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SPATIAL ESTIMATES FOR STOCHASTIC FLOWS IN EUCLIDEAN SPACE

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We study the behavior for large $|x|$ of Kunita-type stochastic flows $\phi(t, \omega, x)$ on R^d, driven by continuous spatial semimartingales. For this class of flows we prove new spatial estimates for large $|x|$, under very mild regularity conditions on the driving semimartingale random field. It is expected that the results would be of interest for the theory of stochastic flows on noncompact manifolds as well as in the study of nonlinear filtering, stochastic functional and partial differential equations. Some examples and counterexamples are given.

1. Introduction. Consider an Itô-type stochastic differential equation (s.d.e.) on $\mathbf{R}^d$

$$\text{(I)} \qquad \begin{aligned} d\phi(t) &= F(\phi(t), dt), \qquad t > s, \\ \phi(s) &= x \end{aligned}$$

drive by a spatial continuous semimartingale $F(x, t) := (F^1(x, t), \dots, F^d(x, t))$, $x \in \mathbf{R}^d$, in the sense of Kunita ([9], Chapter 3). It is known that, under suitable regularity conditions on (the local characteristics of) the driving semimartingale F, equation (I) admits a C^k stochastic flow $\phi_{s,t}$: $\Omega \times \mathbf{R}^d \to \mathbf{R}^d$, $-\infty < s \le t < \infty$, $k \ge 0$ ([9], pages 154–185; cf. [7]).

Regarding the above s.d.e., one might ask the following pertinent questions. Under what conditions on the local characteristics of the driving noise F can one guarantee that the induced flow $\phi_{s,t}(x)$ admits sublinear growth in x and/or globally bounded spatial derivatives almost surely? These questions are interesting from the point of view of the general theory of stochastic flows on noncompact manifolds as well in the study of global solutions of certain variational equations that appear in nonlinear filtering and anticipating stochastic differential equations ([9], pages 322 and 323, [14]). Ongoing work by the authors suggests that successful resolution of the above questions would shed some light on the problem of the existence of infinite-dimensional flows associated with quasilinear stochastic hereditary systems and stochastic partial differential equations (cf. [10, 11, 12, 13, 15]). This article is an attempt to resolve the above two questions in the manner explained below.

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The asymptotic behavior of $\phi_{s,t}$ for large t has been studied by many authors (e.g., [1, 2, 3, 4, 9]). However, few results are available regarding almost-sure spatial estimates on the flow $\phi(t, \omega, x)$ and its derivatives for large values of $|x|$. In fact, the only asymptotic results that we are aware of are the following limits which hold almost surely for any $\varepsilon > 0$:

$$\lim_{|x|\to\infty} \sup_{0\le s\le t\le T} \frac{|\phi_{s,t}(x)|}{(1+|x|)^{1+\varepsilon}} = 0,$$

$$\lim_{|x|\to\infty} \sup_{0\le s\le t\le T} \frac{(1+|x|)^{1-\varepsilon}}{1+|\phi_{s,t}(x)|} = 0,$$

and their derivative counterparts:

$$\lim_{|x|\to\infty} \sup_{0\le s\le t\le T} \frac{|D^{\alpha}\phi_{s,t}(x)|}{(1+|x|^{\varepsilon})} = 0,$$

where

$$\alpha := (\alpha_1, \alpha_2, \dots, \alpha_d), \qquad |\alpha| := \sum_{i=1}^{d} \alpha_i, \qquad D^{\alpha} := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}},$$

with α_i nonnegative integers for $i = 1, \dots, d$. The above limits are given in [14], Lemma 2.1 for s.d.e.'s driven by finite-dimensional Brownian motion, and in [9], pages 163, 176, under general regularity hypotheses on the semimartingale F.

In this article we strengthen the first two bounds on the spatial growth of the stochastic flow (Theorem 1, Section 2). This result shows that the stochastic flow $\phi_{s,t}(x)$ grows slower than $|x|(\log|x|)^{\varepsilon}$ as $|x| \to \infty$ for arbitrary small $\varepsilon > 0$. We show by example that this bound is sharp. In Section 3, we give an example of a one-dimensional s.d.e. with *sublinear* coefficients $F(\cdot, t)$ but with the underlying stochastic flow growing *superlinearly* for large x. In this example the stochastic flow has a.s. unbounded spatial derivatives, even though the driving martingale F has local characteristics with all derivatives globally bounded. An example of Baxendale also demonstrates this point [5]. It is interesting to note that in these two examples the driving noise is *infinite-dimensional*. However, the infinite dimensionality of the driving noise is *not* the crucial factor. The last example in this section is driven by *one-dimensional* Brownian motion and has coefficients with globally bounded derivatives, while its stochastic flow has a.s. unbounded derivatives. This result is surprising since it is in sharp contrast with well-known behavior of deterministic flows driven by vector fields whose derivatives are globally bounded. In the final section (Section 4), we give sufficient conditions on the coefficients of a one-dimensional s.d.e. in order for the stochastic flow to have sublinear growth and a.s. bounded derivatives (Theorem 3). In view of the last example in Section 3, the second assertion of Theorem 3 requires that the drift coefficient be *globally bounded*. We conclude the article by stating a conjecture on sublinear growth of stochastic flows drive by finite-dimensional noise.

The reader may note that, throughout this article, all s.d.e.'s are of Itô type.

2. Basic setting and general results. In this section, we will adopt the notation and terminology of Kunita ([9], pages 79–85).

Let $(\Omega, F, (F_t)_{t \geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions. Suppose $F(\cdot, t)$ is a continuous $C(\mathbf{R}^d, \mathbf{R}^d)$-valued (spatial) semimartingale. Express $F(x, \cdot)$ in the form

$$F(x,t) = V(x,t) + M(x,t), \qquad x \in \mathbf{R}^d, 0 \leq t \leq T,$$

where $V(x, \cdot)$ is a continuous bounded variation process and $M(x, \cdot)$ is a continuous local $(F_t)_{0 \leq t \leq T}$-martingale for each $x \in \mathbf{R}^d$. Let $\langle M(x, \cdot), M(y, \cdot)\rangle$ be the joint quadratic variation of $M(x, \cdot), M(y, \cdot)$ for $x, y \in \mathbf{R}^d$. Assume that F has local characteristics $(a(x, y, t), b(x, t), A_t)$ that satisfy the relations

$$\langle M(x,\cdot), M(y,\cdot)\rangle(t) = \int_0^t a(x,y,u)\, dA(u),$$

$$V(x,t) = \int_0^t b(x,u)\, dA(u), \qquad 0 \leq t \leq T,$$

with the process A $(F_t)_{0 \leq t \leq T}$-adapted, increasing and sample continuous. Further measurability properties of the local characteristics are given in [9], pages 79–85. Note that the local characteristics are not uniquely determined by F ([9], pages 79–85).

In what follows, let Δ denote the diagonal $\Delta := \{(x, x): x \in \mathbf{R}^d\}$ in $\mathbf{R}^d \times \mathbf{R}^d$, and let Δ^c be its complement.

Assume that the local characteristics of F are of class $B_b^{0,1}$ ([9], pages 79–85, 100–101). This means that for all $T > 0$,

$$\int_0^T \|a(t)\tilde{\|}_1\, dA(t) < \infty, \qquad \int_0^T \|b(t)\|_1\, dA(t) < \infty$$

almost surely, where

$$\begin{aligned}\|a(t)\tilde{\|}_1 = {} & \sup_{x,y \in \mathbf{R}^d} \frac{|a(x,y,t)|}{(1+|x|)(1+|y|)} \\ & + \sup\left\{ \frac{|a(x,y,t) - a(x',y,t) - a(x,y',t) + a(x',y',t)|}{(|x-x'|)(|y-y'|)} : \right. \\ & \left. \qquad\qquad (x,x'),(y,y') \in \Delta^c \right\}\end{aligned}$$

and

$$\|b(t)\|_1 = \sup_{x \in \mathbf{R}^d} \frac{|b(x,t)|}{(1+|x|)} + \sup\left\{ \frac{|b(x,t) - b(y,t)|}{|x-y|} : (x,y) \in \Delta^c \right\}.$$

THEOREM 1. *Let F be a continuous spatial semimartingale with local characteristics of class $B_b^{0,1}$. Then the s.d.e.* (I) *has a* (*continuous*) *flow*

$\phi_{s,t}\colon \Omega \times \mathbf{R}^d \to \mathbf{R}^d$, $0 \le s \le t < \infty$, *which satisfies*

$$\lim_{|x|\to\infty} \sup_{0\le s\le t\le T} \frac{|\phi_{s,t}(\cdot, x)|}{|x|(\log|x|)^{\varepsilon}} = 0$$

and

$$\lim_{|x|\to\infty} \sup_{0\le s\le t\le T} \frac{|x|}{|\phi_{s,t}(\cdot, x)|(\log|x|)^{\varepsilon}} = 0,$$

almost surely for every $\varepsilon > 0$ *and* $T > 0$.

