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$\mathscr{E}$-MARTINGALES AND THEIR APPLICATIONS IN MATHEMATICAL FINANCE

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After introducing a new concept, the notion of $\mathscr{E}$-martingale, we extend the well-known Doob inequality (for $1 < p < +\infty$) and the Burkholder–Davis–Gundy inequalities (for $p = 2$) to $\mathscr{E}$-martingales. By means of these inequalities, we give sufficient conditions for the closedness of a space of stochastic integrals with respect to a fixed $\mathbb{R}^d$-valued semimartingale, a question which arises naturally in the applications to financial mathematics. We also provide a necessary and sufficient condition for the existence and uniqueness of the Föllmer–Schweizer decomposition.

1. Introduction. Let $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{0\le t\le T})$ be a filtered probability space and $T \in [0, +\infty)$ be a fixed time horizon. Assume X is a semimartingale which is locally bounded in L^2. Then X is special; that is, it admits the canonical decomposition

$$X = X_0 + M + A,$$

where M is a local martingale and A a predictable finite variation process. We introduce the space Θ of all predictable X-integrable processes θ such that the stochastic integral

$$G(\theta) := \int \theta\, dX := \theta X$$

is in the space $\mathscr{H}^2$ of semimartingales (see Definition 2.7 below). The problem of determining whether the space

$$G_T(\Theta) := \{(\theta X)_T\colon\ \theta \in \Theta\}$$

is closed in L^2 is an important issue in mathematical finance. We refer to the financial introduction of Delbaen, Monat, Schachermayer, Schweizer and Stricker (1997), hereafter referred to as DMSSS, for more details and references.

Let Q be equivalent to P, $Z_T := dQ/dP$ and $Z_t := E^P(Z_T|\mathscr{F}_t)$. The measure Q is called an equivalent local martingale measure if X is a local martingale under Q. The existence of an equivalent local martingale measure is closely related to no arbitrage [see Stricker (1990), Ansel and Stricker (1992) and Delbaen and Schachermayer (1994)]. When X is continuous, the closedness of $G_T(\Theta)$ and its connection to BMO or bmo_2 as well as to reverse Hölder inequalities were completely worked out and clarified by DMSSS. Their proofs

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are based on weighted norm inequalities due to Doléans-Dade and Meyer (1979), Bonami and Lépingle (1979) and Kazamaki (1994). In a subsequent paper Grandits and Krawczyk (1997) extended some of the previous results to the L^p case for $1 < p < +\infty$. In the discontinuous case Monat and Stricker (1994, 1995) provided a sufficient condition for the closedness of $G_T(\Theta)$ that is quite far from being necessary. In order to deal with the discontinuous case, we need an extension of Doob and Burkholder–Davis–Gundy inequalities. This is the central topic of our paper. The first step is to generalize the notion of martingale under Q. This topic was first suggested by P. A. Meyer to Ruiz de Chavez (1984), in order to obtain a more general martingale characterization of Brownian motion, possibly involving signed measures. It is well known that a process Y is a Q-martingale iff YZ is a P-martingale. With this formulation, the probability Q does not appear any more and one could start with some local martingale Z. It is natural to introduce the class of processes Y such that YZ is a martingale. Let us mention that Yoeurp (1982) characterized in his nice thesis the space of semimartingales Y such that YZ is a local martingale, where Z is a given semimartingale. Yet, when $\tau := \inf\{t\colon Z_t = 0\}$, the previous property gives no information on the behavior of Y after τ and there is no hope of obtaining Doob inequality in that case. So we introduce the new concept of $\mathscr{E}$-martingale. Assume $Z = \mathscr{E}(N)$ where $\mathscr{E}(N)$ denotes the stochastic exponential of the local martingale N. Put $T_0 = 0$ and for $n \geq 0$, $T_{n+1} := \inf\{t > T_n\colon \mathscr{E}(N - N^{T_n})_t = 0\}$. Then (T_n) converges stationarily to T. A process Y is called an $\mathscr{E}$-martingale if for any $n \geq 0$, $(Y - Y^{T_n})\mathscr{E}(N - N^{T_n})$ is a martingale and $E(|X_{T_n}\, {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$. Note that, when $\mathscr{E}(N)$ is a strictly positive martingale, our definition coincides with the notion of martingale under measure Q defined by $dQ = \mathscr{E}(N)_T dP$. We call this case the classical case. Our definition of $\mathscr{E}$-martingale is designed in such a way that Doob inequality (for $1 < p < +\infty$) and the Burkholder–Davis–Gundy inequalities (for $p = 2$) hold under additional assumptions on $\mathscr{E}(N)$. These assumptions are also necessary. The results of Doléans-Dade and Meyer (1979) as well as those of the Japanese school [see the book by Kazamaki (1994) in the continuous case] are based on Gehring's lemma, which implies the following key result: if Z is continuous or there is a constant $C > 0$ such that $CZ_- \leq Z \leq (1/C)Z_-$, then Z satisfies the reverse Hölder inequality (R_p) iff there exists $\varepsilon > 0$ such that Z satisfies $(R_{p+\varepsilon})$. A new and very nice approach was discovered by Jawerth (1986) and worked out by Long (1993). This idea allows removing the condition on the jumps of Z. However it seems that this approach does not work when Z is not strictly positive. We present here a third approach, which allows us to deal with $\mathscr{E}$-martingales and which is even simpler than the previous one in the classical case.

This paper is organized as follows. In Section 2 we describe some properties of the stochastic exponential of a semimartingale. Section 3 deals with the new concept of $\mathscr{E}$-martingales. In Section 4 we extend the well-known Doob inequality (for $1 < p < +\infty$) and the Burkholder–Davis–Gundy inequalities (for $p = 2$) to $\mathscr{E}$-martingales. The last section is devoted to the closedness of $G_T(\Theta)$ in L^2 and the Föllmer–Schweizer decomposition.

2. Preliminaries As in the introduction, we consider a probability space $(\Omega,\ \mathscr{F},\ P)$ and a fixed time horizon $T \in [0,\ +\infty)$. We suppose that we have a filtration $(\mathscr{F}_t)_{0\le t\le T}$ on $(\Omega,\ \mathscr{F},\ P)$ satisfying the usual conditions; that is, $(\mathscr{F}_t)_{0\le t\le T}$ is right continuous and complete and we assume moreover that $\mathscr{F} = \mathscr{F}_T$. $\mathscr{M}$ denotes the space of martingales. For $p > 1$, $\mathscr{M}^p$ (resp. $\mathscr{M}_0^p$) denotes the space of martingales M with $E(|M_T|^p) < +\infty$ (resp. $M \in \mathscr{M}^p$ and $M_0 = 0$). If Y is a process, we set $Y_t^* = \sup_{0\le s\le t} |Y_s|$. If Y is a cadlag and adapted process, then so is Y^* and we denote by Y^τ the process Y stopped at τ. Then ${}^\tau Y$ is the process defined by ${}^\tau Y = Y - Y^\tau$. If τ_n is a sequence of stopping times, then $\tau_n \to T$ means stationarily; that is, there exists k (depending on ω), such that $\tau_n = T$ for $n \ge k$. Since we do not care for the precise values of constants in our inequalities, C denotes a numerical constant, which may vary at each ocurrence. If $\mathscr{C}$ is a class of processes, we denote by $\mathscr{C}_{\mathrm{loc}}$ the class of processes Y such that there exists an increasing sequence of stopping times $(\tau_n)_{n\ge 0}$ that converges to T and such that for all n, $Y^{\tau_n} \in \mathscr{C}$. For all unexplained notations, we refer to Dellacherie and Meyer (1980) or to Jacod (1979).

Definition 2.1. Let X be a semimartingale. We denote by $\mathscr{E}(X)$ its stochastic exponential, that is, the unique solution of the stochastic differential equation $dY = Y_- dX$ and $Y_0 = 1$.

There exists a characterization of semimartingales which may be represented as stochastic exponentials.

Proposition 2.2. *Let Z be a semimartingale. There exists a semimartingale X such that $Z = \mathscr{E}(X)$ iff $Z_0 = 1$, $Z_{t-} \neq 0$ a.s. for $0 \le t \le \tau$ and $Z_t = 0$ a.s. for $\tau \le t \le T$ where $\tau = \inf\{t\colon Z_t = 0\}$. In that case we can choose $X := (Z_-^{-1} 1_{[0,\,\tau]}) \cdot Z$. This process X is called the stochastic logarithm of Z and denoted by $\mathscr{L}(Z)$.*

For the proof of Proposition 2.2, we refer to Jacod (1979) where the "only if" part is stated in Proposition 6.5 and the "if" part is stated in Exercise 6.1, page 198.

A family of examples of martingales that can be represented as exponentials is given in the following proposition.

Proposition 2.3. *Assume $1 < p < +\infty$. If $Z \in \mathscr{M}^p$, $Z_0 = 1$ and there exists a constant C such that for all stopping times σ,*

$$E(|Z_T|^p | \mathscr{F}_\sigma) \le C |Z_\sigma|^p \tag{1}$$

then $Z = \mathscr{E}(\mathscr{L}(Z))$.

Proof. The proof mimics the proof of Lemma 3.4 in Delbaen and Schachermayer (1996). Let $\tau_n := \inf\{t\colon |Z_t| \le 1/n\}$ and $\tau := \lim \tau_n$. Put $A := \bigcap_n \{\tau_n < \tau\}$ and notice that $A \in \sigma(\bigcup_n \mathscr{F}_{\tau_n})$, $\{Z_{\tau-} = 0\} = A$ and $1_A E(Z_T | \sigma(\bigcup_n \mathscr{F}_{\tau_n})) = 0$.

