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ASYMPTOTIC BEHAVIOR OF CONDITIONAL LAWS AND MOMENTS OF α-STABLE RANDOM VECTORS, WITH APPLICATION TO UPCROSSING INTENSITIES[1]

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We derive upper bounds for the conditional moment $\mathbf{E}\{|X|^\varrho\,|Y=y\}$ of a strictly α-stable random vector (X, Y) when $\alpha \neq 1$ and $\varrho \leq 2$ and prove weak convergence for the conditional law $(X/u \mid Y=u)$ as $u \to \infty$ when $\alpha > 1$. As an example of application, we derive a new result in crossing theory for α-stable processes.

1. Introduction. Given an $\alpha \in (0,1)\cup(1,2)$, we write $Z \in S_\alpha(\sigma, \beta)$ when Z is a strictly α-stable random variable with Fourier transform (characteristic function)

$$\mathbf{E}\big\{\exp[i\theta Z]\big\} = \exp\big\{-|\theta|^\alpha\sigma^\alpha\big[1+i\beta\tau_\alpha\,\mathrm{sign}(\theta)\big]\big\}, \tag{1.1}$$

where $\tau_\alpha \equiv \tan(\pi(2-\alpha)/2)$. Here the scale $\sigma = \sigma_Z \geq 0$ and the skewness $\beta = \beta_Z \in [-1, 1]$ are "free" parameters.

Given measurable functions f, g: $\mathbb{R} \to \mathbb{R}$, we put $f^{\langle\alpha\rangle} \equiv |f|^\alpha\,\mathrm{sign}(f)$ and define

$$\langle f\rangle \equiv \int_{\mathbb{R}} f(x)\,dx, \qquad \langle f\rangle_\alpha \equiv \langle f^{\langle\alpha\rangle}\rangle,$$
$$\|f\|_\alpha \equiv \langle|f|^\alpha\rangle^{1/\alpha}, \qquad \langle f, g\rangle_{n,\alpha} \equiv \langle f^n g^{\langle\alpha-n\rangle}\rangle$$

(where $n \in \mathbb{N}$ is required if $f \not\geq 0$). Moreover $\mathbb{L}^\alpha(\mathbb{R}) \equiv \{(h\colon \mathbb{R}\to\mathbb{R})\colon \|h\|_\alpha < \infty\}$.

Let $\{\xi(t)\}_{t\in\mathbb{R}}$ be an α-stable Lévy motion with skewness $\beta = -1$, so that $\xi(t)$ has independent stationary increments and $\xi(t) \in S_\alpha(|t|^{1/\alpha}, -\mathrm{sign}(t))$. Assuming that $f,\ g \in \mathbb{L}^\alpha(\mathbb{R})$, it is then well known that the bivariate α-stable random vector $(X, Y) \equiv \big(\int_{\mathbb{R}} f\,d\xi, \int_{\mathbb{R}} g\,d\xi\big)$ satisfies

$$\theta X + \varphi Y \in S_\alpha\left(\|\theta f + \varphi g\|_\alpha, -\frac{\langle\theta f + \varphi g\rangle_\alpha}{\|\theta f + \varphi g\|_\alpha^\alpha}\right). \tag{1.2}$$

Further, each bivariate strictly α-stable vector (X, Y) has this representation in law for some choice of f and g. See, for example, Samorodnitsky and Taqqu (1994) [hereafter denoted (ST) (1994)] Chapters 1–3, on these and other basic properties of α-stable random variables.

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Of course (when needed), random variables and processes that appear in the sequel are assumed to be defined on a common complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

Cioczek-Georges and Taqqu (1994, 1995a) showed that

$$\text{(1.3)} \quad \mathbf{E}\big\{|X|^{\varrho}|Y=y\big\} < \infty \quad \Leftrightarrow \quad \langle |f|, |g|\rangle_{\nu,\alpha} < \infty \quad \begin{array}{l}\text{for some } \nu > \varrho \text{ if } \varrho < 2,\\ \text{for } \nu = \varrho \text{ if } \varrho = 2\end{array}$$

for $\varrho \in (0, 2\wedge(2\alpha+1)]$, thereby sharpening earlier findings of Samorodnitsky and Taqqu (1991) and Wu and Cambanis (1991). Note that the condition on the right-hand side in (1.3) is void when $\varrho < \alpha$.