REMARK. (i) The set of full measure in Theorem 1 may be chosen independently of ε and T.

(ii) The conclusion of Theorem 1 has an obvious counterpart for the stochastic flow $\phi^{\lambda}_{s,t}(x)$ depending on a *finite-dimensional* parameter $\lambda \in \mathbf{R}^n$ and associated with the s.d.e.

$$(\mathrm{I}^{\lambda}) \qquad \begin{aligned} d\phi^{\lambda}(t) &= F(\phi^{\lambda}(t), \lambda, dt), \qquad t > s,\\ \phi^{\lambda}(s) &= x. \end{aligned}$$

The precise conditions on F can be easily stated by applying Theorem 1 to the augmented flow $(\phi^{\lambda}_{s,t}(x), \lambda) \in \mathbf{R}^{d+n}$. It is interesting to note that the conclusion of Theorem 1 does not hold in general if the parameter λ varies in an *infinite-dimensional* Hilbert space (cf. [10], pages 28–29, 144–147).

(iii) From the first example in Section 3, we shall see that the first assertion of Theorem 1 is false for $\varepsilon = 0$.

To prove Theorem 1 we require three lemmas. In view of a remark in [9], page 85, we shall assume, without loss of generality, that the local characteristics of F are of class $B^{0,1}_{ub}$, namely

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{0\le t\le T} \left[\|a(t)\tilde{\|}_1 + \|b(t)\|_1\right] < \infty.$$

In addition and until further notice, we will assume that $A(t) \equiv t$ for all $t \ge 0$.

For fixed $\varepsilon > 0$ we define the following functions

$$f(x) := \frac{x}{(\log x)^{\varepsilon}}, \qquad x \in (1,\infty),$$

$$g(x) := \frac{1}{x(\log x)^{\varepsilon}}, \qquad x \in (1,\infty),$$

$$\hat{x} := xg(|x|), \qquad x \in \mathbf{R}^d, |x| > 1.$$

Recall that $|x|$ denotes the Euclidean norm of $x \in \mathbf{R}^d$.

LEMMA 1. *Let* $\varepsilon > 0$. *Then there exists an integer* $p_0 = p_0(\varepsilon) \ge 2$ *such that for every* $p \ge p_0$ *the following inequality holds*:

$$(2.1) \quad (1+y^2)^p[g(y) - g(x)]^{2p} \le C_1[yg(y) - xg(x)], \qquad 3 \le y \le x < \infty,$$

for some positive constant $C_1 := C_1(p, \varepsilon)$.

PROOF. Inequality (2.1) is trivial for $y = x$.
For $y < x$, consider the function

$$G(x, y) \coloneqq \frac{(1 + y^2)^p [g(y) - g(x)]^{2p}}{yg(y) - xg(x)}, \qquad 1 < y < x. \tag{2.2}$$

Differentiating g yields

$$\frac{dg}{dx} = -\frac{g(x)}{x}\left[1 + \frac{\varepsilon}{\log x}\right].$$

Therefore,

$$\frac{d}{dx}[xg(x)] = -\frac{\varepsilon g(x)}{\log x}.$$

In (2.2), fix $y \geq 3$ and apply the Cauchy mean value theorem to the function $G(\cdot, y)$. This gives a $\xi \in (y, x)$ such that

$$\begin{aligned} G(x, y) &= \frac{2p(1 + y^2)^p [g(y) - g(\xi)]^{2p-1}(-g'(\xi))}{\dfrac{\varepsilon g(\xi)}{\log \xi}} \\ &= \frac{2p}{\varepsilon}(1 + y^2)^p [g(y) - g(\xi)]^{2p-1}\left[\frac{\log \xi}{\xi} + \frac{\varepsilon}{\xi}\right] \\ &\leq \frac{2p}{\varepsilon}(1 + y^2)^p [g(y)]^{2p-1}\left[\frac{\log y}{y} + \frac{\varepsilon}{y}\right] \\ &\leq \frac{p \cdot 2^{p+1}}{\varepsilon}\left\{\frac{1}{(\log y)^{\varepsilon(2p-1)-1}} + \frac{\varepsilon}{(\log y)^{\varepsilon(2p-1)}}\right\} \end{aligned}$$

because the function $\log x/x$ is decreasing for $x \geq 3$. Take p_0 to be such that $\varepsilon(2p_0 - 1) - 1 > 0$, that is, $p_0 > \frac{1}{2}(1 + (1/\varepsilon))$. Then it is easy to check that the right-hand side of the above inequality is bounded in $y \geq 3$ whenever $p \geq p_0$. Hence for each $p \geq p_0$, G is bounded over the set $\{(x, y): 3 \leq y < x < \infty\}$ by a positive constant $C_1 \coloneqq C_1(p, \varepsilon)$. This proves (2.1). □

LEMMA 2. *Let the constants p_0, C_1 be as in Lemma 1. Then*

$$g(x)^{2p}(x - y)^{2p} \leq C_1(yg(y) - xg(x)), \qquad 3 \leq y \leq x < \infty,$$

for every $p \geq p_0$.

PROOF. Note that $xg(x) \leq yg(y)$ for $3 \leq y \leq x < \infty$. Therefore

$$g(x)(x - y) \leq y(g(y) - g(x)) \leq (1 + y^2)^{1/2}(g(y) - g(x))$$

and the assertion of the lemma follows from Lemma 1. □

LEMMA 3. *For every* $p \ge p_0$ *there is a positive constant* $C_2 := C_2(p, \varepsilon)$ *such that*

(i) $\left(1+|y|^2\right)^p [g(|y|) - g(|x|)]^{2p} \le C_2 |\hat{y} - \hat{x}|$, $x, y \in \mathbf{R}^d$, $3 \le |y| \le |x|$,

(ii) $\left(\dfrac{|x-y|}{|x|(\log|y|)^{\varepsilon}}\right)^{2p} \le C_2 |\hat{y} - \hat{x}|, \qquad x, y \in \mathbf{R}^d, 3 \le |y| \le |x|.$

PROOF. By Lemma 1 and the triangle inequality, we have

$$\left(1+|y|^2\right)^p [g(|y|) - g(|x|)]^{2p} \le C_1(|\hat{y}| - |\hat{x}|) \le C_1 |\hat{y} - \hat{x}|.$$

for all $p \ge p_0$, $3 \le |y| \le |x|$. Therefore assertion (i) of the lemma holds with $C_2 = C_1$.

In order to prove assertion (ii), we start by proving the weaker inequality

$$\left(\frac{|x|-|y|}{|x|(\log|y|)^{\varepsilon}}\right)^{2p} \le C_2 |\hat{y} - \hat{x}|, \tag{2.3}$$

for all $x, y \in \mathbf{R}^d$, $3 \le |y| \le |x|$, $p \ge p_0$. Since $(\log|x|)^{\varepsilon} \ge (\log|y|)^{\varepsilon}$, then

$$\frac{|x|-|y|}{|x|(\log|y|)^{\varepsilon}} \le \frac{1}{(\log|y|)^{\varepsilon}} - \frac{|y|}{|x|(\log|x|)^{\varepsilon}} = |y|[g(|y|) - g(|x|)].$$

Therefore part (i) of the lemma implies the assertion (2.3) with $C_2 = C_1$. Now let us prove the inequality in (ii). Set $\beta := 2p$. Define $\overline{C} = \overline{C}(p, \varepsilon)$ by

$$\overline{C} := \sup\left\{\frac{\beta|x||y|(|x|+|y|)^{2(\beta-1)}(\log|x|)^{\varepsilon}}{|x|^{2\beta}(\log|y|)^{2\varepsilon\beta-\varepsilon}} : 3 \le |y| \le |x| < \infty\right\}.$$

Since $\beta \ge 1$, the expression inside the sup can be estimated from above by

$$\frac{\beta|x||y|2^{2(\beta-1)}(\log|x|)^{\varepsilon}}{|x|^{2\beta}(\log|y|)^{2\varepsilon\beta-\varepsilon}} = \beta 2^{2(\beta-1)} \frac{f(|y|)}{f(|x|)} \frac{1}{(\log|y|)^{2\varepsilon(\beta-1)}}$$

for $3 \le |y| \le |x|$. Since f is increasing, it follows that $\overline{C} < \infty$.

