{x,u}\big]\big|\,dx \le \|f\|_1.$$

As in the regular case, we can generalize the result for $p > 1$.

THEOREM 4.2.2. *Let $f \in L^p$ for some $p \in [1,\infty[$ ($f \in L^\infty$ and Borel measurable) and $t \in]0,\infty[$. Then, for almost all $x \in \mathbb{R}^2$, the integral $E[f(X_t^x)]$ converges. Moreover, the function*

$$x \mapsto E\left[f(X_t^x)\right]$$

belongs to L^p (belongs to L^∞ and is Borel measurable) and

$$\left\| E\left[f(X_t^x)\right]\right\|_p \le \|f\|_p, \qquad \left(\|E\left[f(X_t^x)\right]\|_\infty \le \|f\|_\infty\right).$$

5. Existence and uniqueness of the solution. Fix p, q in $[1,\infty[$, such that $1 \le p < 2 < q$. In this section we prove that, if the initial vorticity is in $L^p \cap L^q$, then (1.4) has a unique solution in $BC([0,\infty[, C_0 \times L^p \cap L^q)$. The section is organized as follows. In Section 5.1 we build two maps between the sets $BC([0,\infty[, C_0)$ and $BC([0,\infty[, L^p \cap L^q)$ and write our system in terms of those maps. Then, in Section 5.2, we prove the existence and uniqueness of a local solution (u,ξ) in $BC([0,\tau], C_0 \times L^p \cap L^q)$ using Banach's fixed point theorem. Finally, in Section 5.3 we extend the local solution by the Markov property and the a priori estimate provided by Theorem 4.2.2.

5.1. *The operators S and T.* The space $BC([0,\infty[, C_0)$ endowed with the norm $|||\, u \,|||_\infty = \sup_t \|u(t,\cdot)\|_\infty$ and the space $BC([0,\infty[, L^p \cap L^q)$ with the norm $|||\, \xi \,|||_{p,q} = \sup_t \|\xi(t,\cdot)\|_{p,q}$ are two Banach spaces. In the following we will always refer to these norms.

We now construct the map which will represent the last equation in system (1.4).

Denote by A the closed subspace of $BC([0,\infty[, C_0)$,

$$A = \left\{u \in BC\left([0,\infty[, C_0\right) \mid \operatorname{div}_x u = 0 \text{ in the distributional sense}\right\}.$$

Recall the map defined in Section 2:

$$s\colon L^p \cap L^q \ni f \mapsto -\frac{1}{2}\int_0^\infty \frac{1}{s} E\left[f(x+W_s)W_s{}^\perp\right] ds \in C_0.$$

As we showed, this map is well defined, linear and bounded and, for all $f \in L^p \cap L^q$, $\operatorname{div}_x s(f) = 0$ in the distributional sense. Hence the operator

$$S\colon BC\left([0,\infty[, L^p \cap L^q\right) \to A,$$

defined by

$$S\xi(t,x) = s(\xi(t,\cdot))(x) = -\frac{1}{2}\int_0^\infty \frac{1}{s} E\left[\xi(t, x+W_s)W_s{}^\perp\right] ds,$$

is well defined, linear and bounded.

Next we build a map in order to represent the first equation in (1.4). For each ξ_0 in $L^p \cap L^q$, we define

$$T^{\xi_0}\colon A \to L^\infty\left([0,\infty[, L^p \cap L^q\right)$$

by

$$T^{\xi_0} u(t,x) = P_t^{t,u} \xi_0(x),$$

where $(P_s^{t,u})_{s\in[0,t]}$ are the operators corresponding to the SDE,

$$
\begin{aligned}
dX_s^{x,t,u} &= -u\big(t-s, X_s^{x,t,u}\big)\,ds + \sqrt{2\nu}\,dW_s,\\
X_0^{x,t,u} &= x.
\end{aligned}
\tag{5.1}
$$

By Theorem 4.2.2, the operators T^{ξ_0} are well defined and

$$
|||T^{\xi_0}u|||_{p,q} \le \|\xi\|_{p,q} \qquad \forall\ u\in A. \tag{5.2}
$$

We would like the operators T^{ξ_0} to take values in the domain of S, namely in $BC([0,\infty[, L^p\cap L^q)$, and this actually occurs.

THEOREM 5.1.1. $T^{\xi_0}(A)\subset BC([0,\infty[, L^p\cap L^q)$.

PROOF. We can suppose that $p>1$. Take s and t in $[0,\infty[$. We have to check that the norm in $L^p\cap L^q$ of $|P_t^{t,u}\xi_0(x)-P_s^{s,u}\xi_0(x)|$ goes to 0 as $|t-s|$ tends to 0. Fix a two-dimensional BM (Ω, F, F_t, W_t, P). By Girsanov's formula,

$$
\begin{aligned}
&\big|P_t^{t,u}\xi_0(x)-P_s^{s,u}\xi_0(x)\big|\\
&\quad=\Big|E\big[\xi_0\big(x+\sqrt{2\nu}\,W_t\big)Z_t^{x,t,u}-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)Z_s^{x,s,u}\big]\Big|,
\end{aligned}
$$

where $(Z_r^{x,t,u})_{r\in[0,t]}$ and $(Z_r^{x,s,u})_{r\in[0,s]}$ are the Girsanov densities corresponding, respectively, to the drifts $u(t-r,x)$ and $u(s-r,x)$.