If q is the conjugate of p, we have

$$I_{\{\tau_n<\tau\}} \le E\left(\left|\frac{Z_T}{Z_{\tau_n}}\right| I_{\{\tau_n<\tau\}} | \mathscr{F}_{\tau_n}\right) = E\left(\left|\frac{Z_T}{Z_{\tau_n}}\right| I_{\{\tau_n<\tau\}} I_{A^c} | \mathscr{F}_{\tau_n}\right)$$
$$\le E\left(\left|\frac{Z_T}{Z_{\tau_n}}\right|^p I_{\{\tau_n<\tau\}} | \mathscr{F}_{\tau_n}\right)^{1/p} P(A^c|\mathscr{F}_{\tau_n})^{1/q}.$$

Thus we obtain

$$I_A = I_A I_{\{\tau_n<\tau\}} \le C I_A P(A^c|\mathscr{F}_{\tau_n})^{1/q} \to 0$$

so $P(A) = 0$. Thus $Z_\tau = 0$, so by (1) and by Jensen inequality, $Z_t = 0$ on $\{t > \tau\}$. It follows from Proposition 2.2 that $Z = \mathscr{E}(\mathscr{L}(Z))$. □

Next we recall some properties of the stochastic exponential.

PROPOSITION 2.4. *If X and X' are semimartingales and τ is a stopping time, then* (i) $\mathscr{E}(X + X' + [X,\ X']) = \mathscr{E}(X)\mathscr{E}(X')$, (ii) $\mathscr{E}(X)^\tau = \mathscr{E}(X^\tau)$.

The assertion (i) was shown by Yor (1976) and (ii) is an immediate consequence of the uniqueness of the stochastic exponential.

DEFINITION 2.5. Let Z be a martingale and $1 < p < +\infty$. Then Z belongs to bmo_p if there is a constant C such that

$$E(|Z_T - Z_S|^p|\mathscr{F}_S) \le C^p \tag{2}$$

for all stopping (or equivalently deterministic) times S. The best constant in (2) is denoted by $||Z||_{bmo_p}$.

The Burkholder–Davis–Gundy inequalities yield the next proposition.

PROPOSITION 2.6. *Let Z be a local martingale. Then Z is in bmo_p if and only if there is a constant C, such that*

$$E(([Z]_T - [Z]_S)^{p/2}|\mathscr{F}_S) \le C^p$$

for all stopping (or equivalently deterministic) times S.

DEFINITION 2.7. For a special semimartingale X with canonical decomposition $X = X_0 + M + A$ we define

$$|||X|||^2 = E((X_0)^2) + E([M]_T) + E\left(\left(\int_0^T |dA_t|\right)^2\right)$$

and

$$\mathscr{H}^2 = \{X \mid |||X||| < +\infty\}.$$

For the proof of Proposition 2.8 we refer to Protter (1990).

PROPOSITION 2.8. *For any special semimartingale X we have* (i) $E([X]_T) \leq |||X|||^2$, (ii) $E(X_T^{*2}) \leq C|||X|||^2$.

In the sequel we will need to inverse semimartingales. In general it is not possible, but we have the following.

PROPOSITION 2.9. *If Y is a semimartingale such that* $\inf_t |Y_t| > 0$ *a.s., then* $1/Y$ *is a semimartingale.*

PROOF. Let $\tau_n := \inf\{t: |Y_t| < 1/n\}$ and $\widetilde{Y}^n := Y^{\tau_n} - \Delta Y_{\tau_n} 1_{[\![\tau_n, T]\!]}$. Since $\inf_t |Y_t| > 0$, τ_n converges stationarily to T. Consider a smooth real function f such that $f(x) = 1/x$ for $|x| \geq 1/n$ and apply Itô's formula to $f(\widetilde{Y}^n) = 1/\widetilde{Y}^n$. Since the semimartingale $\widetilde{Y}^n$ coincides with Y on $[\![0, \tau_n[\![$, $1/Y$ is a semimartingale [see page 236 of Dellacherie and Meyer (1980)]. The proof of Proposition 2.9 is complete. □

3. $\mathscr{E}$-martingales. Throughout the paper N denotes a local martingale such that $N_0 = 0$. For any stopping time τ we denote ${}^\tau\mathscr{E} = \mathscr{E}(N - N^\tau)$. In this paper we use the symbol $\mathscr{E}(N)$ (or even $\mathscr{E}$, since N is fixed) for this family of processes, rather than for the process ${}^0\mathscr{E}(N)$.

PROPOSITION 3.1. *For any pair of stopping times σ, τ,* (i) ${}^\sigma\mathscr{E} = {}^\sigma\mathscr{E}_\tau\, {}^\tau\mathscr{E}$ *on* $\{\sigma \leq \tau\}$, (ii) ${}^\sigma\mathscr{E}_\tau = 1$ *on* $\{\sigma \geq \tau\}$.

PROOF. (i) We may assume, that $\tau \geq \sigma$ (otherwise, we take $\sigma \vee \tau$ instead of τ). Since $[N^\tau - N^\sigma, N - N^\tau] = 0$, Proposition 2.4 yields the claim. (ii) Again we may assume $\tau \leq \sigma$. By Proposition 2.4(ii)

$$ {}^\sigma\mathscr{E}^\tau = \mathscr{E}(N^\tau - N^\tau) = \mathscr{E}(0) = 1. \qquad \square$$

DEFINITION 3.2. Let $q \geq 1$. We say that $\mathscr{E}(N)$ satisfies the reverse Hölder inequality (R_q) iff there exists a constant $C \geq 1$ such that for any t,

$$E(|{}^t\mathscr{E}_T|^q | \mathscr{F}_t) \leq C. \tag{3}$$

Note, that if for any t, ${}^t\mathscr{E}$ is a martingale, then by Jensen inequality we have

$$E(|{}^t\mathscr{E}_T|^q | \mathscr{F}_t) \geq 1.$$

Thus the inequality (3) should rather be called the reverse Jensen inequality, but for historical reasons we use Hölder.

PROPOSITION 3.3. *If $\mathscr{E}(N)$ satisfies (R_q), then for any stopping time τ*

$$E(|{}^\sigma\mathscr{E}_T|^q | \mathscr{F}_\tau) \leq C|{}^\sigma\mathscr{E}_\tau|^q. \tag{4}$$

PROOF. Note, that by Proposition 3.1 for $s \geq t$,

$$E(|{}^t\mathscr{E}_T|^q|\mathscr{F}_s) = E(|{}^t\mathscr{E}_s|^q\,|{}^s\mathscr{E}_T|^q|\mathscr{F}_s) = |{}^t\mathscr{E}_s|^q E(|{}^s\mathscr{E}_T|^q|\mathscr{F}_s) \leq C|{}^t\mathscr{E}_s|^q,$$

and for $s \leq t$,

$$E(|{}^t\mathscr{E}_T|^q|\mathscr{F}_s) = E(E(|{}^t\mathscr{E}_T|^q|\mathscr{F}_t)|\mathscr{F}_s) \leq C = C|{}^t\mathscr{E}_s|^q.$$

By standard reasoning we conclude that (4) holds for every pair of simple (i.e., admitting a finite number of values) stopping times σ, τ. Now let τ be simple, σ arbitrary and $\sigma_n \searrow \sigma$, σ_n simple. Since $E(|{}^\tau\mathscr{E}_T|^q) < +\infty$,

$$E(|{}^\tau\mathscr{E}_T|^q|\mathscr{F}_\sigma) = \lim_n E(|{}^\tau\mathscr{E}_T|^q|\mathscr{F}_{\sigma_n}) \leq C \lim_n |{}^\tau\mathscr{E}_{\sigma_n}|^q = C|{}^\tau\mathscr{E}_\sigma|^q.$$

To complete the proof, let τ be arbitrary and let $\tau_n \searrow \tau$, τ_n simple. Since ${}^\tau\mathscr{E}_T = {}^\tau\mathscr{E}_{\tau_n}\,{}^{\tau_n}\mathscr{E}_T$ and ${}^\tau\mathscr{E}_{\tau_n} \to 1$ a.s., ${}^{\tau_n}\mathscr{E}_T \to {}^\tau\mathscr{E}_T$ a.s. Hence by Fatou's lemma,

$$E(|{}^\tau\mathscr{E}_T|^q|\mathscr{F}_\sigma) \leq \liminf E(|{}^{\tau_n}\mathscr{E}_T|^q|\mathscr{F}_\sigma) \leq C \liminf |{}^{\tau_n}\mathscr{E}_\sigma|^q = C|{}^\tau\mathscr{E}_\sigma|^q. \qquad \square$$

DEFINITION 3.4. Throughout the paper, T_n is the increasing sequence of stopping times, defined by $T_0 = 0$, $T_{n+1} = \inf\{t > T_n \mid {}^{T_n}\mathscr{E}_t = 0\} \wedge T$.

In the case when ${}^0\mathscr{E}$ is a positive martingale (we call this case classical), $T_0 = 0$ and $T_n = T$ for $n \geq 1$.

PROPOSITION 3.5. *For every n,*

$${}^{T_n}\mathscr{E} = {}^{T_n}\mathscr{E}^{T_{n+1}}.$$

Moreover, there exists a right continuous version of $({}^s\mathscr{E}_t)_{s,\,t\geq 0}$.

PROOF. The first statement is a straightforward consequence of Proposition 3.1, whereas the second one follows from the equality

$${}^\sigma\mathscr{E}_\tau = 1_{\{\sigma=\tau=T\}} + 1_{\{\sigma>\tau\}} + \sum_p 1_{\{T_p\leq\sigma<T_{p+1},\,\sigma\leq\tau\}} \frac{{}^{T_p}\mathscr{E}_\tau}{{}^{T_p}\mathscr{E}_\sigma}$$

for every pair of stopping times σ and τ.

DEFINITION 3.6. We say that $\mathscr{E}$ is regular, if for any n, ${}^{T_n}\mathscr{E}$ is a martingale.

The next proposition gives a sufficient condition for (R_2) and regularity. However it is far from being necessary.