Now fix $C \ge \max(\sqrt{\overline{C}}, C_1)$. Define $\theta = \theta(x, y) := \langle x, y\rangle/(|x||y|)$ for $x, y \in \mathbf{R}^d \setminus \{0\}$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbf{R}^d$. If in assertion (ii) we replace C_2 with C, the resulting inequality is equivalent to

$$\begin{aligned} &\left(|x|^2 + |y|^2 - 2\theta|x||y|\right)^{\beta} + \frac{2\theta C^2(\log|y|)^{2\varepsilon\beta}|x|^{2\beta}}{(\log|x|)^{\varepsilon}(\log|y|)^{\varepsilon}} \\ &\qquad \le C^2(\log|y|)^{2\varepsilon\beta}|x|^{2\beta}\left(\frac{1}{(\log|y|)^{2\varepsilon}} + \frac{1}{(\log|x|)^{2\varepsilon}}\right). \end{aligned} \tag{2.4}$$

Fix $3 \le |y| \le |x|$ and denote the left-hand side of (2.4) by $\tilde{F}(\theta)$. Since $\beta \ge 1$, then

$$\begin{aligned}
\frac{d\tilde{F}}{d\theta}(\theta) &= -2\beta\left(|x|^2 + |y|^2 - 2\theta|x||y|\right)^{\beta-1}|x||y| + \frac{2C^2|x|^{2\beta}}{(\log|x|)^{\varepsilon}}(\log|y|)^{2\varepsilon\beta-\varepsilon} \\
&\ge -2\beta|x||y|(|x|+|y|)^{2(\beta-1)} + \frac{2C^2|x|^{2\beta}}{(\log|x|)^{\varepsilon}}(\log|y|)^{2\varepsilon\beta-\varepsilon} \\
&\ge 0
\end{aligned}$$

because $C^2 \ge \overline{C}$. This shows that (2.4) is true for all $\theta \in [-1,1]$ iff it is true for $\theta = 1$. In (2.4), the case $\theta = 1$ corresponds to the weaker inequality (2.3) which has already been proved. This completes the proof of Lemma 3. □

PROOF OF THEOREM 1. Define the random fields

$$\xi_{s,t}(\hat{x}) = \begin{cases} g(|x|)\phi_{s,t}(x), & \hat{x} \ne 0, |x| \ge 3, \\ 0, & \hat{x} = 0 \end{cases}$$

and

$$\eta_{s,t}(\hat{x}) = \begin{cases} \dfrac{f(|x|)}{1 + |\phi_{s,t}(x)|}, & \hat{x} \ne 0, |x| \ge 3, \\ 0, & \hat{x} = 0, 0 \le s \le t < \infty. \end{cases}$$

It is easy to see that the map $x \mapsto \hat{x}$ is a homeomorphism

$$\{x \in \mathbf{R}^d : |x| > 1\} \to \mathbf{R}^d \setminus \{0\}$$

onto.

Fix $\gamma > d + 2$, $0 \le \varepsilon \le 1$ and $T > 0$. We will show that there exists $q_0 > 0$ such that for every $q \ge q_0$ there is a positive constant $C = C(q, T)$ so that

$$E\left(|\xi_{s,t}(\hat{x}) - \xi_{s',t'}(\hat{y})|^q\right) \le C\left(|\hat{x} - \hat{y}|^{\gamma} + |t - t'|^{\gamma} + |s - s'|^{\gamma}\right) \tag{2.5}$$

for all $0 \le s \le t \le T$, $0 \le s' \le t' \le T$ and $|\hat{x}|, |\hat{y}| \le \delta$, where $\delta = (\log 3)^{-\varepsilon}$. We will also show that (2.5) holds if ξ is replaced by η. It then follows from the Kolmogorov–Totoki theorem ([9], Theorem 1.4.1, page 31) that ξ and η have modifications which are jointly continuous in $\hat{x}$, s and t. In particular, the assertions of our theorem then follow.

Let us first show the moment inequality (2.5) for ξ. Suppose that $0 < |\hat{x}| \le |\hat{y}| \le \delta$, that is, $|x| \ge |y| \ge 3$. Then

$$\begin{aligned}
E\left(|\xi_{s,t}(\hat{x}) - \xi_{s',t'}(\hat{y})|^q\right) &= E\left(|g(|x|)\phi_{s,t}(x) - g(|y|)\phi_{s',t'}(y)|^q\right) \\
&\le d_q|g(|y|) - g(|x|)|^q E\left(|\phi_{s',t'}(y)|^q\right) \\
&\quad + d_q|g(|x|)|^q E\left(|\phi_{s,t}(x) - \phi_{s',t'}(y)|^q\right),
\end{aligned}$$

where d_q is a positive constant depending on q only. Using Lemma 3 and Lemma 4.5.3 of [9], we get positive constants $\tilde{C}, \tilde{C}_1, q_0$ such that

$$\begin{aligned} |g(|y|) - g(|x|)|^q E\left(|\phi_{s',t'}(y)|^q\right) &\le \tilde{C}|g(|y|) - g(|x|)|^q \left(1 + |y|^2\right)^{q/2} \\ &\le \tilde{C}_1 |\hat{y} - \hat{x}|^\gamma \end{aligned}$$

for all $q \ge q_0$ and $x, y \in \mathbf{R}^d$ with $|x| \ge |y| \ge 3$.

If $q = 2p > 2$, then it follows from Lemma 4.5.6 in [9] that there is a positive constant $\tilde{C}_2$ such that

$$\begin{aligned} &|g(|x|)|^q E\left(|\phi_{s,t}(x) - \phi_{s',t'}(y)|^q\right) \\ &\quad \le \tilde{C}_2 |g(|x|)|^{2p} \left(1 + |x|^2 + |y|^2\right)^p (|t - t'|^p + |s - s'|^p) \\ &\qquad + \tilde{C}_2 |g(|x|)|^{2p} |x - y|^{2p}. \end{aligned}$$

Obviously $|g(x)|^{2p}(1 + |x|^2 + |y|^2)^p$ is bounded. Furthermore, by Lemma 3(ii), we have

$$g(|x|)^{2p}|x - y|^{2p} = \left(\frac{|x - y|}{|x|(\log|x|)^\varepsilon}\right)^{2p} \le \left(\frac{|x - y|}{|x|(\log|y|)^\varepsilon}\right)^{2p} \le C_2|\hat{y} - \hat{x}|.$$

Taking both sides of the above inequality to power γ we see that (2.5) holds for q sufficiently large. The case $\hat{x} = 0$ is treated similarly.