Adding and subtracting $E[\xi_0(x+\sqrt{2\nu}\,W_s)Z_t^{x,t,u}]$ we find

$$
\begin{aligned}
&\Big|E\big[\xi_0\big(X_t^{x,t,u}\big)-\xi_0\big(X_s^{x,s,u}\big)\big]\Big|\\
&\quad\le\Big|E\Big[\Big(\xi_0\big(x+\sqrt{2\nu}\,W_t\big)-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\Big)Z_t^{x,t,u}\Big]\Big|\\
&\qquad+\Big|E\Big[\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\big(Z_t^{x,t,u}-Z_s^{x,s,u}\big)\Big]\Big|
\end{aligned}
\tag{5.3}
$$

We show that the L^p norm and the L^q norm of this sum go to 0 as s tends to t. Concerning the L^p norm of the first addendum, we have

$$
\begin{aligned}
&\Big\|E\Big[\Big(\xi_0\big(x+\sqrt{2\nu}\,W_t\big)-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\Big)Z_t^{x,t,u}\Big]\Big\|_p\\
&\quad\le\Big\|E\Big[\big|\xi_0\big(x+\sqrt{2\nu}\,W_t\big)-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\big|^p\Big]^{1/p}E\Big[\big|Z_t^{x,t,u}\big|^{p'}\Big]^{1/p'}\Big\|_p\\
&\quad\le\left(E\left[\int_{\mathbb{R}^2}\big|\xi_0\big(x+\sqrt{2\nu}\,W_t\big)-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\big|^p\,dx\right]\right)^{1/p}\\
&\qquad\times ct^{1/2}\exp\left\{\frac{1}{p'}c\frac{\|u\|_\infty^{p'}}{(2\nu)^{p'/2}}t^{p'/2}\right\}\to 0,
\end{aligned}
$$

where the convergence is due to the continuity of the shift in L^p. In the same way one can check that

$$
\Big\|E\Big[\Big(\xi_0\big(x+\sqrt{2\nu}\,W_t\big)-\xi_0\big(x+\sqrt{2\nu}\,W_s\big)\Big)Z_t^{x,t,u}\Big]\Big\|_q\to 0.
$$

Consider now the L^p norm of the second addendum in (5.3). Suppose that $s < t$. We have

$$\begin{aligned}
&\left\| E\left[\xi_0\left(x+\sqrt{2\nu}W_s\right)\left(Z_t^{x,t,u}-Z_s^{x,s,u}\right)\right]\right\|_p \\
&\quad \le \left\| E\left[|\xi_0\left(x+\sqrt{2\nu}W_s\right)|^p\right]^{1/p} E\left[|Z_t^{x,t,u}-Z_s^{x,s,u}|^{p'}\right]^{1/p'}\right\|_p \\
&\quad \le \|\xi_0\|_p \sup_x E\left[|Z_t^{x,t,u}-Z_s^{x,s,u}|^{p'}\right]^{1/p'} \\
&\quad \le \|\xi_0\|_p \sup_x \left(E\left[|Z_t^{x,t,u}-Z_s^{x,t,u}|^{p'}\right]^{1/p'} + E\left[|Z_s^{x,t,u}-Z_s^{x,s,u}|^{p'}\right]^{1/p'}\right).
\end{aligned}$$

In view of Lemma 3.0.2, we get

$$\begin{aligned}
&\sup_x E\left[|Z_s^{x,t,u}-Z_s^{x,s,u}|^{p'}\right]^{1/p'} \\
&\quad \le \frac{c}{\sqrt{2\nu}} \sup_{r\in[0,s]} |u(t-r,\cdot)-u(s-r,\cdot)|_\infty s^{1/2} \exp\left\{ c s^{p'/2} \frac{2 \,|||\, u \,|||_\infty^{p'}}{(2\nu)^{p'/2}}\right\} \\
&\quad \to 0,
\end{aligned}$$

and, by Lemma 3.0.1,

$$\begin{aligned}
&E\left[|Z_t^{x,t,u}-Z_s^{x,t,u}|^{p'}\right]^{1/p'} \\
&\quad = E\left[\left|\int_s^t \frac{1}{\sqrt{2\nu}} u\left(t-r, x+\sqrt{2\nu}W_r\right) Z_r^{x,t,u}\, dW_r\right|^{p'}\right]^{1/p'} \\
&\quad \le cE\left[(t-s)^{(p'/2)-1}\int_s^t \left|\frac{1}{\sqrt{2\nu}} u\left(t-r, x+\sqrt{2\nu}W_r\right)\right|^{p'} Z_r^{x,t,u}|^{p'}\, dr\right]^{1/p'} \\
&\quad \le c\frac{|||\, u \,|||_\infty}{\sqrt{2\nu}} (t-s)^{1/2} \exp\left\{\frac{c}{p'}\left(\frac{\|u\|_\infty}{\mu}\right)^{p'} t^{p'/2}\right\} \to 0.
\end{aligned}$$

Therefore,

$$\sup_x E\left[|Z_s^{x,t,u}-Z_s^{x,s,u}|^{p'}\right]^{1/p'} \to 0.$$

It turns out that

$$\left\| E\left[\xi_0\left(x+\sqrt{2\nu}W_s\right)\left(Z_t^{x,t,u}-Z_s^{x,s,u}\right)\right]\right\|_p \to 0.$$

The proof that

$$\left\| E\left[\xi_0\left(x+\sqrt{2\nu}W_s\right)\left(Z_t^{x,t,u}-Z_s^{x,s,u}\right)\right]\right\|_q \to 0$$

can be carried out the same way. We have so proved the continuity of the map

$$\left[0,\infty\right[\ni t \mapsto E\left[\xi_0\left(x+\sqrt{2\nu}W_t\right) Z_t^{x,t,u}\right] \in L^p \cap L^q. \qquad \square$$