PROPOSITION 3.7. *Let $N \in \mathscr{M}_0^2$. If $\langle N\rangle_T \in L^\infty$, then $\mathscr{E}(N)$ is regular and satisfies (R_2).*

PROOF. Let τ be a stopping time. According to Proposition 2.4,

$$({}^{\tau}\mathscr{E}(N))^2 = {}^{\tau}\mathscr{E}(2N + [N]) = {}^{\tau}\mathscr{E}(\widetilde{N})\mathscr{E}(\langle N\rangle - \langle N\rangle^{\tau}),$$

where $\widetilde{N} := (1/(1+\Delta\langle N\rangle))(2N+[N]-\langle N\rangle)$ [see Proposition II-1 in Lépingle and Mémin (1978)]. Since $\langle N\rangle$ is increasing, we have

$$0 < \mathscr{E}(\langle N\rangle - \langle N\rangle^{\tau}) \le \exp(\|\langle N\rangle_T\|_\infty).$$

Observe that ${}^{\tau}\mathscr{E}(\widetilde{N})$ is a nonnegative local martingale; hence it is a nonnegative supermartingale and $0 \le E({}^{\tau}\mathscr{E}(\widetilde{N})_\sigma|\mathscr{F}_\tau) \le {}^{\tau}\mathscr{E}(\widetilde{N})_\tau = 1$ for each stopping time σ. We conclude that

$$E(({}^{\tau}\mathscr{E}(N)_\sigma)^2|\mathscr{F}_\tau) \le \exp(\|\langle N\rangle_T\|_\infty).$$

It follows that the family ${}^{\tau}\mathscr{E}(N)_\sigma$ is uniformly integrable and $\mathscr{E}(N)$ satisfies (R_2). The proof of Proposition 3.7 is complete. □

REMARK 3.8. If N is a continuous martingale with $N_0 = 0$ and $\langle N\rangle_T \in L^\infty$, then a closer look at the previous proof shows that $\mathscr{E}(N)$ satisfies (R_p) for all $p < +\infty$. However, in the discontinuous case, it is easy to construct an example such that $N \in \mathscr{M}_0^2$ with $\langle N\rangle_T \in L^\infty$ but N does not satisfy (R_p) for any $p > 2$.

PROPOSITION 3.9. *Assume $\mathscr{E}$ satisfies (R_p) for some $p > 1$. Then we have the following.*

(i) *If $\mathscr{E}$ is a martingale, then there exists a constant C such that for every stopping time τ,*

$$E\Big(\sup_{\tau\le t}|\mathscr{E}_t|^p|\mathscr{F}_\tau\Big) \le C|\mathscr{E}_\tau|^p. \tag{5}$$

(ii) *$\mathscr{E}$ is regular iff for any stopping time τ, ${}^{\tau}\mathscr{E}$ is a martingale. In that case ${}^{\tau}\mathscr{E} \in \mathscr{M}^p$.*

PROOF. (i) Since $\mathscr{E}$ is a martingale satisfying (R_p), we conclude that $\mathscr{E} \in \mathscr{M}^p$. From the conditional Doob inequality it follows that

$$\begin{aligned} E\Big(\sup_{\tau\le t}|\mathscr{E}_t|^p|\mathscr{F}_\tau\Big) &\le 2^p\Big(E\Big(\sup_{\tau\le t}|\mathscr{E}_t - \mathscr{E}_{t\wedge\tau}|^p|\mathscr{F}_\tau\Big) + |\mathscr{E}_\tau|^p\Big) \\ &\le C(E(|\mathscr{E}_T - \mathscr{E}_\tau|^p|\mathscr{F}_\tau) + |\mathscr{E}_\tau|^p). \end{aligned}$$

Now $\mathscr{E}$ satisfies (R_p) and hence

$$E(|\mathscr{E}_T - \mathscr{E}_\tau|^p|\mathscr{F}_\tau) \le C|\mathscr{E}_\tau|^p.$$

By combining the previous inequalities we get (5).

(ii) Assume $\mathscr{E}$ is regular and σ is a stopping time such that $\sigma \ge \tau$. Since ${}^{T_n}\mathscr{E}$ is a martingale,

$$E({}^{T_n}\mathscr{E}_\tau\,{}^{\tau}\mathscr{E}_T|\mathscr{F}_\sigma) = {}^{T_n}\mathscr{E}_\tau\,{}^{\tau}\mathscr{E}_\sigma.$$

Note that on $A_n := \{T_n \le \tau < T_{n+1}\}$, ${}^{T_n}\mathscr{E}_\tau \neq 0$ and that by the reverse Hölder inequality, ${}^{\tau}\mathscr{E}_T \in L^p$. Hence

$$1_{A_n}\,{}^{\tau}\mathscr{E}_\sigma = 1_{A_n} E({}^{\tau}\mathscr{E}_T|\mathscr{F}_\sigma).$$

Since $\bigcup_n A_n = \{\tau < T\}$ and ${}^{T}\mathscr{E}_T = 1$, we obtain

$${}^{\tau}\mathscr{E}_\sigma = E({}^{\tau}\mathscr{E}_T|\mathscr{F}_\sigma).$$

Therefore ${}^{\tau}\mathscr{E} \in \mathscr{M}$ and by (R_p), ${}^{\tau}\mathscr{E} \in \mathscr{M}^p$. The proof of Proposition 3.9 is complete. □

The next proposition improves Lemma 4.2 of DMSSS.

PROPOSITION 3.10. *If $\mathscr{E}$ is regular and satisfies (R_p) for some $p > 1$, then $N \in bmo_p$.*

PROOF. By Proposition 3.9, ${}^{\tau}\mathscr{E} \in \mathscr{M}^p$. For fixed $s \in [0, T]$ we introduce the following sequence of stopping times:

$$\tau_0 = s, \qquad \tau_{n+1} = \inf\left\{t \ge \tau_n : |{}^{\tau_n}\mathscr{E}_t| \le \tfrac{1}{2}\right\} \wedge T.$$

Since ${}^{\tau_n}\mathscr{E}$ is a martingale, we have

$$1 \le E(|{}^{\tau_n}\mathscr{E}_{\tau_{n+1}}||\mathscr{F}_{\tau_n}) = E(|{}^{\tau_n}\mathscr{E}_{\tau_{n+1}} 1_{\{\tau_{n+1}<T\}}||\mathscr{F}_{\tau_n}) + E(|{}^{\tau_n}\mathscr{E}_T 1_{\{\tau_{n+1}=T\}}||\mathscr{F}_{\tau_n}).$$

The first term is smaller then $\frac{1}{2}P(\tau_{n+1} < T|\mathscr{F}_{\tau_n})$ whereas the second can be estimated from above using Hölder inequality and (R_p). We obtain

$$1 \le \tfrac{1}{2}P(\tau_{n+1} < T|\mathscr{F}_{\tau_n}) + C(1 - P(\tau_{n+1} < T|\mathscr{F}_{\tau_n}))^{1/q}$$

where $(1/p)+(1/q) = 1$. This implies the existence of $\delta < 1$ such that $P(\tau_{n+1} < T|\mathscr{F}_{\tau_n}) \le \delta$. Since for $t \le \tau_{n+1}$, $2|{}^{\tau_n}\mathscr{E}_{t-}| \ge 1$ we have

$$\begin{aligned} E(([N]_{\tau_{n+1}} - [N]_{\tau_n})^{p/2}|\mathscr{F}_{\tau_n}) &\le E\left(\left(\int_{\tau_n}^{\tau_{n+1}} 4({}^{\tau_n}\mathscr{E}_-)^2 d[N]\right)^{p/2}\middle|\mathscr{F}_{\tau_n}\right) \\ &\le CE(|{}^{\tau_n}\mathscr{E}_{\tau_{n+1}} - 1|^p|\mathscr{F}_{\tau_n}). \end{aligned}$$

It follows from the reverse Hölder inequality that

$$E(([N]_{\tau_{n+1}} - [N]_{\tau_n})^{p/2}|\mathscr{F}_{\tau_n}) \le C1_{\{\tau_n<T\}}.$$

Finally we can estimate $(E(([N]_T - [N]_s)^{p/2}|\mathscr{F}_s))^{1/p}$ by the series

$$\begin{aligned} \sum_{n\ge 0}(E(([N]_{\tau_{n+1}} - [N]_{\tau_n})^{p/2}|\mathscr{F}_s))^{1/p} &\le \sum_{n\ge 0} E(([N]_{\tau_{n+1}} - [N]_{\tau_n})^{p/2}|\mathscr{F}_{\tau_n})|\mathscr{F}_s)^{1/p} \\ &\le 2C\sum_{n\ge 0}(E(1_{\{\tau_n<T\}}|\mathscr{F}_s))^{1/p}. \end{aligned}$$

Since

$$E(1_{\{\tau_n<T\}}|\mathscr{F}_s) = E(1_{\{\tau_n<T\}}1_{\{\tau_{n-1}<T\}}|\mathscr{F}_{\tau_{n-1}}|\mathscr{F}_s) \le \delta E(1_{\{\tau_{n-1}<T\}}|\mathscr{F}_s),$$

we find that $E(1_{\{\tau_n<T\}}|\mathscr{F}_s) \le \delta^n$; hence

$$(E(([N]_T - [N]_s)^{p/2}|\mathscr{F}_s))^{1/p} \le C.$$

This completes the proof of Proposition 3.10. □

DEFINITION 3.11. We say that a cadlag process X is an $\mathscr{E}(N)$-martingale (or an $\mathscr{E}$-martingale), if for any n,

$$E(|X_{T_n}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$$

and $({}^{T_n}X\ {}^{T_n}\mathscr{E})$ is a martingale. The class of $\mathscr{E}(N)$-martingales will be denoted by $\mathscr{M}(\mathscr{E})$.

Note that in the classical case, the notion of $\mathscr{E}(N)$-martingale coincides with the notion of martingale under measure $d\widetilde{P} = \mathscr{E}_T dP$.

PROPOSITION 3.12. *Assume that $\mathscr{E}$ is regular.*

(i) *A cadlag, adapted process X is an $\mathscr{E}$-martingale if and only if for any n, $E(|X_{T_n}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$, $E(|X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$ and for any t,*

$$E(X_T\ {}^{T_n}\mathscr{E}_T|\mathscr{F}_t) = E(X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|\mathscr{F}_t) = X_t\ {}^{T_n}\mathscr{E}_t \tag{6}$$

on the set $\{t \in [T_n, T_{n+1}]\}$. Therefore the terminal value X_T of the $\mathscr{E}$-martingale X determines the whole process X.