Now let us show that (2.5) holds if ξ is replaced by η. Again assume that $0 < |\hat{x}| \le |\hat{y}| \le \delta$. Then

$$\begin{aligned} &E\left(|\eta_{s,t}(\hat{x}) - \eta_{s',t'}(\hat{y})|^q\right) \\ &\quad = E\left(\left|\frac{f(|x|) - f(|y|)}{1 + |\phi_{s,t}(x)|} + f(|y|)\left[\frac{1}{1 + |\phi_{s,t}(x)|} - \frac{1}{1 + |\phi_{s',t'}(y)|}\right]\right|^q\right) \\ &\quad \le d_q [f(|x|) - f(|y|)]^q E\left([1 + |\phi_{s,t}(x)|]^{-q}\right) \\ &\qquad + d_q f(|y|)^q E\left(\left[\frac{|\phi_{s',t'}(y) - \phi_{s,t}(x)|}{(1 + |\phi_{s,t}(x)|)(1 + |\phi_{s',t'}(y)|)}\right]^q\right). \end{aligned}$$

By Lemma 4.5.3 in [9], we get

$$[f(|x|) - f(|y|)]^q E\left[\left(1 + |\phi_{s,t}(x)|\right)^{-q}\right] \le \tilde{C}_3 [f(|x|) - f(|y|)]^q \left(1 + |x|^2\right)^{-q/2}$$

for some positive constant $\tilde{C}_3$. Using the fact that $\log|y| \le \log|x|$, it follows that

$$\begin{aligned} f(|x|) - f(|y|) &= \frac{|x|}{(\log|x|)^\varepsilon} - \frac{|y|}{(\log|y|)^\varepsilon} \le |x||y|[g(|y|) - g(|x|)] \\ &\le \left(1 + |x|^2\right)^{1/2} \left(1 + |y|^2\right)^{1/2} [g(|y|) - g(|x|)] \end{aligned}$$

for $|x| \geq |y| \geq 3$. Therefore, in view of Lemma 3(i), we have

$$\left(\frac{f(|x|) - f(|y|)}{\left(1 + |x|^2\right)^{1/2}}\right)^q \leq \left(1 + |y|^2\right)^{q/2}\left[g(|y|) - g(|x|)\right]^q \leq \tilde{C}_4|\hat{y} - \hat{x}|^\gamma$$

for some positive $\tilde{C}_4$, $|x| \geq |y| \geq 3$ and sufficiently large q.

It remains to prove that

$$\begin{aligned} &f(|y|)^q E\left[|\phi_{s',t'}(y) - \phi_{s,t}(x)|^q\left(1 + |\phi_{s,t}(x)|\right)^{-q}\left(1 + |\phi_{s',t'}(y)|\right)^{-q}\right] \\ &\qquad \leq \tilde{C}_5\left(|\hat{x} - \hat{y}|^\gamma + |t - t'|^\gamma + |s - s'|^\gamma\right) \end{aligned} \tag{2.6}$$

for sufficiently large q and some positive $\tilde{C}_5 = \tilde{C}_5(q)$. To show this, we apply Hölder's inequality and Lemmas 4.5.3, 4.5.6 in [9] to the left-hand side of (2.6). Then there is a positive constant $\tilde{C}_6 := \tilde{C}_6(q)$ such that the left-hand side of (2.6) is less than or equal to

$$\begin{aligned} &\tilde{C}_6\left(f(|y|)\right)^q\left[\left(|x - y|^q + \left(1 + |x|^2 + |y|^2\right)^{q/2}\left(|t - t'|^{q/2} + |s - s'|^{q/2}\right)\right]\right. \\ &\qquad \times\left(1 + |x|^2\right)^{-q/2}\left(1 + |y|^2\right)^{-q/2} \end{aligned}$$

for sufficiently large q. As before, the coefficients of $|t - t'|^{q/2}$ and $|s - s'|^{q/2}$ are bounded. To complete the proof of (2.6), it remains to show that

$$\left(\frac{|f(|y|)||x - y|}{\left(1 + |x|^2\right)^{1/2}\left(1 + |y|^2\right)^{1/2}}\right)^{q/\gamma} \leq \tilde{C}_7|\hat{x} - \hat{y}|$$

for q large and some positive constant $\tilde{C}_7$. Now this follows easily from Lemma 3(ii).

Finally, following a time-change argument in [9], page 162, it is easy to see that Theorem 1 is also valid for general F with local characteristics of class $B_b^{0,1}$. This completes the proof of the theorem. □

REMARK. Let F have local characteristics in $B_{ub}^{0,1}$ and $A(t) \equiv t$. Then for a.a. $\omega \in \Omega$, each $\varepsilon, T > 0$, there exists a positive constant $K(\omega, T, \varepsilon)$ such that $K(\cdot, T, \varepsilon) \in L^q(\Omega, \mathbf{R})$ for all $q \geq 1$ and

$$|\phi_{s,t}(\omega, x)| \leq K(\omega, T, \varepsilon)\left[1 + |x|(\log^+|x|)^\varepsilon\right] \tag{2.7}$$

for all $x \in \mathbf{R}^d$, $0 \leq s \leq t \leq T$.

In a similar manner, it follows that for any $\varepsilon > 0$ the expression

$$\sup_{0 \leq s \leq t \leq T,\ x \in \mathbf{R}^d} \frac{|x|}{\left[1 + |\phi_{s,t}(\cdot, x)|\right]\left[1 + (\log^+|x|)^\varepsilon\right]} \tag{2.8}$$

belongs to $L^q(\Omega, \mathbf{R})$ for every $q \geq 1$.

PROOF. We first prove (2.7). Start with inequality (2.5) in the proof of Theorem 1. By [9], Lemma 4.5.3, for any $q \geq 0$, any $x_0 \in \mathbf{R}^d$ and $0 \leq s_0 \leq t_0 \leq T$, we have $E|\phi_{s_0, t_0}(\cdot, x_0)|^q < \infty$. This implies that $E|\xi_{s_0, t_0}(\cdot, \hat{x}_0)|^q < \infty$. Apply the last assertion of Theorem 1.4.1 ([9], pages 31 and 32) to the continuous modification $\xi_{s,t}(\hat{x})$; thus for any $M > 0$ we have

$$E \sup_{0 \leq s \leq t \leq T,\, |\hat{x}| \leq M} \left| \xi_{s,t}(\cdot, \hat{x}) \right|^q < \infty \tag{2.9}$$

for all $q \geq 0$. Hence there exists $N > 0$ such that

$$E \sup_{0 \leq s \leq t \leq T,\, |x| \geq N} \frac{\left| \phi_{s,t}(\cdot, x) \right|^q}{|x|^q (\log |x|)^{q\varepsilon}} < \infty \tag{2.10}$$

for all $q \geq 0$.

Also since $E|\phi_{s_0, t_0}(\cdot, x_0)|^q < \infty$, then applying the last assertion of [9], Theorem 1.4.1, to the modification $\phi_{s,t}(\cdot, x)$ gives

$$E \sup_{0 \leq s \leq t \leq T,\, |x| \leq N} \left| \phi_{s,t}(\cdot, x) \right|^q < \infty \tag{2.11}$$

for all $q \geq 0$. Therefore

$$E \sup_{0 \leq s \leq t \leq T,\, |x| \leq N} \frac{\left| \phi_{s,t}(\cdot, x) \right|^q}{\left[1 + |x| (\log^+ |x|)^{\varepsilon}\right]^q} < \infty \tag{2.12}$$

for all $q \geq 0$. Combining (2.10) and (2.12) gives (2.7).

The proof of (2.8) is similar. □

In s.d.e. (I) consider the estimate

$$\lim_{|x| \to \infty} \sup_{0 \leq s \leq t \leq T} \frac{\left| D^{\alpha} \phi_{s,t}(x) \right|}{(1 + |x|^{\varepsilon})} = 0$$

on the derivatives $D^{\alpha}\phi_{s,t}(x)$ of the flow $\phi_{s,t}(x)$. This estimate is given in [9], Exercises 4.6.8 and 4.6.9. In the above estimate,

$$\alpha := (\alpha_1, \alpha_2, \dots, \alpha_d), \qquad D^{\alpha} := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}}, \qquad |\alpha| := \sum_{i=1}^{d} \alpha_i,$$

for α_i nonnegative integers, $i = 1, \dots, d$, and the spatial semimartingale F has local characteristics of class $B_b^{m,\delta}$ for $m \geq 1 \geq \delta > 0$. This means that the local characteristics $a(x, y, t)$, $b(x, t)$ satisfy

$$\int_0^T \| a(t) \tilde{\|}_{m+\delta}\, dt < \infty \qquad \int_0^T \| b(t) \|_{m+\delta}\, dt < \infty$$

for all $T > 0$, where

$$\|a(t)\tilde{\|}_{m+\delta} := \sup_{x, y \in \mathbf{R}^d} \frac{|a(t, x, y)|}{(1+|x|)(1+|y|)} + \sum_{1 \le |\alpha| \le m} \sup_{x, y \in \mathbf{R}^d} \left| D_x^\alpha D_y^\alpha a(t, x, y) \right| + \sum_{|\alpha| = m} \|D_x^\alpha D_y^\alpha a(t, x, y)\tilde{\|}_\delta,$$

$$\|b(t)\|_{m+\delta} := \sup_{x \in \mathbf{R}^d} \frac{|b(t, x)|}{(1+|x|)} + \sum_{1 \le |\alpha| \le m} \sup_{x \in \mathbf{R}^d} \left| D_x^\alpha b(t, x) \right| + \sum_{|\alpha| = m} \sup_{(x, y) \in \Delta^c} \frac{\left| D_x^\alpha b(t, x) - D_y^\alpha b(t, y) \right|}{|x - y|^\delta}$$

and

$$\|\theta\tilde{\|}_\delta := \sup \left\{ \frac{|\theta(x, y) - \theta(x', y) - \theta(x, y') + \theta(x', y')|}{|x - x'|^\delta |y - y'|^\delta} : (x, x'), (y, y') \in \Delta^c \right\}.$$

for any δ-Hölder continuous function $\theta: \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$.