Due to the above theorem and to inequality (5.2), the operator T^{ξ_0} maps A into the set $B_{\|\xi_0\|_{p,q}} = \{\xi \in BC([0,\infty[, \ L^p \cap L^q\ ) \,|\, |||\xi|||_{p,q} \le \|\xi_0\|_{p,q}\}$. Moreover, by Proposition 2.2.1, S maps $B_{\|\xi_0\|p,q}$ into the closed subset of A,

$$A_{a\|\xi_0\|p,q} = \{u \in A \mid |||u|||_\infty \le a\|\xi_0\|_{p,q}\},$$

where $a = \frac{1}{2}c_{p,q}$. Therefore, by the definition of T^{ξ_0} and S and by the characterizations of their images, we see that a pair (ξ, u) in $BC([0,\infty[, C_0 \times L^p \cap L^q)$ solves system (1.4) if and only if it is a solution of

$$\begin{aligned} (u,\xi) &\in A_{a\|\xi_0\|_{p,q}} \times B_{\|\xi_0\|_{p,q}},\\ u &= S\xi,\\ \xi &= T^{\xi_0}u. \end{aligned} \tag{5.4}$$

5.2. *Existence and uniqueness of a local solution.* A solution of (5.4) is a pair (u,ξ) in $A_{a\|\xi_0\|_{p,q}} \times B_{\|\xi_0\|p,q}$ such that $u = ST^{\xi_0}u$ and $\xi = T^{\xi_0}u$. Since $A_{a\|\xi_0\|_{p,q}}$ is a Banach space, if $S \cdot T^{\xi_0}$ is a contraction, it follows from the Banach fixed point theorem that system (5.4) has a unique solution. So far we know that S is linear and continuous and therefore Lipschitz. What about T^{ξ_0}? According to Lemma 3.0.2,

$$\begin{aligned} &\sup_{t\in[0,\tau]} \|Tu(t,\cdot) - Tv(t,\cdot)\|_{p,q}\\ &\quad \le \|\xi_0\|_{p,q} \sup_{\mathbb{R}^2} E\big[|Z_t^{x,t,u} - Z_t^{x,t,v}|^q\big]^{1/q}\\ &\quad \le \|\xi_0\|_{p,q}\tau^{1/2}\frac{c}{\sqrt{2\nu}}\|u - v\|_\infty \exp\left(\frac{c}{q}\left(\frac{\|u\|_\infty^q + \|v\|_\infty^q}{(2\nu)^{q/2}}\right)\tau^{q/2}\right)\\ &\quad \le \|\xi_0\|_{p,q}\tau^{1/2}\frac{c}{\sqrt{2\nu}}\|u - v\|_\infty \exp\left(\frac{c}{q}\left(\frac{2a\|\xi_0\|_{p,q}^q}{(2\nu)^{q/2}}\right)\tau^{q/2}\right). \end{aligned}$$

That is,

$$\begin{aligned} &\sup_{t\in[0,\tau]} \|Tu(t,\cdot) - Tv(t,\cdot)\|_{p,q}\\ &\quad \le \left(\|\xi_0\|_{p,q}\tau^{1/2}\frac{c}{\sqrt{2\nu}}\exp\left(\frac{c}{q}\left(\frac{2a\|\xi_0\|_{p,q}^q}{(2\nu)^{q/2}}\right)\tau^{q/2}\right)\right)\|u - v\|_\infty. \end{aligned}$$

So the map $T^{\xi_0}\colon A_{a\|\xi_0\|p,q} \to B_{\|\xi_0\|p,q}$ is continuous. Moreover, if we fix τ such that

$$a\|\xi_0\|_{p,q}\tau^{1/2}\frac{c}{\sqrt{2\nu}}\exp\left(\frac{c}{q}\left(\frac{2a\|\xi_0\|_{p,q}^q}{(2\nu)^{q/2}}\right)\tau^{q/2}\right) \le 1, \tag{5.5}$$

and consider the Banach spaces

$$\begin{aligned} A_{\tau, a\|\xi_0\|_{p,q}} = \big\{u \in BC([0,\tau], C_0) \mid \operatorname{div}_x u = 0 \text{ in distribution},&\\ \|u(t,\cdot)\|_\infty \le a\|\xi_0\|_{p,q}\ \ \forall\ t \in [0,\tau]\big\}& \end{aligned}$$

and

$$B_{\tau, \|\xi_0\|_{p,q}} = \{u \in BC([0,\tau], L^p \cap L^q) \mid \|\xi(t,.)\|_{p,q} \leq a\|\xi_0\|_{p,q} \quad \forall\ t \in [0,\tau]\},$$

the map $S \cdot T^{\xi_0}\colon A_{\tau, a\|\xi_0\|_{p,q}} \to A_{\tau, a\|\xi_0\|_{p,q}}$ is a contraction. Hence, if τ satisfies (5.5), then (5.4) has one and only one solution in $[0,\tau]$, and the sequence (u_n, ξ_n) in $A_{\tau, a\|\xi_0\|_{p,q}} \times B_{\tau, \xi_0}$ defined by

$$\begin{aligned} u_0 &= 0, \\ u_{n+1} &= ST^{\xi_0} u_n, \\ \xi_n &= T^{\xi_0} u_{n-1} \end{aligned} \tag{5.6}$$

converges to the solution.