In Section 2 we characterize the unique continuous (wrt. y) regular conditional law $(X \mid Y = y)$ for a strictly α-stable vector (X, Y) by means of specifying its Fourier transform: When [as in (1.3)] making statements about conditional probabilities and expectations, we assume that they are computed according to this law, and so there is no ambiguity concerning what versions these statements refer to.

In Section 3 we use a result of Albin (1997) to investigate the asymptotic behavior of the moment $\mathbf{E}\{X^2 \mid Y = u\}$ as $u \to \infty$ when $\alpha > 1$.

In Section 4 we derive two upper bounds for $\mathbf{E}\{|X|^{\varrho} \mid Y = y\}$: The first bound is valid whenever $\varrho < 2\wedge(\alpha+1)$ and $\mathbf{E}\{|X|^{\varrho} \mid Y = y\} < \infty$, but is not sharp as $|y| \to \infty$. The second bound is sharp for all y but applies when $\alpha > 1$ and $\varrho < \alpha$ only.

Bounds on conditional moments are important because they make it possible to use Markov- and Tjebysjev-like inequalities in multivariate α-stable contexts. Indeed, the bounds of Section 4 are crucial in the proof of weak convergence of conditional α-stable laws with $\alpha > 1$ in Section 5, as well as in the treatment of upcrossing intensities for α-stable processes in Section 6.

In Section 5 we prove weak convergence of $(X/u \mid Y = u)$ as $u \to \infty$ for $\alpha > 1$. From this follows convergence of the moment $\mathbf{E}\{|X/u|^{\varrho} I_{\{X/u>\lambda\}} \mid Y = u\}$ when $\varrho < \alpha$ and, under the additional condition $\langle f, |g|\rangle_{2,\alpha} < \infty$, when $\varrho \in [\alpha, 2)$. We also discuss convergence of probabilities and moments conditioned on $Y > u$.

The expected number of upcrossings of a level u by a stationary and differentiable symmetric α-stable ($S\alpha S$) process $\{\eta(t)\}_{t\in I}$ such that $(\eta'(0), \eta(0))$ possesses a continuous density function $f_{\eta'(0),\eta(0)}$ is given by Rice's formula,

$$\text{(1.4)} \qquad \mu(I;u) = \text{length}(I)\int_0^{\infty} x f_{\eta'(0),\eta(0)}(x,u)\,dx.$$

Michna and Rychlik (1995) proved this result under quite restrictive additional conditions, and Adler and Samorodnitsky (1997) extended it to a virtually optimal setting. See also Marcus (1989) and Adler, Samorodnitsky and Gadrich (1993).

In Section 6 we prove a version of (1.4) that is valid without any requirements about stationarity, symmetry or existence of joint densities. Our proof is based on the counting device for upcrossings described in Leadbetter, Lindgren and Rootzén [(1983), Section 7.2]. Despite the fact that our proof produces a more general result, it is considerably shorter and easier than proofs by previous authors.

2. Conditional distributions for α-stable random vectors. Choose functions $f_1, \dots, f_n, g \in \mathbb{L}^\alpha(\mathbb{R})$ where $\|g\|_\alpha > 0$, and consider the α-stable random vector

$$(2.1)\qquad (X, Y) = (X_1, \dots, X_n, Y) = \left(\int_{\mathbb{R}} f_1\, d\xi, \dots, \int_{\mathbb{R}} f_n\, d\xi, \int_{\mathbb{R}} g\, d\xi\right).$$

In Proposition 1 below we prove the existence of and characterize the unique regular conditional distributions $F_{X|Y}(\cdot \mid y)$ that depend continuously on $y \in \mathbb{R}$.

Let Z be an α-stable random vector in $\mathbb{R}^m$ with spectral measure Γ_Z [as defined in, e.g., ST (1994), Section 2.3]. By Kuelbs and Mandrekar [(1974), Lemma 2.1], the linear dimension $L(dF_Z)$ of the support of the distribution of Z equals that of the support of Γ_Z $L(\Gamma_Z)$. Further, the Fourier transform of Z is integrable (so that Z has a bounded and continuous density function) if $L(\Gamma_Z) = m$ [cf. ST (1994), Lemma 5.1.1]. It follows that if Z has a density [so that $L(dF_Z) = m$], then Z has a bounded and continuous density:

Henceforth we shall without loss assume that *α-stable density functions are bounded and continuous when they exist*. In particular, the component Y of the vector (X, Y) defined in (2.1) has a bounded and continuous density f_Y since $L(\Gamma_Y) = 1$ when $\|g\|_\alpha > 0$ [cf. ST (1994), Example 2.3.3].