We shall say that the spatial semimartingale F has local characteristics of class $B_{ub}^{m, \delta}$ for $m \ge 1 \ge \delta > 0$ if they satisfy

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{0 \le t \le T} \left[\|a(t)\tilde{\|}_{m+\delta} + \|b(t)\|_{m+\delta} \right] < \infty$$

for all $T > 0$.

Theorem 2 is a close parallel to the estimate on $D^\alpha \phi_{s,t}(x)$ given above. It gives a spatial estimate on the inverse of the Jacobian of the flow $\phi_{s,t}(\omega, \cdot)$. For s.d.e.'s driven by finite-dimensional Brownian motion, the theorem was proved in [14], Lemma 2.1, using a Sobolev embedding argument. For completeness we shall present a (different) proof that also covers the case of general semimartingale noise. Our proof is based on a linearization of (I) and the Kolmogorov–Totoki theorem.

THEOREM 2. *Recall the s.d.e.,*

$$\text{(I)} \qquad \begin{aligned} d\phi(t) &= F(\phi(t), dt), \qquad t > s, \\ \phi(s) &= x, \end{aligned}$$

and assume that the local characteristics of F are of class $B_b^{1,\delta}$ for some $\delta \in (0, 1]$. Then the Jacobian $\partial\phi_{s,t}(x)$ is invertible. Define $J_{s,t}(x) := (\partial\phi_{s,t}(x))^{-1}$. Then we have

$$\lim_{|x| \to \infty} \sup_{0 \le s \le t \le T} \frac{\|J_{s,t}(x)\|}{(1 + |x|^\varepsilon)} = 0$$

a.s. for every $\varepsilon > 0$ and $T > 0$, where $\|\cdot\|$ is any matrix norm.

REMARKS. (i) Let F have local characteristics in $B_{ub}^{1,\delta}$ and $A(t) \equiv t$. Then for every $\varepsilon > 0$, the expression

$$\sup_{0 \le s \le t \le T,\ x \in \mathbf{R}^d} \frac{\| J_{s,t}(x) \|}{(1 + |x|^{\varepsilon})}$$

has moments of all orders. This follows by an argument similar to the one used in the remark following the proof of Theorem 1.

(ii) In Section 3 we will give a one-dimensional example with F having local characteristics of class $B_b^{0,1}$, and for which $\sup_{x \in \mathbf{R}} |J_{s,t}(x)| = \infty$ a.s. Thus Theorem 2 fails for $\varepsilon = 0$.

PROOF OF THEOREM 2. Without loss of generality, we can and will restrict ourselves to the case where the local characteristics of F belong to the class $B_{ub}^{1,\delta}$ for some $\delta \in (0,1]$ and $A(t) \equiv t$. In this case the $d \times d$-matrix equation

$$J_{s,t}(x) = I - \int_s^t J_{s,u}(x) \frac{\partial F}{\partial x}(\phi_{s,u}(x), du) + \int_s^t J_{s,u}(x)\, \tilde{a}(\phi_{s,u}(x), \phi_{s,u}(x), u)\, du$$

has a unique solution for every $s \ge 0$, $x \in \mathbf{R}$, where

$$\tilde{a}^{ij}(x, y, t) := \sum_l \frac{\partial^2 a^{il}(x, y, t)}{\partial x^j\, \partial y^l}.$$

Using the fact that

$$\partial\phi_{s,t}(x) = I + \int_s^t \frac{\partial F}{\partial x}(\phi_{s,u}(x), du)\, \partial\phi_{s,u}(x)$$

([9], page 174), it follows from Itô's formula that $J_{s,t}(x)\, \partial\phi_{s,t}(x) = I$ a.s. Hence J *is* the inverse of $\partial\phi$ (cf. [9], Exercise 4.6.8 and Lemma 4.4.3).

Next we show that for all $p > 1$, $T > 0$ there exists $K_1 > 0$ such that

$$E\left(\| J_{s,t}(x) \|^{2p}\right) \le K_1 \tag{2.13}$$

for all $x \in \mathbf{R}^d$, $0 \le s \le t \le T$;

$$E\left(\| J_{s,t}(x) - J_{s',t'}(y) \|^{2p}\right) \le K_1\left(|s - s'|^p + |t - t'|^p + |x - y|^{2p}\right) \tag{2.14}$$

for all $x, y \in \mathbf{R}^d$, $0 \le s \le t \le T$, $0 \le s' \le t' \le T$.

First observe that

$$J_{s,t}(x) = I + \int_s^t G_{s,x}(J_{s,u}(x), du),$$

where

$$G_{s,x}(y, t) := -y \int_s^t \frac{\partial F}{\partial x}(\phi_{s,u}(x), du) + y \int_s^t \tilde{a}(\phi_{s,u}(x), \phi_{s,u}(x), u)\, du.$$

Next observe that the norms of the characteristics of $G_{s,x}$ are bounded uniformly with respect to s, x, ω, t. From [9], Lemma 4.5.3 and its proof, it follows that $E[(1 + \| J_{s,t}(x)\|^2)^p] \le K_1$. Thus (2.13) holds. Observe that the constant C in [9], Lemma 4.5.3 depends only on the driving semimartingale through an upper bound on the growth rates of the characteristics.

In [9], Lemma 4.6.1 put $G_3 \equiv 0$, $\lambda = x$, $q(\lambda) \equiv I$ and $\gamma = \frac{1}{2}$. This gives (2.14).

Define

$$\tilde{\eta}_{s,t}(x^*) := \begin{cases} \dfrac{\| J_{s,t}(x)\|}{|x|^{\varepsilon}}, & x^* \neq 0, |x^*| \le 1, \\ 0, & x^* = 0, \end{cases}$$

where $x^* := x/(|x|^{1+\varepsilon})$. Then for $1 \le |y| \le |x|$, the following inequalities are easy to check:

$$\begin{aligned} |\tilde{\eta}_{s,t}(x^*) - \tilde{\eta}_{s,t}(y^*)| \le {} & \frac{\| J_{s,t}(x) - J_{s',t'}(y)\|}{|x|^{\varepsilon}} \\ & + \| J_{s',t'}(y)\| \left| \frac{1}{|x|^{\varepsilon}} - \frac{1}{|y|^{\varepsilon}} \right|, \end{aligned} \tag{2.15}$$

$$\frac{1}{|x|^{\varepsilon}}(|x-y| \wedge 1) \le K_2 |x^* - y^*|^{\gamma}, \tag{2.16}$$

$$\big||x|^{-\varepsilon} - |y|^{-\varepsilon}\big|^{2p} \le |x^* - y^*|^{2p}, \tag{2.17}$$

for some $K_2, \gamma > 0$. Now take $2p$th moments in (2.15) for sufficiently large p and use (2.13), (2.14), (2.16), (2.17) and the Kolmogorov–Totoki theorem to complete the proof of the theorem. □

3. Superlinear growth of flows. Examples.

EXAMPLE 1. We give an example of a one-dimensional s.d.e.:

$$\begin{aligned} d\phi(t) &= F(\phi(t), dt), \qquad t > 0, \\ \phi(0) &= x \in \mathbf{R} \end{aligned} \tag{II}$$

with F having local characteristics of class $B_b^{0,1}$, and such that the corresponding stochastic flow ϕ satisfies

$$\limsup_{|x|\to\infty} \sup_{0 \le t \le 1} \frac{|\phi(t,\omega,x)|}{|x| \cdot h(x)} = \infty \tag{3.1}$$

a.s., where $h(x) = \exp(\sqrt{\log\log|x|})$. Note that, for every $q > 0$, $h(x) \ge (\log\log|x|)^q$ for sufficiently large $|x|$.