5.3. *Existence and uniqueness of a global solution.* Let $u \in A$ and $0 \leq s \leq t$ and consider the map

$$P^u_{s,t}\colon L^p \cap L^q \to L^p \cap L^q,$$

defined by

$$P^u_{s,t}\varphi(x) = E[\varphi(X^{s,x,u}_t)] \qquad \forall\ \varphi \in L^p \cap L^q,$$

where $(X^{s,x,u}_r)_{r \geq s}$ is a solution of the SDE,

$$\begin{aligned} dX^{s,x,u}_r &= u(r, X^{s,x,u}_r)\, dr + \mu\, dW_r, \\ X^{s,x,u}_s &= x. \end{aligned}$$

The maps $P^u_{s,t}$ are linear and bounded, and, for all $0 \leq s < t$, $\|P^u_{t,s}\|_{L(L^p \cap L^q)} < 1$. We prove that these operators satisfy a Markov property.

Let $\{u_n\}$ be a sequence in $BC([0,\infty[, C^1_b) \cap A$ that converges to u in $BC([0,T], C_b)$ for each $T > 0$. Using Girsanov's formula and Lemma 3.0.2, we obtain that, for all $0 \leq s < t$,

$$P^{u_n}_{s,t} \to P^u_{s,t} \text{ in } L(L^p \cap L^q).$$

For each $n \in \mathbb{N}$, the family $\{P^{u_n}_{s,t} \mid 0 \leq s \leq t\}$ satisfies the Kolmogorov–Chapman condition, namely,

$$P^{u_n}_{s,t} = P^{u_n}_{s,r} P^{u_n}_{r,t} \qquad \forall\ 0 \leq s < r < t.$$

By passing to the limits as $n \to \infty$, we find that this condition holds even for the operators $\{P^u_{s,t} \mid 0 \leq s \leq t\}$. In fact, we have

$$\begin{aligned} &\|P^{u_n}_{s,r} P^{u_n}_{r,t} - P^u_{s,r} P^u_{r,t}\| \\ &\quad \leq \|P^{u_n}_{s,r}(P^{u_n}_{r,t} - P^u_{r,t})\| + \|(P^{u_n}_{s,r} - P^u_{s,r}) P^u_{r,t}\| \\ &\quad \leq \|P^{u_n}_{s,r}\| \cdot \|P^{u_n}_{r,t} - P^u_{r,t}\| + \|P^{u_n}_{s,r} - P^u_{s,r}\| \cdot \|P^u_{r,t}\| \\ &\quad \leq \|P^{u_n}_{r,t} - P^u_{r,t}\| + \|P^{u_n}_{s,r} - P^u_{s,r}\| \to 0. \end{aligned}$$

In the last inequality above we have used the fact that all the operators $P^{u_n}_{r,s}$ and $P^u_{r,s}$ are contractions. We can conclude that, for all $u \in A$,

$$P^u_{s,t} = P^u_{s,r} P^u_{r,t} \qquad \forall\ 0 \leq s < r < t.$$

Applying this property we will prove inductively the existence and uniqueness of a long-time solution.

THEOREM 5.3.1. *Let* $\xi_0 \in L^p \cap L^q$, *for some* $1 \le p < 2 < 2$. *Then the system*

$$u = S\xi, \qquad \xi = T^{\xi_0} u \tag{5.7}$$

has a unique global solution.

PROOF. Choose τ so that (5.5) is verified. We already know that (5.7) admits one and only one solution in $[0, \tau]$.

Consider now the system

$$\begin{aligned} u(t, x) &= S\xi(t, x) \qquad \forall (t, x) \in [0, 2\tau] \times \mathbb{R}^2, \\ \xi(t, x) &= T^{\xi_0} u(t, x) \qquad \forall (t, x) \in [0, 2\tau] \times \mathbb{R}^2. \end{aligned} \tag{5.8}$$

For all $t \in [\tau, 2\tau]$ we have

$$\begin{aligned} Tu^{\xi_0}(t, x) &= E\big[\xi_0\big(X_t^{x,t,u}\big)\big] \\ &= P_{0,t}^{u(t-\cdot,\cdot)}\xi_0(x) \\ &= P_{0,t-\tau}^{u(t-\cdot,\cdot)} P_{t-\tau,t}^{u(t-\cdot,\cdot)}\xi_0(x) \\ &= \big(P_{0,t-\tau}^{u(t-\cdot,\cdot)}\xi(\tau,\cdot)\big)(x) \\ &= \big(P_{0,t-\tau}^{\tilde{u}(t-\tau-\cdot,\cdot)}\xi(\tau,\cdot)\big)(x), \end{aligned}$$

where we used the equivalence $P_{t-\tau,t}^{u(t-\cdot,\cdot)} = P_{0,\tau}^{u(\tau-\cdot,\cdot)}$ and the notation $\tilde{u}(s, \cdot) = u(s + \tau, \cdot)$ $\forall s \in [0, \tau]$. That yields

$$T^{\xi_0} u(t, x) = T^{\xi(\tau,\cdot)}\tilde{u}(t - \tau, x).$$

Set $\tilde{\xi}(s, \cdot) = \xi(s + \tau, \cdot)$ $\forall\ s \in [0, \tau]$. Clearly $(u, \xi)_{|[0,2\tau]\times\mathbb{R}^2}$ satisfies (5.8) if and only if $(u, \xi)_{|[0,\tau]\times\mathbb{R}^2}$ satisfies (5.7) in $[0, \tau]$ and $(\tilde{u}, \tilde{\xi})_{|[0,2\tau]\times\mathbb{R}^2}$ solves

$$\begin{aligned} \tilde{u}(t, x) &= S\tilde{\xi}(t, x) \qquad \forall\ (t, x) \in [0, \tau] \times \mathbb{R}^2, \\ \tilde{\xi}(t, x) &= T^{\xi_0}\tilde{u}(t, x) \qquad \forall (t, x) \in [0, \tau] \times \mathbb{R}^2. \end{aligned} \tag{5.9}$$