PROPOSITION 1. *Consider the α-stable random vector (X, Y) in $\mathbb{R}^{n+1}$ given by* (2.1) *where $\alpha \in (0,1)\cup(1,2)$ and $\|g\|_\alpha > 0$. Then there exists a unique family of distribution functions $\{F_{X|Y}(\cdot \mid y)\}_{y\in \mathrm{int}(\mathrm{supp}(Y))}$ on $\mathbb{R}^n$ with the properties that*

$$(2.2)\qquad \int_{x\in\mathbb{R}^n} h(x)\, dF_{X|Y}(x \mid y) \quad \textit{is a version of } \mathbf{E}\{h(X) \mid Y = y\}$$

for each measurable map h: $\mathbb{R}^n \to \mathbb{R}$ satisfying $\mathbf{E}\{|h(X)|\} < \infty$, and that

$$(2.3)\qquad F_{X|Y}(\cdot \mid y) \to_d F_{X|Y}(\cdot \mid y_0) \quad \textit{as } y \to y_0 \text{ (continuity)}.$$

Further, writing $\langle\langle\theta, x\rangle\rangle = \theta_1 x_1 + \cdots + \theta_n x_n$, the law $F_{X|Y}(\cdot \mid y)$ has Fourier transform

$$(2.4)\quad \phi_{X|Y}(\theta \mid y) \equiv \frac{1}{2\pi f_Y(y)} \int_{\mathbb{R}} \exp(-i\varphi y)\mathbf{E}\big\{\exp\big[i\big(\langle\langle\theta, X\rangle\rangle + \varphi Y\big)\big]\big\}\, d\varphi$$

for $\theta \in \mathbb{R}^n$, and if (X, Y) possesses a density function $f_{X,Y}(x, y)$, then

$$(2.5)\qquad F_{X|Y}(\cdot \mid y) \quad \textit{has density function } f_{X|Y}(\cdot \mid y) \equiv f_{X,Y}(\cdot, y)/f_Y(y).$$

Proposition 1 allows us to refer to conditional probabilities and expectations for α-stable random vectors in the same easygoing manner as for Gaussian vectors. The result does not seem to have been observed previously, but a related discussion is given in ST [(1994), Section 5.1]: *in the sequel, probabilities and expectations conditioned on the event that $Y = y$ are assumed to be computed according to the law $F_{X|Y}(\cdot \mid y)$ specified by* (2.4).

PROOF. As is well known [e.g., Breiman (1968), Section 4.3], there exists a so-called regular family of distributions $\{F^{\mathrm{reg}}_{X|Y}(\cdot \mid y)\}_{y\in\mathrm{int}(\mathrm{supp}(Y))}$ such that

$$\int_{x\in\mathbb{R}^n} h(x)\,dF^{\mathrm{reg}}_{X|Y}(x \mid y) \quad \text{is a version of } \mathbf{E}\{h(X) \mid Y = y\}, \tag{2.6}$$

whenever $\mathbf{E}\{|h(X)|\} < \infty$. Further observe the fact that [cf. (1.1) and (1.2)]

$$\begin{aligned}\big|\mathbf{E}\big\{\exp\big[i\big(\langle\langle\theta, X\rangle\rangle + \varphi Y\big)\big]\big\}\big| &= \exp\big\{-\big\|\langle\langle\theta, f\rangle\rangle + \varphi g\big\|_\alpha^\alpha\big\}\\ &\le \exp\big\{-\big|\big\|\langle\langle\theta, f\rangle\rangle\big\|_\alpha - |\varphi|\|g\|_\alpha\big|^\alpha\big\}.\end{aligned}$$