(ii) *Put ${}^*\mathscr{E}_T := \sup_{0\le t\le T} |{}^t\mathscr{E}_T|$ and let H be a random variable such that $H^*\mathscr{E}_T \in L^\infty$. Then the process X defined by $X_t := E(H^t\mathscr{E}_T|\mathscr{F}_t)$ is an $\mathscr{E}$-martingale.*

(iii) *If $\mathscr{E}$ satisfies (R_q) and $H \in L^p$ then there exists $X \in \mathscr{M}(\mathscr{E})$ such that $X_T = H$.*

PROOF. (i) The first equality in (6) is obvious since ${}^{T_n}\mathscr{E}_T = 0 = {}^{T_n}\mathscr{E}_{T_{n+1}}$ on the set $\{T > T_{n+1}\}$. By the definition of ${}^{T_n}X$,

$${}^{T_n}X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}} = X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}} - X_{T_n}\ {}^{T_n}\mathscr{E}_{T_{n+1}}.$$

The claim immediately follows.

(ii) Since $H^*\mathscr{E}_T \in L^\infty$, there exists a cadlag version of the process X and for each stopping time τ we have $X_\tau = E(H^\tau\mathscr{E}_T|\mathscr{F}_\tau)$. Now the regularity of $\mathscr{E}$ implies that for each integer n, $E(|X_{T_n}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$ and $E(|X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) < +\infty$. Moreover, on the set $\{t \in [T_n, T_{n+1}]\}$, equality (6) holds. Hence X is an $\mathscr{E}$-martingale.

(iii) Set

$$X_t := \frac{E(H\ {}^{T_n}\mathscr{E}_T|\mathscr{F}_t)}{{}^{T_n}\mathscr{E}_t} \quad \text{on } \{t \in [T_n, T_{n+1}[\}.$$

Thus X admits a cadlag version. Moreover by (R_q), $E(|X_{T_n}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) \le CE(|X_{T_n}|) \le C\|H\|_p$ and $E(|X_{T_{n+1}}\ {}^{T_n}\mathscr{E}_{T_{n+1}}|) = E(|H\ {}^{T_n}\mathscr{E}_T|) \le C\|H\|_p$. Thus by (i), $X \in \mathscr{M}(\mathscr{E})$. □

REMARK 3.13. If $\mathscr{E}$ is regular, then any process of the form $X_t = Y, 1_{\{T_k<T,\, t\geq T_k\}}$, where Y is bounded and $\mathscr{F}_{T_k}$-measurable is an $\mathscr{E}$-martingale (indeed, for any $n \geq k$, ${}^{T_n}X = 0$ and for $n < k$, $1_{\{T_k<T\}}\ {}^{T_n}\mathscr{E}_{T_k} = 0$; regularity guarantees integrability). In particular, if $P(T_1 < T) > 0$, there exist nonconstant increasing $\mathscr{E}$-martingales. At the end of our paper we shall give another striking example of $\mathscr{E}$-martingales.

DEFINITION 3.14. We say that a cadlag process is an $\mathscr{E}$-local martingale if for any n ${}^{T_n}X$, ${}^{T_n}\mathscr{E}$ is a local martingale.

Note that in the classical case the notion of $\mathscr{E}$-local martingale coincides with the notion of local martingale under $\widetilde{P}$.

PROPOSITION 3.15. *A cadlag process X is an $\mathscr{E}$-local martingale iff it is a semimartingale such that $X + [X, N]$ is a local martingale.*

PROOF. First assume that X is an $\mathscr{E}$-local martingale. Note that since $\inf_{t\in[T_n,\, T_{n+1})} |{}^{T_n}\mathscr{E}_t| > 0$, by Proposition 2.9, $1/({}^{T_n}\mathscr{E})1_{[T_n,\, T_{n+1})}$ is a semimartingale, and since ${}^{T_n}X\ {}^{T_n}\mathscr{E}$ is a local martingale, the product ${}^{T_n}X1_{[T_n,\, T_{n+1})}$ is a semimartingale and hence so is $X1_{[T_n,\, T_{n+1})} = {}^{T_n}X1_{[T_n,\, T_{n+1})} + X_{T_n}1_{[T_n,\, T_{n+1})}$. By summing, we conclude that $X1_{[0,T_n]}$ is a semimartingale for any n, and hence X is a semimartingale.

Now let $X = X_0 + M + A$ be an arbitrary semimartingale (not necessarly an $\mathscr{E}$-local martingale), where M is a local martingale and A is a finite variation process. Then the integration by parts formula yields

$$
\begin{aligned}
{}^{T_n}X\ {}^{T_n}\mathscr{E} &= ({}^{T_n}X_-)\cdot({}^{T_n}\mathscr{E}) + ({}^{T_n}\mathscr{E}_-)\cdot({}^{T_n}X) + [{}^{T_n}X, {}^{T_n}\mathscr{E}] \\
&= \text{local martingale} + {}^{T_n}\mathscr{E}_-\cdot({}^{T_n}X) + {}^{T_n}\mathscr{E}_-\cdot[{}^{T_n}X, {}^{T_n}N] \qquad (7)\\
&= \text{local martingale} + {}^{T_n}\mathscr{E}_-\cdot({}^{T_n}X + {}^{T_n}[X, N]).
\end{aligned}
$$

Thus if $({}^{T_n}X)({}^{T_n}\mathscr{E})$ is a local martingale, then $Y := {}^{T_n}\mathscr{E}_-\cdot({}^{T_n}X + {}^{T_n}[X, N])$ is a local martingale and since $\theta_t = (1/{}^{T_n}\mathscr{E}_{t-})1_{[0,T_{n+1}]}$ is caglad, $\theta\cdot Y = {}^{T_n}X^{T_{n+1}} + {}^{T_n}[X, N]^{T_{n+1}}$ is a local martingale. Thus for any k,

$$
{}^0X^{T_k} + {}^0[X, N]^{T_k} = \sum_{0\leq n<k} {}^{T_n}X^{T_{n+1}} + {}^{T_n}[X, N]^{T_{n+1}}
$$

is a local martingale, hence $X + [X, N]$ is a local martingale. Conversely, if X is an arbitrary semimartingale such that $X + [X, N]$ is a local martingale, then by (7), ${}^{T_n}X\ {}^{T_n}\mathscr{E}$ is a local martingale for any n, and hence X is an $\mathscr{E}$-local martingale. □

The subsequent corollary is stated in Yoeurp (1982).

COROLLARY 3.16. *Let X be a special semimartingale with canonical decomposition $X = X_0 + M + A$. Then X is an $\mathscr{E}$-local martingale iff $[M, N]$ is locally integrable and $A = -\langle M, N\rangle$.*

PROOF. By Proposition 3.15, X is an $\mathcal{E}$-local martingale iff $X+[X,N]$ is a local martingale, or equivalently $A+[M,N]$ is a local martingale. □

From the definition of an $\mathcal{E}$-local martingale and the previous corollary we easily deduce the next corollary.

COROLLARY 3.17. *If $X = X_0 + M - \langle M, N\rangle$ is a special semimartingale, such that for any n we have $E(X_T^*({}^{T_n}\mathcal{E})_T^*) < +\infty$, then X is an $\mathcal{E}$-martingale.*

4. Inequalities for $\mathcal{E}(N)$-martingales.

We begin this section by extending the well-known Doob inequality for martingales to $\mathcal{E}$-martingales.

THEOREM 4.1. *Let $p \in (1,+\infty)$, q be its conjugate and assume $\mathcal{E}$ is regular. The following assertions are equivalent.*

(i) *$\mathcal{E}$ satisfies (R_q).*
(ii) *There exists a constant C such that for any $X \in \mathcal{M}(\mathcal{E})$,*

$$||X_T^*||_p \le C||X_T||_p. \tag{8}$$

(ii′) *There exists a constant C such that for any $X \in \mathcal{M}(\mathcal{E})$ and any $\lambda > 0$,*

$$\lambda^p P(X_T^* > \lambda) \le CE(|X_T|^p 1_{\{X_T^*>\lambda\}}). \tag{9}$$

(iii) *There exists a constant C such that for any bounded $\mathcal{E}$-martingale X and any stopping time τ,*

$$P(|X_\tau| \ge 1) \le C||X_T||_p^p. \tag{10}$$

For the proof we need the following easy, but very useful, lemma.

LEMMA 4.2. *If $(A_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ are positive cadlag processes, (A_t) is increasing and adapted, U is a random variable, such that for any $s \in [0,T]$,*

$$E\Big(\sup_{t\ge s} B_t|\mathcal{F}_s\Big) \le E(U|\mathcal{F}_s),$$

then

$$E\Big(\sup_t (A_tB_t)\Big) \le E(A_TU).$$

PROOF. Since (A_tB_t) is cadlag,

$$E\big(\sup(A_tB_t)\big) = \sup_{\pi-\text{finite}\subset T} E\Big(\sup_{t\in\pi}(A_tB_t)\Big).$$

Let us take an arbitrary finite set π, but for the simplicity of notation let us assume that $\pi = \{0,1,\dots,T\}$. Let $(D_n)_{n=0}^T$ be any measurable partition of Ω.