Since we have the a priori estimate $\|\xi(\tau, \cdot)\|_{p,q} \le \|\xi_0\|_{p,q}$, this system admits a unique solution. Therefore (5.7) has exactly one solution in $[0, \tau]$. So one can prove that it has one and only one global solution. □

6. Invariance of Lebesgue measure with respect to certain SDE. Let ξ_0 belong to $L^p \cap L^\infty$ for some $p \in [1, 2[$. Then the fluid velocity is "quasi-Lipschitz" continuous in x. That result relies on the following proposition.

PROPOSITION 6.0.1. *If $f \in L^p \cap L^\infty$ with $1 \le p < 2$, then the function*

$$u(x) = -\frac{1}{2}\int_0^\infty \frac{1}{s} E\left[f(x+W_s) W_s^{\perp} \right] ds$$

satisfies the "quasi-Lipschitz" estimate,

$$|u(x) - u(y)| \le b \|f\|_{p,\infty} \varphi(|x-y|) \qquad \forall\ x, y \in \mathbb{R}^2,$$

where

$$\varphi(r) = \begin{cases} r, & \text{if } r > 1, \\ r(1 - \log r), & \text{if } r \le 1 \end{cases},$$

and b is a constant which does not depend on f.

(For the proof see [2]).

Since ξ_0 belongs to $L^p \cap L^\infty$, by Theorem 4.2.2, the vorticity is in $L^\infty([0,\infty[, L^p \cap L^\infty)$. Hence, in view of the preceding proposition, the velocity u satisfies the following "quasi-Lipschitz" estimate:

$$|u(t,x) - u(t,y)| \le c\varphi(|x-y|) \qquad \forall\ x, y \in \mathbb{R}^2, \tag{6.1}$$

where $c = b \sup_t \|\xi\|_{p,\infty}$.

Now consider the SDE,

$$\begin{aligned} dX_t^x &= u(t, X_t^x)\, dt + \mu\, dW_t, \\ X_0^x &= x, \end{aligned} \tag{6.2}$$

where μ is a positive constant and the drift u belongs to $BC([0,\infty[, C_b)$ and satisfies (6.1). In this section we show that, for each given two-dimensional Brownian motion (Ω, F, F_t, W_t, P), (6.2) has a continuous solution X_t^x, and, if u is divergence free in the distributional sense, then for P-almost all fixed ω, the maps $x \mapsto X_t^x(\omega)$ preserve Lebesgue measure; that is, for all fixed t,

$$\int_{\mathbb{R}^2} f(X_t^x(\omega))\, dx = \int_{\mathbb{R}^2} f(x)\, dx \qquad \forall\ f \in L^1.$$

We first prove it for small t and then for all times.

THEOREM 6.0.2. *Suppose that u belongs to $BC([0,\infty[, C_b)$ and satisfies* (6.1); *choose τ such that*

$$\tau \le \frac{1}{2(\sup_r \|u_r\|_\infty + c)}. \tag{6.3}$$

Consider the SDE,

$$\begin{aligned} dX_t^{s,x} &= u(t, X_t^{s,x})\, dt + \mu\, dW_t, \qquad t \ge s, \\ X_s^{s,x} &= x. \end{aligned} \tag{6.4}$$

For all fixed two-dimensional Brownian motion (Ω, F, F_t, W_t, P), (6.4) *has a local solution $(X_r^{s,x})_{s \le r \le s+\tau}$ such that, for P-almost all ω, the map*

$$[s, s+\tau] \times \mathbb{R}^2 \ni (t,x) \to X_t^x(\omega) \in \mathbb{R}^2$$

is continuous.

PROOF. Let $\{\rho_n\}$ be a sequence of mollifiers and set $u_n = u * \rho_n \ \forall \ n$. The sequence $\{u_n\}$ belongs to $BC([0, +\infty[, C_0^\infty)$ and, for all n in $\mathbb{N}$, $\|u_n\|_\infty \le \|u\|_\infty$. Furthermore, the functions u_n satisfy estimate (6.1),

$$|u_n(t, x) - u_n(t, y)| = \int_{\mathbb{R}^2} |u(t, x - z) - u(t, y - z)| \rho_n(z)\, dz$$

$$\le c\varphi(|x - y|) \int_{\mathbb{R}^2} \rho_n(z)\, dz = c\varphi(|x - y|).$$

Moreover, since $\{u_t \mid t \in [s, s + \tau]\}$ is uniformly equicontinuous,

$$u_n \to u \quad \text{in } C([s, s + \tau], C_b).$$

For simplicity, throughout this proof we use the notation

$$\|| v \||_{\infty, \tau} = \sup_{t \in [s, s + \tau]} \|v(t, \cdot)\|_\infty \qquad \forall \ v \in BC([0, +\infty[, C_b).$$