Since the density f_Y is continuous and locally bounded away from zero, it follows that $\phi_{X|Y}(\theta \mid y)$ is a well-defined continuous function of $(\theta, y) \in \mathbb{R}^{n+1}$ and that

$$\begin{aligned}\mathbf{E}\big\{\exp(i\langle\langle\theta, X\rangle\rangle)I_{\{Y\in[a,b]\}}\big\} &= \int_{(x,y)\in\mathbb{R}^n\times[a,b]} \exp(i\langle\langle\theta, x\rangle\rangle)\,dF_{X,Y}(x, y)\\ &= \int_a^b \phi_{X|Y}(\theta \mid y)f_Y(y)\,dy\end{aligned}$$

for $-\infty < a < b < \infty$. Hence $\phi_{X|Y}(\theta \mid y)$ is a version of $\mathbf{E}\{\exp[i\langle\langle\theta, X\rangle\rangle] \mid Y = y\}$, and in view of (2.6) we therefore conclude that

$$\phi_{X|Y}(\theta \mid y) = \int_{x\in\mathbb{R}^n} \exp(i\langle\langle\theta, x\rangle\rangle)\,dF^{\mathrm{reg}}_{X|Y}(x \mid y) \quad \text{for } \theta \in \mathbb{Q}^n, \tag{2.7}$$

for almost all $y \in \mathrm{int}(\mathrm{supp}(Y))$. By continuity in θ, (2.7) extends to all θ, and so $\phi_{X|Y}(\theta \mid y)$ is the Fourier transform of some distribution $F_{X|Y}(\cdot \mid y)$ for almost all y. By continuity in y, this statement in turn extends to all y, and (2.3) must hold. Further, since by (2.7), $F_{X|Y}(\cdot \mid y) =_d F^{\mathrm{reg}}_{X|Y}(\cdot \mid y)$ a.e., (2.6) implies (2.2).

If a density $f_{X,Y}(x, y)$ is continuous in y, then the fact that f_Y is continuous and locally bounded away from zero and the theorem by Scheffé (1947) show that

$$\int_{x\in A} f_{X|Y}(x \mid y)\,dx \quad \text{is continuous in } y \text{ for every measurable } A \subseteq \mathbb{R}^n.$$

Hence the laws with densities $\{f_{X|Y}(\cdot \mid y)\}_{y\in\mathrm{int}(\mathrm{supp}(Y))}$ satisfy (2.2) and (2.3), and thus coincide with the laws $\{F_{X|Y}(\cdot \mid y)\}_{y\in\mathrm{int}(\mathrm{supp}(Y))}$ specified through (2.4). □

3. Conditional second moments. Let $\alpha > 1$ and take functions f and g in (1.2) such that $\langle f, |g|\rangle_{2,\alpha} < \infty$ and either $g \ge 0$ or $g \le 0$ a.e. Extending results by Wu and Cambanis (1991), and complementing results by Cioczek-Georges and Taqqu (1995b), it was shown in Albin (1997), Theorem 1, that

$$\begin{aligned}&\mathbf{E}\{X^2 \mid Y = y\}\\ &\quad = (\alpha - 1)\bigg(\frac{\langle f, |g|\rangle_{2,\alpha}}{\|g\|_\alpha^\alpha} - \frac{\langle f, g\rangle_{1,\alpha}^2}{\|g\|_\alpha^{2\alpha}}\bigg)\int_{|y|}^\infty \frac{zf_Y(z)}{f_Y(y)}\,dz + \frac{\langle f, g\rangle_{1,\alpha}^2}{\|g\|_\alpha^{2\alpha}}y^2.\end{aligned} \tag{3.1}$$

Note a minor error in Albin (1997): in (2.3) and (2.4) (as well as in later occurrences) $\int_y^\infty z f_Y(z)\,dz$ should be corrected to $\int_{|y|}^\infty z f_Y(z)\,dz$.

In Theorem 1 and Corollary 1 below we use (3.1) to determine the asymptotic behavior of $\mathbf{E}\{X^2 \mid Y = u\}$ as $u \to \infty$.

THEOREM 1. *Consider the α-stable random variable (X, Y) given by* (1.2), *where $\alpha \in (1, 2)$, $\|g\|_\alpha > 0$ and $\langle f, |g|\rangle_{2,\alpha} < \infty$. Further, define the sets*

$$G^+ \equiv \{x \in \mathbb{R}: g(x) > 0\} \quad \textit{and} \quad G^- \equiv \{x \in \mathbb{R}: g(x) < 0\}.$$

(i) *Suppose that $g \geq 0$ a.e., so that $\beta_Y = -1$. Then we have*

$$\lim_{u\to\infty} u^{-2}\mathbf{E}\{X^2 \mid Y = u\} = \langle f, g\rangle_{1,\alpha}^2 / \langle g\rangle_\alpha^2.$$