Put $\varepsilon_n = 1_{D_n}$, $\Delta A_n = A_n - A_{n-1}$ for $N \geq 1$ and $\Delta A_0 = A_0$. We have

$$\begin{aligned}
E\biggl(\sum_n \varepsilon_n A_n B_n\biggr) &= E\biggl(\sum_n \varepsilon_n B_n\biggl(\sum_{k\leq n} \Delta A_k\biggr)\biggr) = E\biggl(\sum_k \Delta A_k\biggl(\sum_{n\geq k} \varepsilon_n B_n\biggr)\biggr)\\
&= E\biggl(\sum_k \Delta A_k E\biggl(\sum_{n\geq k} \varepsilon_n B_n|\mathscr{F}_k\biggr)\biggr) \leq E\biggl(\sum_k \Delta A_k E\Bigl(\sup_{n\geq k} B_n|\mathscr{F}_k\Bigr)\biggr)\\
&\leq E\biggl(\sum_k \Delta A_k E(U|\mathscr{F}_k)\biggr) = E\biggl(\sum_k \Delta A_k U\biggr) = E(A_T U).
\end{aligned}$$

Since the set π and the partition D_n were arbitrary, we get the claim. □

PROOF OF THEOREM 4.1. We begin with the proof of (i) ⇒ (ii). It is the most difficult implication in this theorem. In the classical case this result was obtained by Doléans-Dade and Meyer (1979) under an additional assumption on jumps and by Jawerth (1986) without this additional assumption. The reader is encouraged to analyze our proof in the classical case, in which it is less technical (there are no problems with T_n) and simpler than proofs in Doléans-Dade and Meyer (1979) or Jawerth (1986).

Obviously, we can assume $E(|X_T|^p) < +\infty$. Let us fix n and denote $Z_t = {}^{T_n}\mathscr{E}_t$. By regularity, this is a martingale and by Proposition 3.5, $Z_{T_{n+1}} = Z_T$. We define a measure Q, absolutely continous with respect to P by the formula

$$dQ = |Z_T|^q\, dP.$$

By (R_q) the measure Q is finite. Let

$$Y_T := 1_{\{T_n<T\}} \frac{X_{T_{n+1}}}{|Z_T|^{q-1}} \operatorname{sign}(Z_T) 1_{\{Z_T\neq 0\}}.$$

Since we obviously have

$$E(|Y_T|^p |Z_T|^q) \leq E(|X_T|^p) < +\infty,$$

we conclude that $Y_T \in L^p(Q)$. Let

$$Y_t = E^Q(Y_T|\mathscr{F}_t).$$

Here Y is a Q-martingale, hence for any $A \in \mathscr{F}_t$,

$$\int_A Y_T |Z_T|^q\, dP = \int_A Y_t |Z_T|^q\, dP = \int_A Y_t E(|Z_T|^q|\mathscr{F}_t)\, dP,$$

and therefore

$$N_t = Y_t E(|Z_T|^q|F_t) \tag{11}$$

is a martingale and

$$N_T = 1_{\{T_n<T\}} X_{T_{n+1}} Z_T.$$

Since $X \in \mathscr{M}(\mathscr{E})$, by Proposition 3.12,

$$E(X_{T_{n+1}} Z_T|\mathscr{F}_t) = X_t Z_t$$

on $\{t \in [T_n, T_{n+1}]\}$ and hence on this set

$$N_t = 1_{\{T_n<T\}} X_t Z_t. \tag{12}$$

By the definition of T_{n+1}, $Z_t \neq 0$ for $t < T_{n+1}$ and therefore by (11), (12) and (R_q), for $t \in [T_n, T_{n+1})$,

$$|X_t| = \left| \frac{E(|Z_T|^q | \mathscr{F}_t) Y_t}{Z_t} \right| \leq C|Y_t|\ |Z_t|^{q-1}.$$

Hence

$$\begin{aligned} E\Big(\sup_{t\in[T_n,\, T_{n+1})} |X_t|^p\Big) &\leq CE\Big(\sup_{t\in[T_n,\, T_{n+1})} (|Y_t|\ |Z_t|^{q-1})^p\Big) \\ &\leq CE\Big(\sup_t |Y_t|^p |Z_t|^q\Big). \end{aligned} \tag{13}$$

Since by the regularity assumption, Z is a martingale, the (conditional) Doob inequality yields for any s,

$$E\Big(\sup_{t\geq s} |Z_t|^q\ |\mathscr{F}_s\Big) \leq CE(|Z_T|^q\ |\mathscr{F}_s).$$

By Lemma 4.2 (for $A = Y^{*p}$, $B = |Z|^q$ and $U = C|Z_T|^q$),

$$E\Big(\sup_t |Y_t|^p |Z_t|^q\Big) \leq CE(Y_T^{*p} |Z_T|^q). \tag{14}$$

Since Y is a Q-martingale, by the Doob inequality under measure Q we have

$$\begin{aligned} E(Y_T^{*p}|Z_T|^q) = E^Q(Y_T^{*p}) \leq CE^Q(|Y_T|^p) &= CE(|X_{T_{n+1}}|^p 1_{\{T_n<T,\, |Z_T|>0\}}) \\ \leq CE(|X_{T_{n+1}}|^p 1_{\{T_n<T,\, T_{n+1}=T\}}) &= CE(|X_T|^p 1_{\{T_n<T,\, T_{n+1}=T\},}), \end{aligned} \tag{15}$$

where we used that $Z_T = 0$ on $\{T_{n+1} < T\}$. Combining (13), (14) and (15), we get

$$E\Big(\sup_{t\in[T_n,\, T_{n+1})} |X_t|^p\Big) \leq CE(|X_T|^p 1_{\{T_n<T,\, T_{n+1}=T\}}).$$

Hence

$$\begin{aligned} E\Big(\sup_{t\in[0,\, T]} |X_t|^p\Big) &\leq E\Big(\sum_n \sup_{t\in[T_n,\, T_{n+1})} |X_t|^p\Big) + E(|X_T|^p) \\ &\leq C\sum_n E(|X_T|^p 1_{\{T_n<T,\, T_{n+1}=T\}}) + E(|X_T|^p) \\ &\leq CE(|X_T|^p), \end{aligned}$$

since events $\{T_n < T, T_{n+1} = T\}$ are disjoint.

This completes the proof of (i) $\Rightarrow$ (ii).

Obviously (ii) $\Rightarrow$ (iii).

Now we are going to prove (iii) $\Rightarrow$ (i). Fix $0 < \alpha < \beta$. Let τ be a stopping time, $A' \in \mathscr{F}_\tau$ and ${}^*\mathscr{E}_T := \sup_{0 \le t \le T} |{}^t\mathscr{E}_T|$. Put

$$A'' := \{E(|{}^\tau\mathscr{E}_T|^q 1_{\{{}^*\mathscr{E}_T \le \beta\}} | \mathscr{F}_\tau) \ge \alpha\}, \qquad A := A' \cap A'',$$

$$H := 1_A 1_{\{{}^*\mathscr{E}_T \le \beta\}} \frac{|{}^\tau\mathscr{E}_T|^{q-1} \operatorname{sign}({}^\tau\mathscr{E}_T)}{E(|{}^\tau\mathscr{E}_T|^q 1_{\{{}^*\mathscr{E}_T \le \beta\}} | \mathscr{F}_\tau)}, \qquad X_t := E(H\,{}^t\mathscr{E}_T | \mathscr{F}_t).$$

By Proposition 3.12, X is a bounded $\mathscr{E}$-martingale such that $X_\tau = 1_A$ and hence

$$P(A) = P(|X_\tau| \ge 1) \le CE(|X_T|^p) = CE\left(1_A \frac{1}{(E(|{}^\tau\mathscr{E}_T|^q 1_{\{{}^*\mathscr{E}_T \le \beta\}} | \mathscr{F}_\tau))^{p-1}}\right).$$

It follows that

$$1_{A''} E(|{}^\tau\mathscr{E}_T|^q 1_{\{{}^*\mathscr{E}_T \le \beta\}} | \mathscr{F}_\tau) \le C.$$

Since $0 < \alpha < \beta$ are arbitrary, we get the claim.

Therefore (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Now we prove (ii) $\Rightarrow$ (ii′). Assume $X_T 1_{\{X_T^* > \lambda\}} \in L^p$. For $\lambda > 0$, set $\tau_\lambda := \{t \colon |X_t| > \lambda\}$ and observe that $\{X_T^* > \lambda\} = \{\tau_\lambda < T\} \cup \{|X_{\tau_\lambda}| > \lambda\}$ belongs to $\mathscr{F}_{\tau_\lambda}$. By Proposition 3.12 there exists $Y \in \mathscr{M}(\mathscr{E})$ such that $Y_T := X_T 1_{\{X_T^* > \lambda\}}$ and $Y_{\tau_\lambda} = X_{\tau_\lambda}$ on $\{X_T^* > \lambda\}$. From (ii) applied to Y it follows that

$$E(|X_{\tau_\lambda}|^p 1_{\{X_T^* > \lambda\}}) \le CE(|X_T|^p 1_{\{X_T^* > \lambda\}}). \tag{16}$$

As $\lambda^{-1}|X_{\tau_\lambda}| \ge 1$ on $\{X_T^* > \lambda\}$, we obtain

$$\lambda^p P(X_T^* > \lambda) \le CE(|X_T|^p 1_{\{X_T^* > \lambda\}}). \tag{17}$$

By combining (16) and (17) we conclude (ii) $\Rightarrow$ (ii′).

Finally, the implication (ii′) $\Rightarrow$ (iii) is obvious.

The proof of Theorem 4.1 is complete. □

Now we are going to give a necessary and sufficient condition for the equivalence of the three following norms: $|||X|||$, $\|X_T^*\|_2$ and $\|[X]_T^{1/2}\|_2$. This generalizes some results of DMSSS.

THEOREM 4.3. *Assume $N \in \mathscr{M}^2_{0,\mathrm{loc}}$. The following assertions are equivalent:*

(i) $\widetilde{N} := (1/\sqrt{1 + \Delta\langle N\rangle}) \cdot N \in bmo_2$.