Define the sequences $\delta_n = \exp(2n\sqrt{\log n})$, $b_n := \delta_n \exp(\sqrt{2\log n})$, for $n = 2, 3, 4, \ldots$. Let $\sigma\colon \mathbf{R} \to [0, \infty)$ be globally Lipschitz and such that

$$\sigma(x) = x - \delta_n$$

for $x \in [\delta_n, b_n]$, $n \geq 2$. Observe that $\delta_n < b_n < \delta_{n+1}$, for $n \geq 2$. Let W_n, $n \geq 1$ be independent standard Brownian motions, and define

$$F(x,t) = \begin{cases} \sigma(x) W_n(t), & x \in [\delta_n, \delta_{n+1}],\ n \geq 2, \\ \sigma(x) W_1(t), & x \leq \delta_2. \end{cases}$$

If $x \in [\delta_n, b_n]$, $n \geq 2$, then the flow $\phi(t, \omega, x)$ of the s.d.e. (II) is given by

$$\phi(t,\omega,x) = (x - \delta_n)\exp\{W_n(t,\omega) - t/2\} + \delta_n, \qquad 0 < t < \tau_n,$$

where τ_n is the first time that $\phi(t, \cdot, x)$ hits b_n. With the above choice of F, the s.d.e. (II) has a globally defined flow on $\mathbf{R}$ because F has local characteristics of class $B_b^{0,1}$.

First, observe that $(e+1)\delta_n \in (\delta_n, b_n]$ for all sufficiently large n. Next, we estimate the probability that the process $\phi(t, \omega, (e+1)\delta_n)$ hits b_n before time 1. This probability is estimated from below as follows:

$$\begin{aligned} &P\Big(\delta_n + e\delta_n \sup_{0 \leq t \leq 1} \exp\{W_n(t) - \tfrac{1}{2}t\} \geq b_n\Big) \\ &\geq P\left(\sup_{0 \leq t \leq 1} W_n(t) \geq \log\left[\frac{b_n}{e\delta_n}\right] + \frac{1}{2}\right) \\ &\geq P\Big(\sup_{0 \leq t \leq 1} W_n(t) \geq \sqrt{2\log n}\Big) \\ &= 2P\big(W_n(1) \geq \sqrt{2\log n}\big) \\ &= \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2\log n}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \\ &\geq \frac{K}{n\sqrt{\log n}} \end{aligned}$$

for $n \geq 2$, where K is a positive constant. The last inequality follows easily from l'Hôpital's rule.

Now by the independence of the W_n's and the second Borel–Cantelli lemma, we have

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{0 \leq t \leq 1} \frac{|\phi(t, \omega, (e+1)\delta_n)|}{(e+1)\delta_n h[(e+1)\delta_n]} \\ &\geq \limsup_{n\to\infty} \frac{b_n}{(e+1)\delta_n h[(e+1)\delta_n]} \end{aligned} \tag{3.2}$$

for a.a. ω. But $h[(e+1)\delta_n] = \exp(\sqrt{\log\{\log(e+1) + 2n\sqrt{\log n}\}})$. Therefore the right-hand side of (3.2) is equal to

$$\frac{1}{e+1} \limsup_{n\to\infty} \exp\Big(\sqrt{2\log n} - \sqrt{\log\{\log(e+1) + 2n\sqrt{\log n}\}}\Big) = \infty.$$

REMARKS. (i) In Example 1, the local characteristic $a(x, y, t)$ of F is given by

$$a(x, y, t) = \begin{cases} \sigma(x)\sigma(y), & \text{if } x, y \in (\delta_n, \delta_{n+1}],\ n \geq 2, \text{ or } x, y \in (-\infty, \delta_2], \\ 0, & \text{otherwise.} \end{cases}$$

Then a is of class $B_b^{0,1}$ but is not of class $B_b^{1,\delta}$ for any $\delta \in (0, 1]$ because α is not C^1. However, it is easy to smooth out σ in a sufficiently small neighborhood of each δ_n so that it becomes C^∞, remains globally Lipschitz and the estimate of the hitting probability still holds. Moreover, in this case we have $\sup_{x \in \mathbf{R}} |D_3 \phi(t, \omega, x)| = \infty$ for almost all $\omega \in \Omega$ and each $t \geq 0$.

(ii) The following example was communicated to us by Peter Baxendale [5]. The example also shows that, in the case of *infinite-dimensional noise*, it is impossible to seek almost sure global bounds on the derivatives of the flow. Let $f\colon \mathbf{R} \to [0, 1]$ be smooth with $f \not\equiv 0$ and with compact support in $(0, 1)$. Let $\phi_t^W(\cdot, x)$ denote the flow in $\mathbf{R}$ given by

$$\phi_t^W(\cdot, x) = x + \int_0^t f\big(\phi_s^W(\cdot, x)\big)\, dW(s).$$

Then $\phi_t^W(\omega, \cdot)$ fixes $\mathbf{R} \setminus [0, 1]$ for a.a. ω. Define $f_n(x) = f(x - n)$, $x \in \mathbf{R}$, and consider the flow $\tilde{\phi}_t$ given by

$$\tilde{\phi}_t(x) = x + \sum_{n \in \mathbf{Z}} \int_0^t f_n\big(\tilde{\phi}_s(x)\big)\, dW^n(s).$$

For $x \in [n, n+1]$ we have $\tilde{\phi}_t(x) = \phi_t^{W^n}(\cdot, x - n) + n$. So if the $\{W^n\colon n \in \mathbf{Z}\}$ are independent Brownian motions, then the $\{\tilde{\phi}_t|_{[n, n+1]}\colon n \in \mathbf{Z}\}$ are independent flows. This means that any global bound on the derivative of the flow $\tilde{\phi}_t$ is the supremum of infinitely many independent bounds on each of the intervals $[n, n+1]$. Since $\mathbf{P}\{\sup_{0 \leq x \leq 1} |D\phi_t^W(x)| > K\} > 0$ for all $K < \infty$ it follows that $\sup_{x \in \mathbf{R}} |D\tilde{\phi}_t(x)| = \infty$ a.s.

EXAMPLE 2. Next we give an example driven by *one-dimensional* white noise for which the spatial derivative of the flow is almost surely unbounded. More specifically, the example gives a one-dimensional s.d.e.:

$$\text{(III)} \qquad \begin{aligned} d\phi(t) &= h(\phi(t))\, dt + g(\phi(t))\, dW(t), \qquad t > 0, \\ \phi(0) &= x \in \mathbf{R}, \end{aligned}$$

driven by a one-dimensional Brownian motion W, and with coefficients h, g whose derivatives are all globally bounded, but the derivative $D_3\phi(t, \omega, x)$ of its stochastic flow $\phi(t, \omega, x)$ is almost surely *unbounded* in x for each $t > 0$.

We will use the following proposition.

PROPOSITION. *Let $W(t)$, $t \geq 0$, be one-dimensional standard Brownian motion. Consider the s.d.e.*

$$\text{(IV)} \qquad \begin{aligned} d\psi(t) &= b\, dt + \sin(\psi(t))\, dW(t), \qquad t > 0, \\ \psi(0) &= x \in \mathbf{R}, \end{aligned}$$

where $b > 0$ *is a parameter. Denote the corresponding stochastic flow by* $\psi^{(b)}(t, x)$*. Then for every* $t > 0$ *and all* $x \in \mathbf{R}$*, we have*

$$\sup_{n \geq 1} \frac{\partial \psi^{(2^n)}}{\partial x}(t, x) = \infty \quad a.s.$$

PROOF. We drop the parameter b whenever there is no danger of confusion. By [9], page 173, $\psi(t, x)$ exists and is differentiable with respect to $x \in \mathbf{R}$ for each $t \geq 0$. Furthermore,

$$\frac{\partial \psi}{\partial x}(t, x) = 1 + \int_0^t \cos[\psi(s, x)] \frac{\partial \psi}{\partial x}(x, s)\, dW(s).$$

Therefore,

$$\frac{\partial \psi}{\partial x}(t, x) = \exp\left(\int_0^t \cos[\psi(s, x)]\, dW(s) - \frac{1}{2} \int_0^t \cos^2[\psi(s, x)]\, ds \right).$$

To prove the proposition, it is sufficient to show that

$$\sup_{n \geq 1} \int_0^t \cos\left[\psi^{(2^n)}(s, x)\right] dW(s) = \infty$$

for every $t > 0$, a.s.