When $\langle f, g\rangle_{1,\alpha} = 0$ we further have

$$\lim_{u\to\infty} u^{(2-\alpha)/(\alpha-1)}\mathbf{E}\{X^2 \mid Y = u\} = (\alpha - 1)\left(\frac{\alpha\langle g\rangle_\alpha^{2-\alpha}}{\cos((\pi/2)(2-\alpha))}\right)^{1/(\alpha-1)} \langle f, g\rangle_{2,\alpha}.$$

If $\langle f, g\rangle_{1,\alpha} = \langle f, g\rangle_{2,\alpha} = 0$, then we have $f = 0$ a.e., so that $X = 0$ a.s.

(ii) *Suppose that $\langle g^-\rangle_\alpha > 0$, so that $\beta_Y > -1$. Then we have*

$$\lim_{u\to\infty} u^{-2}\mathbf{E}\{X^2 \mid Y = u\} = \langle f I_{G^-}, g^-\rangle_{2,\alpha} / \langle g^-\rangle_\alpha.$$

When $\langle f I_{G^-}, g^-\rangle_{2,\alpha} = 0$ but $\langle f I_{G^+}, g^+\rangle_{1,\alpha} > 0$ we further have

$$\lim_{u\to\infty} u^{\alpha-2}\mathbf{E}\{X^2 \mid Y = u\} = \frac{\Gamma(2\alpha-1)\langle f I_{G^+}, g^+\rangle_{1,\alpha}^2}{\Gamma(\alpha-1)\cos((\pi/2)(2-\alpha))\langle g^+\rangle_\alpha},$$

while $\langle f I_{G^-}, g^-\rangle_{2,\alpha} = \langle f I_{G^+}, g^+\rangle_{1,\alpha} = 0$ but $\langle f I_{G^+}, g^+\rangle_{2,\alpha} > 0$ implies that

$$\lim_{u\to\infty} \mathbf{E}\{X^2 \mid Y = u\} = 2(\alpha-1)\big[\langle f I_{G^+}, g^+\rangle_{2,\alpha} / \langle g^+\rangle_\alpha\big]\mathbf{E}\{[S_\alpha(\|g^+\|_\alpha, -1)^+]^2\}.$$

If $\langle f I_{G^-}, g^-\rangle_{2,\alpha} = \langle f I_{G^+}, g^+\rangle_{2,\alpha} = 0$, then we have $f = 0$ a.e., so that $X = 0$ a.s.

PROOF OF (i). By, for example, ST [(1994), Chapter 1], we have

$$f_{S_\alpha(\sigma,-1)}(u) \sim A_\alpha \sigma^{-1}(u/\sigma)^{(2-\alpha)/(2(\alpha-1))}\exp\{-B_\alpha(u/\sigma)^{\alpha/(\alpha-1)}\} \tag{3.2}$$

as $u \to \infty$, where $A_\alpha > 0$ and $B_\alpha = (\alpha-1)[\cos((\pi/2)(2-\alpha))/\alpha^\alpha]^{1/(\alpha-1)}$ are constants. Defining $w = w(u) \equiv [\alpha\langle g\rangle_\alpha / \cos((\pi/2)(2-\alpha))]^{1/(\alpha-1)} u^{-1/(\alpha-1)}$, (3.2) and easy calculations show that $(u + xw) f_Y(u + xw) / (u f_Y(u)) \to e^{-x}$ and

$$\begin{aligned} \int_u^\infty y f_Y(y)\,dy &= uw f_Y(u) \int_0^\infty \frac{(u + xw) f_Y(u + xw)}{u f_Y(u)}\,dx \\ &\sim uw(u) f_Y(u). \end{aligned} \tag{3.3}$$

Using that $uw(u) = o(u^2)$ and $g = |g|$ a.e., (3.1) now yields the statement (i).