(ii) *There exists a constant C such that for any $M \in \mathscr{M}^2_0$ we have*

$$\left\| \int_0^T |d\langle M, N\rangle| \right\|_2 \le C \big\| \big[\sqrt{1 + \Delta\langle N\rangle} \cdot M\big]_T^{1/2} \big\|_2. \tag{18}$$

(iii) *There exists a constant C such that for any $\mathscr{E}$-local martingale X we have*

$$|||X||| \le C \|[X]_T^{1/2}\|_2. \tag{19}$$

(iv) *There exists a constant C such that for any $\mathscr{E}$-local martingale X we have*

$$\|X_T\|_2 \le C\|[X]_T^{1/2}\|_2. \tag{20}$$

(v) *There exists a constant C such that for any $\mathscr{E}$-local martingale X we have*

$$|||X||| \le C\|X_T^*\|_2. \tag{21}$$

For the proof of Theorem 4.3 we will need the following lemmas.

LEMMA 4.4. *There exists a constant C such that*

$$\forall\ X \in \mathscr{H}^2, \qquad E([X]_T) \le C|||X|||\,||X_T^*||_2.$$

PROOF. Let $X \in \mathscr{H}^2$ and $X = X_0 + M + A$ be its canonical decomposition. We have

$$E([X]_T) = E(X_T^2) - 2E((X_- \cdot X)_T) = E(X_T^2) - 2E((X_- \cdot M)_T) - E((X_- \cdot A)_T).$$

We estimate each of the three terms separately:

$$E(X_T^2) \le |||X|||\,||X_T^*||_2$$

is obvious,

$$\begin{aligned} E((X_- \cdot M)_T) &\le CE([X_- \cdot M]_T^{1/2}) \le CE((X_-^2 \cdot [M])_T^{1/2}) \le CE(X_T^*[M]_T^{1/2}) \\ &\le C||X_T^*||_2(E([M]_T))^{1/2} \end{aligned}$$

by the Burkholder–Davis–Gundy inequality, and

$$E((X_- \cdot A)_T) \le E\Big(X_T^* \int |dA|\Big) \le ||X_T^*||_2 \Big|\Big| \int |dA| \Big|\Big|_2.$$

Putting these estimations together, we get

$$E([X]_T) \le ||X_T^*||_2\Big(|||X||| + C(E([M]_T))^{1/2} + \Big|\Big|\int |dA|\Big|\Big|_2\Big) \le C||X_T^*||\,|||X|||. \qquad \square$$

The second lemma is a more general formulation of Theorem 3.3 of DMSSS. It extends the Fefferman inequality [see, for instance, Pratelli (1976) and Yor (1985)].

LEMMA 4.5. *Assume $N \in \mathscr{M}^2_{\mathrm{loc}}$. Then $N \in bmo_2$ iff there is a constant C such that for each square integrable martingale M we have*

$$E\Big(\Big(\int |d\langle M, N\rangle|\Big)^2\Big) \le CE(\langle M\rangle_T). \tag{22}$$

PROOF. Assume first $N \in bmo_2$ and put $\langle M, N\rangle = A$. Without loss of generality we can assume that A is increasing (otherwise we multiply dM by the sign of $d\langle M, N\rangle$). Since A is predictable, by a stopping argument we may assume that it is bounded. Since for any increasing process A we have $E(A_T^2) \le 2E((A\cdot A)_T)$, we get the following inequalities:

$$
\begin{aligned}
E(A_T^2) &\le 2E\Big(\int |A\,d\langle M, N\rangle|\Big) = 2E\Big(\int |d\langle A\cdot M, N\rangle|\Big) \\
&\le C\|N\|_{bmo_2} E((\langle A\cdot M\rangle)_T^{1/2}),
\end{aligned} \tag{23}
$$

where the last inequality follows from Fefferman inequality [see Pratelli (1976)]. Since A is increasing we have

$$
\begin{aligned}
E((\langle A\cdot M\rangle)_T^{1/2}) &= E((A^2\cdot\langle M\rangle)_T^{1/2}) \le E(A_T\langle M\rangle_T^{1/2}) \\
&\le (E(A_T^2))^{1/2}(E(\langle M\rangle_T))^{1/2}.
\end{aligned} \tag{24}
$$

To complete the proof, it is enough to combine (23), (24) and divide both sides of the obtained inequality by $(E(A_T^2))^{1/2}$, which is finite.

Now we are going to prove the converse. Assume (22) holds. By a stopping argument we may assume that $N \in \mathscr{M}^2$. Fix $t \in [0, T]$, $A \in \mathscr{F}_t$ and let $M := 1_A(N - N^t)$. Thus by (22),

$$
\begin{aligned}
E((1_A(\langle N\rangle_T - \langle N\rangle_t))^2) &\le CE(1_A(\langle N\rangle_T - \langle N\rangle_t)) \\
&\le C(P(A))^{1/2}[E((1_A(\langle N\rangle_T - \langle N\rangle_t))^2)]^{1/2}.
\end{aligned}
$$

Dividing by $[E((1_A(\langle N\rangle_T - \langle N\rangle_t))^2)]^{1/2}$ we obtain

$$E((1_A(\langle N\rangle_T - \langle N\rangle_t)^2)) \le CP(A).$$

It follows that

$$[E(\langle N\rangle_T - \langle N\rangle_t|\mathscr{F}_t)]^2 \le E((\langle N\rangle_T - \langle N\rangle_t)^2|\mathscr{F}_t) \le C$$

and so N is in bmo_2. □

The next lemma improves Lemma 3.8 of DMSSS.

LEMMA 4.6. *If B_t is a cadlag predictable process of finite variation, $B_0 = 0$, and $\eta > 0$ is fixed, then there exists a predictable process ε, taking values in $\{-1, 1\}$, such that*

$$\sup_t |(\varepsilon\cdot B)_t| \le \sup_t |\Delta B_t| + \eta.$$

PROOF. We may assume, that B is increasing (otherwise multiply dB_t by its sign). We define an increasing sequence of stopping times by the formula

$$S_0 = 0, \qquad S_{n+1} = \inf\left\{t \ge S_n : B_t - B_{S_n} \ge \left|\sum_{k=0}^{n-1}(-1)^k(B_{S_{k+1}} - B_{S_k})\right| + \eta\right\},$$

and set

$$\varepsilon = \sum_{k\geq 0}(-1)^k 1_{]\!]S_k,\, S_{k+1}]\!]}.$$

Now it is easy to check that

$$\sup_t |(\varepsilon\cdot B)_t| \leq \sup_t |\Delta B_t| + \eta. \qquad \square$$

PROOF OF THEOREM 4.3. First note that we may always assume that the $\mathscr{E}$-local martingale X is locally square integrable. Then X is a special semimartingale with canonical decomposition $X := X_0 + M - \langle M, N\rangle$, $M \in \mathscr{M}^2_{0,\,\mathrm{loc}}$.

The equivalence (i) $\Leftrightarrow$ (ii) is a straightforward application of Lemma 4.5 since

$$\int_0^T |d\langle M, \widetilde{N}\rangle| = \int_0^T |d\langle (1+\Delta\langle N\rangle)^{-1/2}\cdot M, N\rangle|.$$

Next we are going to prove (ii) $\Rightarrow$ (iii). Let $X := X_0 + M - \langle M, N\rangle$. By a stopping argument we may assume that $M \in \mathscr{M}^2_0$. According to the Galtchouk–Kunita–Watanabe decomposition there exits a predictable process λ such that $E((\lambda^2\cdot[N])_T) < +\infty$ and $M = \lambda\cdot N + L$ where $L \in \mathscr{M}^2_0$ and $\langle L, N\rangle = 0$. Since $\langle M, N\rangle = \lambda\cdot\langle N\rangle$, we get $[\langle M, N\rangle] = (\Delta\langle N\rangle)\cdot\langle \lambda\cdot N\rangle$. Now we have

$$E([X]_T) = E(X_0^2 + [M]_T + [\langle M, N\rangle]_T) \geq E\big(\big[\sqrt{1+\Delta\langle N\rangle}\cdot(\lambda\cdot N)\big]_T\big).$$

By (18) we have

$$E\bigg(\bigg(\int_0^T |d\langle M, N\rangle|\bigg)^2\bigg) = E\bigg(\bigg(\int_0^T |d\langle \lambda\cdot N, N\rangle|\bigg)^2\bigg) \leq CE\big(\big[\sqrt{1+\Delta\langle N\rangle}\cdot(\lambda\cdot N)\big]_T\big).$$

Thus we obtain

$$|||X|||^2 := E\bigg(X_0^2 + [M]_T + \bigg(\int_0^T |d\langle M, N\rangle|\bigg)^2\bigg) \leq CE([X]_T).$$

Since $\|X_T\| \leq C|||X|||$, (iii) $\Rightarrow$ (iv).

Next we are going to prove (iv) $\Rightarrow$ (ii). Let $M \in \mathscr{M}^2_0$. According to the Galtchouk–Kunita–Watanabe decomposition there exists a predictable process λ such that $E((\lambda^2\cdot[N])_T) < +\infty$ and $M = \lambda\cdot N + L$ where $L \in \mathscr{M}^2$ and $\langle L, N\rangle = 0$. Since $\langle M, N\rangle = \lambda\cdot\langle N\rangle$, we get $[\langle M, N\rangle] = (\lambda^2\Delta\langle N\rangle)\cdot\langle N\rangle$. There exists a predictable process ε taking its values in $\{-1, 1\}$ such that $|d\langle M, N\rangle| = \varepsilon d\langle M, N\rangle = \varepsilon d\langle \lambda\cdot N, N\rangle$. Let $X := (\varepsilon\lambda)\cdot N - \langle(\varepsilon\lambda)\cdot N, N\rangle$. Now recall that

$$\begin{aligned}E([X]_T) &= E([\lambda\cdot N]_T + [\langle \lambda\cdot N, N\rangle]_T) = E\big(\big[\sqrt{1+\Delta\langle N\rangle}\cdot(\lambda\cdot N)\big]_T\big)\\ &\leq E\big(\big[\sqrt{1+\Delta\langle N\rangle}\cdot M\big]_T\big).\end{aligned}$$

Observe that

$$E(((\varepsilon\lambda)\cdot N)_T^2) = E([(\varepsilon\lambda)\cdot N]_T) = E([\lambda\cdot N]_T) \leq E([X]_T).$$

Next we have

$$E\left(\left(\int_0^T |d\langle M,N\rangle|\right)^2\right) = E\left(\left(\int_0^T d\langle(\varepsilon\lambda)\cdot N,N\rangle\right)^2\right)$$
$$\le C(\|X_T\|_2^2 + E(((\varepsilon\lambda)\cdot N)_T^2) \le CE([X]_T),$$

where the last inequality follows from (20). Therefore we get (18).