The intuitive idea behind our proof is as follows. Consider the family of random variables

$$X_n := \int_0^t \cos\left[\psi^{(2^n)}(s, x)\right] dW(s), \qquad n \geq 1.$$

Roughly speaking, and for large n, this family is *almost independent* and has *almost uniform variance*. Using known limit theorems, this implies that for each fixed m and large n, the sequence $\{X_n, X_{n+1}, \dots, X_{n+m}\}$ is approximately multivariate Gaussian, and the result follows.

For the rest of the proof, fix $x \in \mathbf{R}$ and write $\psi^{(b)}(s, x) = \psi^{(b)}(s)$ for $s > 0$. We begin by showing that the following is true.

For every integer $m \geq 1$ and $t > 0$, we have the convergence in law

$$L\left(\int_0^t \exp\left[\psi^{(2^n)}(s)\right] dW(s), \dots, \int_0^t \cos\left[\psi^{(2^{n+m})}(s)\right] dW(s) \right) \to N\left(0, \frac{t}{2} I_{m+1}\right), \tag{3.3}$$

as $n \to \infty$, where I_{m+1} is the $(m+1) \times (m+1)$-identity matrix.

To show (3.3) we use Theorem 8.1.9 [8], page 417. For this purpose, we will establish the following limits in probability for each $t > 0$:

$$\int_0^t \cos^2\left[\psi^{(2^n)}(s)\right] ds \to \frac{t}{2}, \tag{3.4}$$

$$\int_0^t \cos\left[\psi^{(2^k)}(s)\right] \cos\left[\psi^{(2^n)}(s)\right] ds \to 0 \tag{3.5}$$

as k (and n) $\to \infty$ with $k < n$.

Let us check (3.5) first. Let $b = 2^k$, fix $t > 0$ and choose a nonnegative integer $m \geq 1$ such that $(2\pi m/b) \leq t < (2\pi(m+1)/b)$. Then

$$\int_0^t \cos\left[\psi^{(2^k)}(s)\right] \cos\left[\psi^{(2^n)}(s)\right] ds$$
$$= \frac{1}{2} \sum_{j=0}^{m-1} \int_{2\pi j/b}^{2\pi(j+1)/b} \Big\{ \cos\left[\psi^{(2^k)}(s) - \psi^{(2^n)}(s)\right]$$
$$+ \cos\left[\psi^{(2^k)}(s) + \psi^{(2^n)}(s)\right] \Big\} ds + o(1).$$

Now, from (IV) it follows that for $j \in \{0, \ldots, m-1\}$, we have a.s.

$$\int_{2\pi j/b}^{2\pi(j+1)/b} \cos\left[\psi^{(2^k)}(s) \pm \psi^{(2^n)}(s)\right] ds$$
$$= \int_{2\pi j/b}^{2\pi(j+1)/b} \Bigg\{ \cos\Bigg[\psi^{(2^k)}\left(\frac{2\pi j}{b}\right) \pm \psi^{(2^n)}\left(\frac{2\pi j}{b}\right)$$
$$+ 2^k\left(s - \frac{2\pi j}{b}\right) \pm 2^n\left(s - \frac{2\pi j}{b}\right)$$
$$+ \int_{2\pi j/b}^{s} \left(\sin\left[\psi^{(2^k)}(u)\right] \pm \sin\left[\psi^{(2^n)}(u)\right]\right) dW(u) \Bigg]\Bigg\} ds$$

Setting

$$A_j^{\pm}(s) := \psi^{(2^k)}\left(\frac{2\pi j}{b}\right) \pm \psi^{(2^n)}\left(\frac{2\pi j}{b}\right) + 2^k\left[s - \frac{2\pi j}{b}\right] \pm 2^n\left[s - \frac{2\pi j}{b}\right]$$

and

$$B_j^{\pm}(s) := \int_{(2\pi j)/b}^{s} \left(\sin\left[\psi^{(2^k)}(u)\right] \pm \sin\left[\psi^{(2^n)}(u)\right]\right) dW(u),$$

we get

$$\int_{2\pi j/b}^{2\pi(j+1)/b} \cos\left[\psi^{(2^k)}(s) \pm \psi^{(2^n)}(s)\right] ds$$
$$= \int_{2\pi j/b}^{2\pi(j+1)/b} \left\{ \cos A_j^{\pm}(s) \cos B_j^{\pm}(s) - \sin A_j^{\pm}(s) \sin B_j^{\pm}(s) \right\} ds$$
$$= \int_{2\pi j/b}^{2\pi(j+1)/b} \cos A_j^{\pm}(s)\, ds - \int_{2\pi j/b}^{2\pi(j+1)/b} \cos A_j^{\pm}(s)\left(1 - \cos B_j^{\pm}(s)\right) ds$$
$$- \int_{2\pi j/b}^{2\pi(j+1)/b} \sin A_j^{\pm}(s) \sin B_j^{\pm}(s)\, ds.$$

On the right-hand side of the above relation, the first of the three integrals is zero. To estimate the other two integrals, we use

$$E\left[B_j^{\pm}(s)\right]^2 \leq \frac{8\pi}{b}$$

for $(2\pi j/b) \le s \le (2\pi(j+1)/b)$. Since $1 - \cos x \le |x|$ and $|\sin x| \le |x|$ for all $x \in R$, we have

$$E\left(\int_0^t \cos\left[\psi^{(2^k)}(s)\right] \cos\left[\psi^{(2^n)}(s)\right] ds\right)^2 \le m^2 \left(\frac{2\pi}{b}\right)^2 \frac{8\pi}{b} + o(1) = o(1).$$

This proves (3.5).

The convergence (3.4) is proved analogously:

$$\frac{1}{2}\int_0^t \cos\left[\psi^{(2^n)}(s) - \psi^{(2^n)}(s)\right] ds = \frac{t}{2}$$

and $\frac{1}{2}\int_0^t \cos[\psi^{(2^n)}(s) + \psi^{(2^n)}(s)]\, ds$ converges to zero in probability as above.

It remains to show that (3.3) implies the proposition. Fix $\varepsilon > 0$ an integer $M \ge 1$ and $t > 0$. Let N_j, $j \ge 1$, be independent $N(0, \frac{1}{2}t)$-distributed random variables and choose a positive integer m such that

$$P\left(\max(N_1, \ldots, N_m) > M\right) \ge 1 - \frac{\varepsilon}{2}.$$

By (3.3) there exists a positive integer n_0 such that

$$P\left(\max\left\{\int_0^t \cos\left[\psi^{(2^n)}(s)\right] dW(s), \ldots, \int_0^t \cos\left[\psi^{(2^{(n+m)})}(s)\right] dW(s)\right\} > M\right)$$
$$\ge 1 - \varepsilon$$

for all $n \ge n_0$. Since $\varepsilon > 0$ and $M \ge 1$ are arbitrary, we get

$$P\left(\sup_{n \ge 1} \int_0^t \cos\left[\psi^{(2^n)}(s)\right] dW(s) = \infty\right) = 1.$$

This completes the proof of the proposition. □

Using the proposition we construct a C^∞ globally Lipschitz function h: $\mathbf{R} \to \mathbf{R}$ such that the s.d.e.

$$\text{(III}'\text{)} \qquad \begin{aligned} d\phi(t) &= h(\phi(t))\, dt + \sin(\phi(t))\, dW(t), \qquad t > 0, \\ \phi(0) &= x \in \mathbf{R} \end{aligned}$$

has a flow $\phi(t, x)$ with derivative $(\partial\phi/\partial x)(t, x)$ that is a.s. unbounded in x for each $t > 0$.