PROOF OF (ii). Writing $C_\alpha \equiv \alpha(\alpha-1)/[\Gamma(2-\alpha)\cos((\pi/2)(2-\alpha))]$, we have

$$\lim_{u\to\infty} u^{\alpha+1} f_{S_\alpha(\sigma,\beta)}(u) = \tfrac{1}{2} C_\alpha (1+\beta)\sigma^\alpha \tag{3.4}$$

[e.g., ST (1994), Chapter 1]. First assume that $g \le 0$ a.e., so that $\beta_Y = 1$ and $|g| = g^-$ a.e. Using (3.4) in an easy calculation, we then obtain

$$\int_u^\infty y f_Y(y)\,dy \sim (\alpha-1)^{-1} u^2 f_Y(u) \quad \text{as } u \to \infty. \tag{3.5}$$

Inserting (3.5) in (3.1), and observing that since $\langle f, |g|\rangle_{2,\alpha} < \infty$ we must have $f = 0$ a.e. when $\langle f I_{G^-}, g^-\rangle_{2,\alpha} = 0$, the statement (ii) of the theorem follows.

Now suppose that $\beta_Y < 1$ and let

$$(X_-, X_+) \equiv \left(\int_{\mathbb{R}} f_-\,d\xi, \int_{\mathbb{R}} f_+\,d\xi\right) \quad \text{and} \quad (Y_-, Y_+) \equiv \left(\int_{\mathbb{R}} (-g^-)\,d\xi, \int_{\mathbb{R}} g^+\,d\xi\right),$$

where $f_+ \equiv I_{G^+} f$ and $f_- \equiv I_{G^-} f$. Then we have

$$\begin{aligned}\mathbf{E}\{X^2 \mid Y = u\} = \int_{\mathbb{R}} \big(&\mathbf{E}\{X_+^2 \mid Y_+ = x\} + \mathbf{E}\{X_-^2 \mid Y_- = u-x\}\\ &+ 2\mathbf{E}\{X_+ \mid Y_+ = x\}\mathbf{E}\{X_- \mid Y_- = u-x\}\big)\\ &\times \frac{f_{Y_+}(x) f_{Y_-}(u-x)}{f_Y(u)}\,dx\end{aligned}$$

[since $\langle f, |g|\rangle_{2,\alpha} < \infty$ implies that $I_{\mathbb{R}\setminus(G^+\cup G^-)} f = 0$ a.e.]. Using the formulas for linear regression [cf. ST (1994), equation 5.2.27],

$$\begin{aligned}\mathbf{E}\{X_+ \mid Y_+ = y\} &= \langle f_+, g^+\rangle_{1,\alpha}\, y/\langle g^+\rangle_\alpha,\\ \mathbf{E}\{X_- \mid Y_- = y\} &= \langle f_-, g^-\rangle_{1,\alpha}\, y/\langle g^-\rangle_\alpha\end{aligned}$$

together with (3.1), we therefore obtain

$$\begin{aligned}\mathbf{E}&\{X^2 \mid Y = u\}\\ &= (\alpha-1)\left(\frac{\langle f_+, g^+\rangle_{2,\alpha}}{\langle g^+\rangle_\alpha} - \frac{\langle f_+, g^+\rangle_{1,\alpha}^2}{\langle g^+\rangle_\alpha^2}\right)\\ &\quad \times \int_{\mathbb{R}}\left[\int_{|x|}^\infty \frac{z f_{Y_+}(z) f_{Y_-}(u-x)}{f_Y(u)}\,dz\right]dx\\ &\quad + (\alpha-1)\left(\frac{\langle f_-, g^-\rangle_{2,\alpha}}{\langle g^-\rangle_\alpha} - \frac{\langle f_-, g^-\rangle_{1,\alpha}^2}{\langle g^-\rangle_\alpha^2}\right)\\ &\quad \times \int_{\mathbb{R}}\left[\int_{|u-x|}^\infty \frac{z f_{Y_-}(z) f_{Y_+}(x)}{f_Y(u)}\,dz\right]dx\\ &\quad + \frac{\langle f_+, g^+\rangle_{1,\alpha}^2}{\langle g^+\rangle_\alpha^2}\int_{\mathbb{R}} \frac{x^2 f_{Y_+}(x) f_{Y_-}(u-x)}{f_Y(u)}\,dx\\ &\quad + \frac{\langle f_-, g^-\rangle_{1,\alpha}^2}{\langle g^-\rangle_\alpha^2}\int_{\mathbb{R}} \frac{(u-x)^2 f_{Y_+}(x) f_{Y_-}(u-x)}{f_Y(u)}\,dx\\ &\quad + \frac{2\langle f_+, g^+\rangle_{1,\alpha}\langle f_-, g^-\rangle_{1,\alpha}}{\langle g^+\rangle_\alpha \langle g^-\rangle_\alpha}\int_{\mathbb{R}} \frac{x f_{Y_+}(x)(u-x) f_{Y_-}(u-x)}{f_Y(u)}\,dx.\end{aligned} \tag{3.6}$$