Fix a two-dimensional Brownian motion $(\Omega, F, (F_t)_{t \ge 0}(W_t)_{t \ge 0}, P)$ and consider the SDEs,

$$\begin{aligned} dX_r^{s, x, n} &= u_n(r, X_r^{s, x, n})\, dt + \mu\, dW_r, \qquad r \ge s, \\ X_s^{s, x, n} &= x. \end{aligned} \tag{6.5}$$

Since the noise is additive and the drifts u_n are uniformly Lipschitz in x, we can choose a version W_t of our fixed BM such that all the paths are continuous, and then construct, for all n, a continuous solution $(X_r^{s, x, n})_{r \ge s}$ such that, for all ω,

$$X_r^{s, x, n}(\omega) = x + \int_s^r u_n(l, X_l^{s, x, n}(\omega))\, dl + \mu(W_r(\omega) - W_s(\omega)) \tag{6.6}$$

$$\forall \ r \ge s \ge 0, \ \forall \ x \in \mathbb{R}^2$$

(we can use the same technique applied in the proof of Theorem 4.1.1).

Fix $\omega \in \Omega$. In order to prove that the sequence $(X_t^{s, x, n}(\omega))_n$ converges uniformly in $\mathbb{R}^2 \times [s, s + \tau]$, we show now that, for $t \in [s, s + \tau]$, we can control the distance $|X_t^{s, x, n}(\omega) - X_t^{s, x, m}(\omega)|$ with $\|u_n - u_m\|_{\infty, \tau}$. For all $t \ge s$, set

$$S_{n, m}^{x, t}(\omega) = \sup\{l \in [s, t] \mid |X_l^{s, x, n}(\omega) - X_l^{s, x, m}(\omega)| \le \|u_n - u_m\|_{\infty, \tau}\}.$$

Note that $S_{n, m}^{x, t}$ is not a stopping time: it depends on the future. Nevertheless, in view of (6.6), we have

$$\begin{aligned} X_t^{s, x, n}(\omega) - X_t^{s, x, m}(\omega) &= X_{S_{n, m}}^{s, x, n}(\omega) - X_{S_{n, m}}^{s, x, m}(\omega) \\ &\quad + \int_{S_{n, m}}^t u_n(l, X_l^{s, x, n}(\omega)) - u_m(l, X_l^{s, x, m}(\omega))\, dl. \end{aligned}$$

That yields

$$
\begin{aligned}
&\left|X_t^{s,x,n}(\omega)-X_t^{s,x,m}(\omega)\right| \\
&\qquad \le \left|X_{S_{n,m}}^{s,x,n}(\omega)-X_{S_{n,m}}^{s,x,m}(\omega)\right| \\
&\qquad\qquad +\int_{S_{n,m}}^t \left|u_n(l,X_l^{s,x,n}(\omega))-u_m(l,X_l^{s,x,m}(\omega))\right| dl \\
&\qquad \le \|u_n-u_m\|_{\infty,\tau}+\int_{S_{n,m}}^t \left|u_n(l,X_l^{s,x,n}(\omega))-u_n(l,X_l^{s,x,m}(\omega))\right| dl \\
&\qquad\qquad +\int_{S_{n,m}}^t \|u_n-u_m\|_\infty \, dl \\
&\qquad \le \|u_n-u_m\|_{\infty,\tau}(t-s+1) \\
&\qquad\qquad +\int_{S_{n,m}}^t c\varphi\left(|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)|\right) dl.
\end{aligned}
$$

Noting that

$$
\left|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)\right| \le 2\|u\|_\infty \tau < 1 \qquad \forall\ l\in[s,s+\tau],
$$

we get

$$
\begin{aligned}
&\left|X_t^{s,x,n}(\omega)-X_t^{s,x,m}(\omega)\right| \\
&\qquad \le \|u_n-u_m\|_{\infty,\tau}(t-s+1) \\
&\qquad\qquad +\int_{S_{n,m}}^t c|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)| \\
&\qquad\qquad\qquad \times \left(1-\log|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)|\right) dl \\
&\qquad \le \|u_n-u_m\|_{\infty,\tau}(t-s+1) \\
&\qquad\qquad +\int_{S_{n,m}}^t c|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)|\left(1-\log\|u_n-u_m\|_{\infty,\tau}\right) dl.
\end{aligned}
$$

Since the sequence $\{u_n\}$ converges uniformly in $[s,s+\tau]\times\mathbb{R}^2$, there exists $\bar{n}$ such that

$$
\forall\ n_1,n_2>\bar{n}, \qquad \|u_{n_1}-u_{n_2}\|_{\infty,\tau}<1.
$$

We take $n,m>\bar{n}$. So we have $\|u_n-u_m\|_{\infty,\tau}<1$ and $1-\log\|u_n-u_m\|_{\infty,\tau}>1$. Hence, for all $t\in[s,s+\tau]$, the inequality

$$
\begin{aligned}
&\left|X_t^{s,x,n}(\omega)-X_t^{s,x,m}(\omega)\right| \\
&\qquad \le \|u_n-u_m\|_{\infty,\tau}(t-s+1) \\
&\qquad\qquad +\int_s^t c|X_l^{s,x,n}(\omega)-X_l^{s,x,m}(\omega)|\left(1-\log\|u_n-u_m\|_{\infty,\tau}\right) dl
\end{aligned}
$$

holds and, applying Gronwall's lemma, we get

$$\begin{aligned}
&\left| X_t^{s,x,n}(\omega) - X_t^{s,x,m}(\omega) \right| \\
&\quad \le \|u_n - u_m\|_{\infty,\tau}(t-s+1)\exp\big(c(1-\log\|u_n-u_m\|_{\infty,\tau})(t-s)\big) \\
&\quad \le \|u_n - u_m\|_{\infty,\tau}(\tau+1)e^{c\tau}\left(\frac{1}{\|u_n-u_m\|_{\infty,\tau}}\right)^{c\tau} \\
&\quad = \|u_n - u_m\|_{\infty,\tau}^{1-c\tau}(\tau+1)e^{c\tau}.
\end{aligned}$$