Since $\sup_{0 \le t \le n} \psi^{(n)}(t, 0) < \infty$ a.s. for each $n \ge 1$, we can choose a sequence of positive integers $\{k_n\}_{n=1}^\infty$ such that

$$P\left(\sup_{0 \le t \le n} \psi^{(n)}(t, 0) \ge 2\pi k_n\right) \le 2^{-n}.$$

By induction set

$$\begin{aligned} l_1 &= 2\pi k_1, \\ l_n &= l_{n-1} + (k_n + 1)2\pi, \qquad n \ge 2. \end{aligned}$$

Let $\tilde{b}$: $[0, 2\pi] \to [0, 1]$ be a C^∞ function with $\tilde{b}(0) = 0, \tilde{b}(2\pi) = 1$ and all derivatives at 0 and 2π equal to zero. Then define

$$h(x) = \begin{cases} n, & x \in [l_{n-1} + 2\pi, l_n],\ n \geq 2, \\ 1, & x \leq l_1, \\ n + \tilde{b}(x - l_n), & x \in [l_n, l_n + 2\pi],\ n \geq 1. \end{cases}$$

Then h has all derivatives globally bounded. By the above proposition and (the easy half of) the Borel–Cantelli lemma, it follows that

$$\sup_{x \in \mathbf{R}} \frac{\partial \phi}{\partial x}(t, x) \geq \sup_{n \geq 1} \frac{\partial \phi}{\partial x}(t, l_n + 2\pi) = \sup_{n \geq 1} \frac{\partial \psi^{(n+1)}}{\partial x}(t, 0) = \infty$$

for each $t > 0$ almost surely. □

REMARK. In this example the drift h is *unbounded* (cf. Theorem 3). Furthermore, by letting the k_n go to infinity arbitrarily fast, we could arrange that h goes to infinity arbitrarily slowly.

4. One-dimensional noise. Linear growth. In Example 1 of Section 3, we showed superlinear growth of the stochastic flow under infinite-dimensional noise. By contrast, we shall show here that the case of one-dimensional noise gives stochastic flows that satisfy a global linear growth property provided that the driving vector fields have the same property.

Let h: $\mathbf{R} \to \mathbf{R}$ be a locally Lipschitz map satisfying a global linear growth condition

$$h(x) \leq K(1 + |x|)$$

for some positive constant K and all $x \in \mathbf{R}$. Suppose g: $\mathbf{R} \to \mathbf{R}$ is a C^1 globally Lipschitz map with its first derivative Dg locally Lipschitz. Consider the one-dimensional Itô stochastic differential equation

$$\text{(V)} \qquad \begin{aligned} d\phi(t) &= h(\phi(t))\, dt + g(\phi(t))\, dW(t), \qquad t > 0, \\ \phi(0) &= x \in \mathbf{R}, \end{aligned}$$

where W is one-dimensional standard Brownian motion on a filtered probability space $(\Omega, F, (F_t)_{t \geq 0}, P)$.

Under the above conditions, the flow of (V) satisfies the following linear growth property. This result is a corollary of the Doss–Sussman representation for stochastic flows.

THEOREM 3. *Let h, g satisfy the above conditions. Suppose ϕ: $\mathbf{R} \times \Omega \times \mathbf{R} \to \mathbf{R}$ is the continuous stochastic flow of* (V). *Then for almost all $\omega \in \Omega$*

and every $T > 0$, *there exists* $K(T, \omega) > 0$ *such that*

$$\sup_{0 \le t \le T} |\phi(t, \omega, x)| \le K(T, \omega)(1 + |x|)$$

for all $x \in \mathbf{R}$.

Furthermore, if h *is* C^1 *and* g *is* C^2 *with* $h, Dh, D^2 g$ *all globally bounded, then for a.a.* ω *and all* $t > 0$, $\phi(t, \omega, \cdot)$ *is* C^1, *and*

$$\sup\{|D_3 \phi(t, \omega, x)| : 0 \le t \le T, x \in \mathbf{R}\} < \infty$$

for almost all $\omega \in \Omega$.

PROOF. We use the Doss–Sussman construction [16, 6].

Let $\psi: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the C^1 deterministic flow of the ordinary differential equation

$$\text{(VI)} \qquad \begin{aligned} \psi'(t) &= g(\psi(t)), \qquad t > 0, \\ \psi(0) &= x \in \mathbf{R}. \end{aligned}$$

Let D_1, D_2 denote the partial derivatives of $\psi(t, x)$ with respect to t and x, respectively. Since g is C^1 and is globally Lipschitz, then $Dg(x)$ is globally bounded in $x \in \mathbf{R}$. Therefore $\psi(t, x)$ is C^2 in t and C^1 in x, with $D_2 \psi(t, x)$ and $[D_2 \psi(t, x)]^{-1}$ globally bounded in $x \in \mathbf{R}$, for t in a compact set in $\mathbf{R}$.

Define $H: \mathbf{R} \times \Omega \times \mathbf{R} \to \mathbf{R}$ by

$$\begin{aligned} &H(t, \omega, x) \\ (4.1) \qquad &:= \left[D_2 \psi(W(t, \omega), x)\right]^{-1} \{h(\psi(W(t, \omega), x)) \\ &\qquad - \tfrac{1}{2} Dg(\psi(W(t, \omega)) g(\psi(W(t, \omega), x))\} \end{aligned}$$

for all $(t, x) \in \mathbf{R} \times \mathbf{R}$, $\omega \in \Omega$. Let $z(t, \omega, x)$ denote the unique solution of the random family of ordinary differential equations

$$\text{(VII)} \qquad \begin{aligned} z'(t) &= H(t, \omega, z(t)), \qquad t > 0, \\ z(0) &= x \in \mathbf{R} \end{aligned}$$

for each $\omega \in \Omega$. It is easy to see from (4.1) that $H(t, \omega, x)$ is continuous in (t, x), locally Lipschitz in x, and has global linear growth in x uniformly with respect to t in bounded sets. Therefore $z(t, \omega, x)$ satisfies similar properties.

Finally, by a simple application of Itô's formula, we have

$$\text{(4.2)} \qquad \phi(t, \omega, x) = \psi(W(t, \omega), z(t, \omega, x)), \qquad t > 0, \omega \in \Omega, x \in \mathbf{R}.$$

The linear growth property for ϕ now follows immediately from the above relation. The final assertion of the theorem follows by differentiating (4.2) with respect to x and using the boundedness of the corresponding spatial derivatives of ψ and z. This completes the proof of the theorem. □

REMARKS. (i) It is important to note that in the last assertion of Theorem 3, the *global boundedness* condition on the drift *h cannot be dropped* (Example 2, Section 3). On the other hand, the s.d.e. (III′) of Example 2 satisfies the hypotheses of the first assertion of Theorem 3. Therefore the flow of (III′) has sublinear growth almost surely.

(ii) The conclusion of Theorem 3 holds if g (and h) are replaced by a *time-dependent* vector-field $g\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that $g(t, x)$ is jointly C^1 in (t, x) and globally Lipschitz on $\mathbf{R}^2$. On the other hand, it appears that the conclusion of Theorem 3 may not hold if $g(t, x)$ is merely *continuous* in t or $D_1 g(t, x)$ is globally unbounded in t and $g(t, x)$ has global linear growth in x uniformly in t.

(iii) For s.d.e.'s on $\mathbf{R}^d$, driven by finite-dimensional Brownian motion and with globally Lipschitz coefficients, we conjecture that the stochastic flow $\phi_{s,t}(\cdot, x)$ satisfies a linear growth property

$$\sup_{0 \le s \le t \le T} |\phi_{s,t}(\omega, x)| \le K(T, \omega)(1 + |x|)$$

for almost all $\omega \in \Omega$, all $x \in \mathbf{R}^d$ and every $T > 0$. The linear growth property is easily checked in the special case when the driving noise is finite-dimensional Brownian motion and the vector fields generate a finite-dimensional solvable Lie algebra. This follows from Kunita's decomposition theorem ([9], Theorem 4.9.10, page 212).

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