Here applications of (3.2)–(3.5) in straightforward calculations reveal that

$$
\begin{aligned}
&\frac{1}{f_Y(u)}\int_{\mathbb{R}}\left[\int_{|x|}^{\infty} z f_{Y_+}(z) f_{Y_-}(u-x)\,dz\right]dx \\
&\qquad \to \frac{2\langle g^-\rangle_\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}\int_{\mathbb{R}}\left[\int_{|x|}^{\infty} z f_{Y_+}(z)\,dz\right]dx, \\
&\frac{u^{-2}}{f_Y(u)}\int_{\mathbb{R}}\left[\int_{|u-x|}^{\infty} z f_{Y_-}(z) f_{Y_+}(x)\,dz\right]dx \\
&\qquad \to \frac{1}{\alpha-1}\frac{2\langle g^-\rangle_\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}, \\
&\frac{u^{\alpha-2}}{f_Y(u)}\int_{\mathbb{R}} x^2 f_{Y_+}(x) f_{Y_-}(u-x)\,dx \\
&\qquad \to \frac{2\langle g^+\rangle_\alpha\langle g^-\rangle_\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha} C_\alpha \int_0^\infty \frac{x^{1-\alpha}}{(1+x)^{\alpha+1}}\,dx, \\
&\frac{u^{-2}}{f_Y(u)}\int_{\mathbb{R}} (u-x)^2 f_{Y_+}(x) f_{Y_-}(u-x)\,dx \\
&\qquad \to \frac{2\langle g^-\rangle_\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}, \\
&\frac{u^{-1}}{f_Y(u)}\int_{\mathbb{R}} x f_{Y_+}(x)(u-x) f_{Y_-}(u-x)\,dx \\
&\qquad \to \frac{2\langle g^-\rangle_\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}\mathbf{E}\{Y_+\}.
\end{aligned}
\tag{3.7}
$$

Note that

$$\int_{\mathbb{R}}\left[\int_{|x|}^{\infty} z f_{Y_+}(z)\,dz\right]dx = 2\mathbf{E}\{[S_\alpha(\|g^+\|_\alpha, -1)^+]^2\},$$

$(1+\beta_Y)\|g\|_\alpha^\alpha = 2\langle g^-\rangle_\alpha$ and

$$\int_0^\infty \frac{x^{1-\alpha}}{(1+x)^{\alpha+1}}\,dx = \Gamma(2-\alpha)\Gamma(2\alpha-1)/\Gamma(\alpha+1).$$

Inserting (3.7) in (3.6) and using that $\langle f I_{G^-}, g^-\rangle_{2,\alpha} = 0$ and $\langle f I_{G^+}, g^+\rangle_{2,\alpha} = 0$ implies $\langle f_-, g^-\rangle_{1,\alpha} = 0$ and $\langle f_+, g^+\rangle_{1,\alpha} = 0$, respectively, the statement (ii) now follows. □

An inspection of the proof of Theorem 1 shows that a version of the theorem applies in the case often encountered when X depends on u, that is, when

$$(X, Y) = (X_u, Y) \equiv \left(\int_{\mathbb{R}} f_u\,d\xi, \int_{\mathbb{R}} g\,d\xi\right) \quad \text{with } f_u(\cdot), g \in \mathbb{L}^\alpha(\mathbb{R}). \tag{3.8}$$

COROLLARY 1. *Consider the α-stable random variable (X_u, Y) given by* (3.8), *where $\alpha \in (1, 2)$, $\|g\|_\alpha > 0$ and $\limsup_{u\to\infty}\langle f_u, |g|\rangle_{2,\alpha} < \infty$.*

(i) *Suppose that* $g \geq 0$ *a.e. and that* $\liminf_{u\to\infty} |\langle f_u, g\rangle_{1,\alpha}| > 0$. *Then we have*

$$\mathbf{E}\{X_u^2 \mid Y = u\} \sim u^2 \langle f_u, g\rangle_{1,\alpha}^2 / \langle g\rangle_\alpha^2 \quad \text{as } u \to \infty.$$