That is,

$$\left| X_t^{s,x,n}(\omega) - X_t^{s,x,m}(\omega) \right| \le (\tau+1)e^{c\tau}\|u_n - u_m\|_{\infty,\tau}^{1-c\tau}.$$

By (6.3), $1 - c\tau > 0$, and therefore $\{X_t^{s,x,n}(\omega)\}$ converges uniformly in $[s, s+\tau] \times \mathbb{R}^2$. Denote

$$X_t^{s,x}(\omega) = \lim_{n\to\infty} X_t^{s,x,n}(\omega) \qquad \forall\ x \in \mathbb{R}^2,\ \forall\ t \in [s, s+\tau],\ \forall\ \omega \in \Omega.$$

That process is clearly continuous. Moreover, according to the dominated convergence theorem,

$$\int_s^t u_n(l, X_l^{s,x,n}(\omega))\,dl \to \int_s^t u(l, X_l^{s,x}(\omega))\,ds \qquad \forall\ t \in [s, s+\tau],$$

so that $X_t^{s,x}$ solves (6.4) in $[s, s+\tau]$. □

THEOREM 6.0.3. *Suppose that u and τ satisfy the hypotheses of the above theorem and that u is divergence free in the distributional sense. Fix a two-dimensional Brownian motion (Ω, F, F_t, W_t, P). Let $(X_r^{s,x})_{s\le r\le s+\tau}$ be a continuous solution of SDE (6.4) with respect to this Brownian motion. Then, for P-almost all ω, $x \mapsto f(X_t^{s,x}(\omega))$ is in L^1 and*

$$\int_{\mathbb{R}^2} f(X_t^{s,x}(\omega))\,dx = \int_{\mathbb{R}^2} f(x)\,dx \qquad \forall\ f \in L^1(\mathbb{R}^2),\ \forall\ t \in [s, s+\tau[.$$

PROOF. To prove the theorem we define a sequence $\{u_n\}$ and construct the corresponding solutions $\{X_t^{s,x,n}\}$ and the limit process $X_t^{s,x}$ as in the preceding proof. Note that, since u is divergence free in the distributional sense, for all n, $\operatorname{div}_x u_n = 0$. Let $\varphi \in C_K^+(\mathbb{R}^2)$. Since $\varphi^{1/2} \in L^1$, by Theorem 4.1.2 we get

$$\int_{\mathbb{R}^2} \varphi^{1/2}(X_t^{s,x,n}(\omega))\,dx = \|\varphi^{1/2}\|_1 \qquad \forall\ n.$$

So, by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left| \varphi(X_t^{s,x,n}) - \varphi(X_t^{s,x,m}) \right| dx \\
&\qquad \le 2\|\varphi^{1/2}\|_1 \sup_{x\in\mathbb{R}^2} \left| \varphi^{1/2}(X_t^{s,x,n}) - \varphi^{1/2}(X_t^{s,x,m}) \right|,
\end{aligned}$$

and, by consequence, $\{\varphi(X_t^{s,x,n})\}$ converges in L^1.

On the other hand, in view of the continuity of φ,

$$\varphi(X_t^{s,x,n}) \to \varphi(X_t^{s,x}) \quad \text{in } L^\infty.$$

Therefore, $\varphi(X_t^{s,x})$ belongs to L^1, the sequence $\{\varphi(X_t^{s,x,n}(\omega))\}$ converges to $\varphi(X_t^{s,x}(\omega))$ in L^1 and

$$\int_{\mathbb{R}^2} \varphi(X_t^{s,x})\,dx = \int_{\mathbb{R}^2} \varphi(x)\,dx.$$

We extend the result for $\varphi \in C_K$, by noting that $(\varphi(X_t^{s,x}))^+ = \varphi^+(X_t^{s,x})$ and $(\varphi(X_t^{s,x}))^- = \varphi^-(X_t^{s,x})$. Finally, using a theorem about monotone classes, we obtain that the preceding equality holds for each $f \in L^1$. □

THEOREM 6.0.4. *Suppose that u is in $BC([0,\infty[, C_b)$ and satisfies* (6.1). *Fix a two-dimensional Brownian motion (Ω, F, F_t, W_t, P) and consider the SDE,*

$$\begin{aligned} dX_t^x &= u(t, X_t^x)\,dt + \mu\,dW_t, \\ X_0^x &= x. \end{aligned} \tag{6.7}$$

Then (6.7) *has a continuous solution X_t^x. Moreover, if u is divergence free in the distributional sense, then, for P-almost all ω, we have*

$$\int_{\mathbb{R}^2} f(X_t^x(\omega))\,dx = \int_{\mathbb{R}^2} f(x)\,dx \qquad \forall\ f \in L^1\ \forall\ t > 0.$$

In the proof we need the next lemma.