Next we are going to prove (iii) $\Rightarrow$ (v). Inequality (19) and Lemma 4.4 imply that

$$|||X|||^2 \le CE([X]_T) \le C|||X|||\|X_T^*\|_2. \tag{25}$$

In order to prove (iii) $\Rightarrow$ (v), we may assume that $\|X_T^*\|_2 < +\infty$. By a stopping argument we may assume that $|||X||| < +\infty$. Dividing by $|||X|||$ in (25) we obtain (21).

It remains to prove (v) $\Rightarrow$ (iii). Let $X = X_0 + M - \langle M,N\rangle$ be the canonical decomposition of X. Fix $\eta > 0$ and put $B := \langle M,N\rangle$. Lemma 4.6 tells us that

$$\|(\varepsilon\cdot X)_T^*\|_2 \le \|(\varepsilon\cdot M)_T^*\|_2 + \|(\varepsilon\cdot B)_T^*\|_2$$
$$\le C(\|[M]_T^{1/2}\|_2 + \|(\Delta B)_T^*\|_2 + \eta) \le C(\|[X]_T^{1/2}\|_2 + \eta),$$

where the last inequality follows from

$$E([X]_T) = E(X_0^2 + [M]_T + [\langle M,N\rangle]_T).$$

Since $\varepsilon\cdot X$ is an $\mathscr{E}$-local martingale, by (21) we have $|||X||| = |||\varepsilon\cdot X||| \le \|(\varepsilon\cdot X)_T^*\|_2$ and thus we get $|||X||| \le \|[X]_T^{1/2}\|_2$. The proof of Theorem 4.3 is now complete. $\square$

The next corollary, which extends for $p = 2$ the Burkholder–Davis–Gundy inequalities to $\mathscr{E}$-martingales, is an obvious consequence of Theorem 4.3, since by Proposition 2.8 there exists a constant C such that for each special semimartingale X, $C|||X||| \ge \|X_T^*\|_2$ and $C|||X||| \ge \|[X]_T^{1/2}\|_2$.

COROLLARY 4.7. *Assume $N \in \mathscr{M}^2_{0,\mathrm{loc}}$. The following assertions are equivalent.*

(i) $\widetilde{N} := 1/\sqrt{1+\Delta\langle N\rangle}\cdot N \in bmo_2$.

(ii) *There exists a constant $C > 0$ such that for any $\mathscr{E}$-local martingale we have*

$$C^{-1}E([X]_T) \le E((X_T^*)^2) \le CE([X]_T). \tag{26}$$

REMARK 4.8. Note that if there exists an increasing sequence of stopping times $(\tau_n)_n$ that converges stationarily to τ and, if for all n, X^{τ_n} satisfies one of the inequalities (19), (20), (21) or (26), then X^τ also satisfies the same inequality. Hence we may replace in Theorem 4.3 and Corollary 4.7 the assertion “for any $\mathscr{E}$-local martingale” by “for any stopping time τ and any $X \in \mathscr{M}(\mathscr{E}^\tau)$.” Indeed, if $X := X_0 + M - \langle M,N\rangle$ where $M,N \in \mathscr{M}^2_{0,\mathrm{loc}}$, there exists a sequence of stopping times $(\tau_p)_p$ that converges stationarily to T such that for all p, $M^{\tau_p} \in \mathscr{M}^2$ and $\langle N \rangle_{\tau_p} \in L^\infty$. It follows from Proposition 3.7 that for all p and for all n, ${}^{T_n}\mathscr{E}^{\tau_p} \in \mathscr{M}^2$ and from Corollary 3.17 that X^{τ_p} is an $\mathscr{E}^{\tau_p}$-martingale. Hence if one of the inequalities (19), (20), (21) or (26) is satisfied for all stopping times τ and any $X \in \mathscr{M}(\mathscr{E}^\tau)$, then the same inequality holds for any $M \in \mathscr{M}^2_{0,\mathrm{loc}}$ and $X := X_0 + M - \langle M, N \rangle$. Conversely, any $\mathscr{E}^\tau$-martingale that is locally square integrable admits the canonical decomposition $X := X_0 + M - \langle M, N^\tau \rangle$ where $M \in \mathscr{M}^2_{0,\mathrm{loc}}$. Now assume one of the inequalities (19), (20), (21) or (26) is satisfied for any $M \in \mathscr{M}^2_{0,\mathrm{loc}}$ and $X := X_0 + M - \langle M, N \rangle$. By combining the Burkholder–Davis–Gundy inequalities for the local martingale $X - X^\tau$ and the previous inequality for X^τ, the same inequality holds for all stopping times τ and any $X \in \mathscr{M}(\mathscr{E}^\tau)$.

Now we are going to give a necessary and sufficient condition for the equivalence of the norms $\|[X]_T^{1/2}\|_2$, $\|X_T\|_2$ and $|||X|||$.

THEOREM 4.9. *Assume $\mathscr{E}$ is regular and $N \in \mathscr{M}_{0,\mathrm{loc}}$. Then the following assertions are equivalent.*

(i) *$\mathscr{E}$ satisfies (R_2).*

(ii) *There exists a constant $C > 0$ such that for any stopping time τ and any $X \in \mathscr{M}(\mathscr{E}^\tau)$ we have*

$$C^{-1}\|[X]_T^{1/2}\|_2 \le \|X_T\|_2 \le C\|[X]_T^{1/2}\|_2. \tag{27}$$

(iii) *There exists a constant C such that for any stopping time τ and any $X \in \mathscr{M}(\mathscr{E}^\tau)$ we have*

$$|||X||| \le C\|X_T\|_2. \tag{28}$$

PROOF. If $\mathscr{E}$ satisfies (R_2), then Proposition 3.9 shows that, for any stopping time τ, $\mathscr{E}^\tau$ satisfies (R_2) with a constant C independent of τ. Now we are going to prove (i) $\Rightarrow$ (ii). By Proposition 3.10, $N \in bmo_2$ and thus by combining Corollary 4.7, Theorem 4.1 and Theorem 4.3 we get (ii). The implication (ii) $\Rightarrow$ (iii) is a straightforward application of Theorem 4.3 and Remark 4.8.

Now since $\|X_T^*\|_2 \le |||X|||$, we conclude from Theorem 4.1 that (iii) $\Rightarrow$ (i). The proof of Theorem 4.9 is complete. □

5. Applications. Throughout this section we fix two locally square integrable martingales M and N. Suppose the first one is $\mathbb{R}^d$ valued whereas the second one is real valued and $M_0 = 0$, $N_0 = 0$. We put $X := M - \langle M, N \rangle = M + A$. Therefore X is an $\mathscr{E}(N)$-local martingale. According to the Galtchouk–Kunita–Watanabe decomposition there exists a predictable $\mathbb{R}^d$-valued process λ and a locally square integrable $\mathbb{R}^d$-valued martingale L such that $\int_0^T \lambda' d\langle M \rangle \lambda < +\infty$ where $'$ denotes transposition, $N = N_0 + \lambda \cdot M + L$ and $\langle L, M \rangle = 0$. It turns out that the developed model here coincides with that

of Schweizer (1994) who assumed that the so-called mean-variance tradeoff process K defined by

$$K_t = \int_0^t \lambda' d\langle M\rangle \lambda$$

is finite for all $t \in [0, T]$; that is, the structure condition holds for X. The existence of λ as well as finiteness of K_T is related to arbitrage properties as shown by Delbaen and Schachermayer (1995). When X is a bounded process admitting a bounded equivalent martingale measure, it follows from Choulli and Stricker (1996) that X satisfies the structure condition. In the case where X is continuous, the structure condition is a necessary condition for the existence of an equivalent local martingale measure. Also in the case where X is continuous, the finiteness of K_T is independent of the choice of probability measure, as shown in Delbaen and Shirakawa (1996) or Choulli and Stricker (1996). For the interpretation of K we refer to Schweizer (1994).

A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0\le t\le T}$ belongs to $L^2(M)$ if

$$E\left(\int_0^T \theta_t' d\langle M\rangle_t \theta_t\right) < +\infty.$$

We define on the space $L^2(M)$ the norm $\|\cdot\|_{L^2(M)}$ by

$$\|\theta\|^2_{L^2(M)} := \|(\theta\cdot M)_T\|_2^2 = E\left(\int_0^T \theta_t' d\langle M\rangle_t \theta_t\right).$$

A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0\le t\le T}$ belongs to $L^2(A)$ if the process

$$\left(\int_0^t |\theta_s' dA_s|\right)_{0\le t\le T} \text{ is square integrable.}$$

We define on the space $L^2(A)$ the norm $\|\cdot\|_{L^2(A)}$ by

$$\|\theta\|_{L^2(A)} := \left\|\int_0^T |\theta_s' dA_s|\right\|_2.$$

Finally, Θ is the space defined by $\Theta := L^2(M)\cap L^2(A)$; $\theta\in\Theta$ is called a $\mathscr{L}^2$-strategy.

If the structure condition holds, then clearly

$$\|\theta\|^2_{L^2(A)} = E\left[\left(\int_0^T |\theta_s' d\langle M\rangle_s \lambda_s|\right)^2\right].$$

Strictly speaking, the Banach space $L^2(M)$ is the space of equivalence classes of predictable processes θ with finite $L^2(M)$-norm modulo the subspace of predictable processes θ for which the process $\theta\cdot M$ vanishes almost surely. But we use the usual identification of processes with the associated equivalence class if no confusion can arise. A similar remark applies to $L^2(A)$ and Θ. Set

$$G_T(\Theta) := \{(\theta\cdot X)_T \colon \theta\in\Theta\}.$$

Let us recall one of the main results of DMSSS.