LEMMA 6.0.1. *Let $u \in BC([0,\infty[, C_b)$ and satisfy* (6.1), $s > 0$, $\eta \in L^2(\Omega, F_s)$ *and* $0 \le \tau \le 1/2(|||u|||_\infty + c)$. *Fix a two-dimensional Brownian motion (Ω, F, F_t, W_t, P) and consider the SDE,*

$$\begin{aligned} dX_r^{s,\eta} &= u(r, X_r^{s,\eta})\,dr + \mu\,dW_r, \\ X_s^{s,\eta} &= \eta. \end{aligned} \tag{6.8}$$

Then (6.8) *has a solution in $[s, s+\tau]$. Moreover, for all $\eta_1, \eta_2 \in L^2(\Omega, F_s)$, we have*

$$\begin{aligned} &|X_t^{s,\eta_1}(\omega) - X_t^{s,\eta_2}(\omega)| \\ &\quad\le \begin{cases} |\eta_1(\omega) - \eta_2(\omega)|^{1-c(t-s)} e^{c(t-s)}, & \text{if } |\eta_1(\omega) - \eta_2(\omega)| < 1, \\ |\eta_1(\omega) - \eta_2(\omega)| e^{c(t-s)}, & \text{otherwise}, \end{cases} \end{aligned}$$

for all $t \in [s, s+\tau]$.

PROOF. We proved that, when η is constant, system (6.8) has a continuous solution in $[s, s+\tau[$. Using exactly the same technique (approaching u by suitable u_n), one can check that, for all $\eta \in L^2(\Omega, F_s)$, (6.8) has a continuous solution in $[s, s+\tau[$. If we fix η_1 and η_2 in $L^2(\Omega, F_s)$, then we have

$$\begin{aligned} &|X_t^{s,\eta_1}(\omega) - X_t^{s,\eta_2}(\omega)| \\ &\quad\le |\eta_1(\omega) - \eta_2(\omega)| \\ &\qquad + \int_{S(\omega)}^t |u(r, X_r^{s,\eta_1}(\omega)) - u(r, X_r^{s,\eta_2}(\omega))|\,dr \quad P\text{-a.e.}, \end{aligned}$$

where $S(\omega) = \sup\{r \in [s,t] \mid |X_r^{s,\eta_1}(\omega) - X_r^{s,\eta_2}(\omega)| \le |\eta_1(\omega) - \eta_2(\omega)|\}$.

Therefore, for P-almost all ω,

$$
\begin{aligned}
&|X_t^{s,\eta_1}(\omega) - X_t^{s,\eta_2}(\omega)| \\
&\quad \le |\eta_1(\omega) - \eta_2(\omega)| + c\int_{S(\omega)}^t \varphi(|X_r^{s,\eta_1}(\omega) - X_r^{s,\eta_2}(\omega)|)\,dr \\
&\quad \le |\eta_1(\omega) - \eta_2(\omega)| \\
&\qquad + c\int_s^t |X_r^{s,\eta_1}(\omega) - X_r^{s,\eta_2}(\omega)|\,dr \max\big((1 - \log|\eta_1(\omega) - \eta_2(\omega)|), 1\big)
\end{aligned}
$$

and, in view of Gronwall's lemma, we get the result. □

PROOF OF THEOREM 6.0.4. Fix $\tau \le 1/2(|||u|||_\infty + c)$. We prove, by induction, the following claim: (6.7) has in $[0, n\tau]$ a continuous solution $(X_t^x)_{t\in[0,n\tau]}$.

This holds for $n = 1$ by Theorem 6.0.3. Let $(X_r^x)_{r\in[0,n\tau]}$ be a process satisfying the claim for n and consider the SDE,

$$
\begin{aligned}
dX_r^{n\tau, X_{n\tau}^x} &= u\big(r, X_r^{n\tau, X_{n\tau}^x}\big)\,dr + \mu\,dW_r, \\
X_{n\tau}^{n\tau, X_{n\tau}^x} &= X_{n\tau}^x.
\end{aligned}
\tag{6.9}
$$

By the preceding lemma, for all x, there exists in $[n\tau, (n+1)\tau]$ a continuous solution $(X_r^{n\tau, X_{n\tau}^x})_{r\in[n\tau,(n+1)\tau]}$, and

$$
\big|X_{n\tau+t}^{n\tau, X_{n\tau}^x} - X_{n\tau+t}^{n\tau, X_{n\tau}^y}\big| \le \max\big(|X_{n\tau}^x - X_{n\tau}^y|^{1-ct}, |X_{n\tau}^x - X_{n\tau}^y|\big)e^{ct}.
$$

It follows that the process

$$
Y_r^x = \begin{cases} X_r^x, & r \in [0, n\tau], \\ X_r^{n\tau, X_{n\tau}^x}, & r \in]n\tau, (n+1)\tau], \end{cases}
$$

is a continuous solution of (6.7) in $[0, (n+1)\tau]$. Suppose now that u is divergence free in the distributional sense. By the continuity of $X_t^x(\omega)$ in (t, x), we get

$$
X_{n\tau+t}^x(\omega) = X_{n\tau+t}^{n\tau, X_{n\tau}^x(\omega)}(\omega) \quad P\text{-a.e.}
$$

Therefore, if $f \in L^1$ and $t \in [0, (n+1)\tau]$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} f(X_{n\tau+t}^x(\omega))\,dx &= \int_{\mathbb{R}^2} f\big(X_{n\tau+t}^{n\tau, X_{n\tau}^x(\omega)}(\omega)\big)\,dx \\
&= \int_{\mathbb{R}^2} f(X_{n\tau+t}^{n\tau, y}(\omega))\,dy = \int_{\mathbb{R}^2} f(z)\,dz
\end{aligned}
$$
□

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