THEOREM 5.1. *Let X denote a continuous semimartingale such that there is an equivalent local martingale measure with square integrable density. Then $G_T(\Theta)$ is closed in L^2 iff there is a local martingale measure Q with density satisfying (R_2).*

Example 3.9 in DMSSS shows that for processes with jumps, Theorem 5.1 no longer holds. Actually, there is a bounded process $X = (X_0, X_1, X_2)$ admitting a bounded equivalent martingale measure such that we have the following:

1. $X = M + \lambda \cdot \langle M \rangle$ where $M \in \mathscr{M}_0^2$, $\lambda \in L^2(M)$ and $\lambda \cdot M \notin bmo_2$;
2. $G_2(\Theta)$ is closed in L^2.

Now assume that there is $N \in \mathscr{M}^2_{0,\,\mathrm{loc}}$ such that X is an $\mathscr{E}(N)$-local martingale and $\mathscr{E}(N)$ satisfies (R_2). From Proposition 1.64 of Jacod and Shiryaev (1987) we deduce that $\mathscr{E}(N)$ is regular. Then by Proposition 3.10, $N \in bmo_2$. Since X is a bounded $\mathscr{E}(N)$-local martingale, X is a special semimartingale and $\langle N, M \rangle = \lambda \cdot \langle M \rangle$. It follows that $N = \lambda \cdot M + L$ where $L \in \mathscr{M}^2_{0,\,\mathrm{loc}}$ and $\langle L, M \rangle = 0$. Thus $\lambda \cdot M \in bmo_2$. This is a contradiction and $\mathscr{E}(N)$ does not satisfy (R_2).

When $\langle N \rangle_T \in L^\infty$, Monat and Stricker (1995) show that $G_T(\Theta)$ is closed in L^2. By Proposition 3.7, the boundedness of $\langle N \rangle_T$ implies that $\mathscr{E}(N)$ satisfies (R_2) and is regular. Thus the next theorem extends Theorem 2.4 of Monat and Stricker (1995) and the "if" part of Theorem 5.1 in the discontinuous case.

THEOREM 5.2. *Assume $\mathscr{E}$ is regular and satisfies (R_2). Then for any σ-algebra $\mathscr{G}_0 \subset \mathscr{F}_0$, $L^2(\mathscr{G}_0) + G_T(\Theta)$ and $G_T(\Theta)$ are closed in L^2.*

PROOF. Let $(X_0^n + (\theta^n \cdot X)_T)_n$ be a sequence in $L^2(\mathscr{G}_0) + G_T(\Theta)$ that converges to Y in L^2. By Theorem 4.9, the sequence $(X_0^n + \theta^n \cdot X)_n$ converges in $\mathscr{H}^2$ equipped with the norm $|||\ |||$. Since $L^2(\mathscr{G}_0), (L^2(M), \|\ \|_{L^2(M)})$ and $(L^2(A), \|\ \|_{L^2(A)})$ are Banach spaces, there exist $Y_0 \in L^2(\mathscr{G}_0)$ and $\theta \in \Theta$ such that $Y = Y_0 + (\theta \cdot X)_T$. Therefore $L^2(\mathscr{G}_0) + G_T(\Theta)$ and $G_T(\Theta)$ are closed and the proof of Theorem 5.2 is complete. $\square$

EXAMPLE 5.3. Let X be a standard Poisson process with intensity λ, stopped at time T, and set $N_t := -X_t + t \wedge T$ for $t \in [0, T]$. Then by Proposition 3.7, $\mathscr{E}$ is regular and satisfies (R_2), by Proposition 3.17, X is an $\mathscr{E}$-martingale and for any bounded predictable process θ, $(\theta \cdot X)_T$ belongs to $G_T(\Theta)$. Moreover, by Theorem 5.2, $G_T(\Theta)$ is closed in Ł^2.

For the second application, which involves the Föllmer–Schweizer decomposition, we assume that $N := -\lambda \cdot M$ where $\lambda \in L^2(M)$. We extend some results of Schweizer (1994), Monat and Stricker (1995) and DMSSS where the continouous case was completely solved, and prove that X admits a Föllmer–Schweizer decomposition if and only if $\mathscr{E}(-\lambda \cdot M)$ is regular and satisfies (R_2). Since $\mathscr{E}$ may vanish, we should add to the definition of the Föllmer–Schweizer decomposition proposed in DMSSS (1997) an additional assumption. However, when $T_1 = T$ our definition coincides with that of DMSSS.

DEFINITION 5.4. (i) Given the semimartingale X, we say that a random variable $H \in L^2$ admits a Föllmer–Schweizer decomposition if it can be written

$$H = H_0 + (\theta \cdot X)_T + L_T, \tag{29}$$

where H_0 is an $\mathscr{F}_0$-measurable random variable, $\theta \in \Theta$ and $L \in \mathscr{M}_0^2$ with $\langle M, L\rangle = 0$.

(ii) The semimartingale X admits a Föllmer–Schweizer decomposition if there are unique continuous projections π_0, π_1, π_2 and π_3^n for $n \geq 1$: $L^2(\Omega, \mathscr{F}, P) \to L^2(\Omega, \mathscr{F}, P)$ such that every $H \in L^2(\Omega, \mathscr{F}, P)$ admits a Föllmer–Schweizer decomposition

$$H = \pi_0(H) + \pi_1(H) + \pi_2(H) = H_0 + (\theta \cdot X)_T + L_T,$$

$$\pi_3^n(H) = H_0 + (\theta \cdot X)_{T_n} + L_{T_n},$$

where $H_0 \in L^2(\Omega, \mathscr{F}_0, P)$, $\theta \in \Theta$ and $L \in \mathscr{M}_0^2$ with $\langle M, L\rangle = 0$.

If for any n ${}^{T_n}\mathscr{E} \in \mathscr{M}^2$, then $Y := H_0 + \theta \cdot X + L$ is an $\mathscr{E}$-martingale in $\mathscr{H}^2$ since $Y = Y_0 + (\theta \cdot M + L) - \langle \theta \cdot M + L, N\rangle$ and $E(Y_T^*({}^{T_n}\mathscr{E})_T^*) < +\infty$. Therefore, if for any n, ${}^{T_n}\mathscr{E} \in \mathscr{M}^2$, then $H \in L^2$ admits a Föllmer–Schweizer decomposition iff it is the terminal value of an $\mathscr{E}$-martingale Y in $\mathscr{H}^2$.

THEOREM 5.5. *X admits a Föllmer–Schweizer decomposition iff $\mathscr{E}$ is regular and satisfies (R_2).*

PROOF. We first prove the "if" part. Assume $\mathscr{E}$ is regular and satisfies (R_2). Since for any n, ${}^{T_n}\mathscr{E} \in \mathscr{M}^2$, if the Föllmer–Schweizer decomposition exists, by Proposition 3.12(i) it is unique. Let $H \in L^2$. By Proposition 3.12(iii) there exists $Y \in \mathscr{M}(\mathscr{E})$ such that $Y_T = H$. By Theorem 4.9, $Y \in \mathscr{H}^2$, $Y = Y_0 + I - \langle I, N\rangle = Y_0 + \theta \cdot X + L$, where $I = \theta \cdot M + L$ is the Galtchouk–Kunita–Watanabe decomposition. Thus, by Theorem 4.9, $\|Y_0\|_2 \leq C\|Y_T\|_2$, $\|L_T^*\|_2 \leq C\|Y_T\|_2$ and hence also $\|(\theta \cdot X)_T^*\|_2 \leq C\|Y_T\|_2$. It follows that π_0, π_1, π_2 and π_3^n are well defined and continuous. This proves that X admits a Föllmer–Schweizer decomposition.

Now we prove the "only if" part. Suppose that X admits a Föllmer–Schweizer decomposition and denote by π_0, π_1, π_2 and π_3^n the corresponding projections in L^2. Let $(\tau_n)_{n\geq 0}$ be an increasing sequence of stopping times converging stationarily to T and such that for any $n \geq 0$, K_{τ_n} is uniformly bounded. By Proposition 3.7 $\mathscr{E}$ is regular and satisfies (R_2). It follows from the "if" part or from Schweizer (1994) and Monat and Stricker (1995) that for every k and every $H \in L^2(\Omega, \mathscr{F}_{\tau_k}, P)$ there is a Föllmer–Schweizer decomposition $H = H_0 + (\theta \cdot X)_T + L_T$ such that the following formulas are valid for any n:

$$H_0 = \pi_0(H) = E(H\mathscr{E}_{\tau_k}|\mathscr{F}_0), \tag{30}$$

$$H_0 + (\theta \cdot X)_{T_n} + L_{T_n} = E(H^{T_n}\mathscr{E}_{\tau_k}|\mathscr{F}_{T_n}) \text{ for } T_n \le \tau_k. \tag{31}$$

As by assumption, π_0 and π_3^k are continuous on L^2, we obtain that for any n, ${}^{T_n}\mathscr{E} \in \mathscr{M}^2$. Therefore, Proposition 3.12 shows that for any $\mathscr{E}$-martingale Y the terminal value Y_T determines the whole process Y. Hence the map j: $(\mathscr{H}^2 \cap \mathscr{M}(\mathscr{E}),\ |||\ |||) \to L^2$ defined by $j(Y) = Y_T$ is continuous and one-to-one. Proposition 3.12 shows that $(\mathscr{H}^2 \cap \mathscr{M}(\mathscr{E}),\ |||\ |||)$ is a Banach space. By the Banach isomorphism theorem we obtain that there is a constant C such that for any $Y \in \mathscr{H}^2 \cap \mathscr{M}(\mathscr{E})$ we have $|||Y||| \le C\|Y_T\|_2$. By a localizing argument, $|||Y||| \le C\|Y_T\|_2$ for any $Y \in \mathscr{M}(\mathscr{E})$. It follows from Theorem 4.9 that $\mathscr{E}$ satisfies (R_2). The proof of the theorem is complete. □

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