(ii) *Suppose that* $\langle g^-\rangle_\alpha > 0$ *and that* $\liminf_{u\to\infty}\langle f_u I_{G^-}, g^-\rangle_{2,\alpha} > 0$. *Then we have*

$$\mathbf{E}\{X_u^2 \mid Y = u\} \sim u^2 \langle f_u I_{G^-}, g^-\rangle_{2,\alpha} / \langle g^-\rangle_\alpha \quad \text{as } u \to \infty.$$

4. Upper bounds on conditional moments. In Theorems 2 and 3 and Corollaries 2 and 3 below we employ Fourier transforms to derive bounds on the moment $\mathbf{E}\{|X|^\varrho \mid Y = y\}$. The usefulness of Fourier techniques when dealing with conditional α-stable moments was discovered by Samorodnitsky and Taqqu (1991) and Wu and Cambanis (1991), and subsequently developed by Cioczek-Georges and Taqqu (1994, 1995a, b).

The bounds in Theorem 2 and Corollary 2(i) apply whenever $\mathbf{E}\{|X|^\varrho \mid Y = y\}$ is finite and $\varrho < 2 \wedge (\alpha + 1)$, but are not sharp for large values of $|y|$. In Theorem 3 and Corollary 2(ii) these results are improved to bounds that possess the right rate as $|y| \to \infty$ in the particular case when $\varrho < \alpha$ and $\alpha > 1$.

THEOREM 2. *Consider the α-stable random variable* (X, Y) *given by* (1.2) *where* $\alpha \in (0,1) \cup (1,2)$ *and* $\|g\|_\alpha > 0$. *Suppose that* $\varrho \in [0, \nu \wedge (\alpha+1))$ *for some* $\nu \in [\alpha, 2)$ *such that* $\langle |f|, |g|\rangle_{\nu,\alpha} < \infty$. *Then we have*

$$\mathbf{E}\{|X|^\varrho I_{\{|X|>\lambda\}} \mid Y = y\} \leq \frac{K_{\alpha,\nu}\langle |f|, |g|\rangle_{\nu,\alpha}\|g\|_\alpha^{\nu-\alpha-1}}{(\nu-\varrho)\lambda^{\nu-\varrho} f_Y(y)} \exp(2\|f\|_\alpha^\alpha \lambda^{-\alpha})$$

for $\lambda > 0$ *and* $y \in \operatorname{int}(\operatorname{supp}(Y))$, *where* $K_{\alpha,\nu} > 0$ *is a constant that depends on* α *and* ν *only*.

Observe that it is sufficient to prove the theorem in the case when $\lambda = 1$.

PROOF OF THEOREM 2 WHEN $\alpha > 1$. There exist constants $K_\alpha^{(1)}$, $K_{\alpha,\nu}^{(2)} > 0$ such that

$$\begin{matrix} \left|\langle tg\rangle_\alpha - \langle tg + sf\rangle_\alpha\right| \\ \left|\|tg\|_\alpha^\alpha - \|tg + sf\|_\alpha^\alpha\right| \end{matrix} \leq K_\alpha^{(1)}\left(\langle |f|, |g|\rangle_{1,\alpha} s t^{\alpha-1} + \|f\|_\alpha^\alpha s^\alpha\right) \tag{4.1}$$

and

$$\begin{matrix} \left|2\langle tg\rangle_\alpha - \langle tg + sf\rangle_\alpha - \langle tg - sf\rangle_\alpha\right| \\ \left|2\|tg\|_\alpha^\alpha - \|tg + sf\|_\alpha^\alpha - \|tg - sf\|_\alpha^\alpha\right| \end{matrix} \leq K_{\alpha,\nu}^{(2)} \langle |f|, |g|\rangle_{\nu,\alpha} s^\nu t^{\alpha-\nu} \tag{4.2}$$

for s, $t > 0$: The proofs of these inequalities only use the elementary fact that

$$1 + \alpha x \leq |1 + x|^\alpha \leq 1 + \alpha x + K_{\alpha,\nu}^{(0)} \min\{|x|^\alpha, |x|^\nu\} \quad \text{for } x \in \mathbb{R},$$

for some constant $K^{(0)}_{\alpha,\nu} > 0$. To prove (4.2), for example, one observes that

$$\begin{aligned}\bigl|2(tg)^{\langle\alpha\rangle} - (tg+sf)^{\langle\alpha\rangle} - (tg-sf)^{\langle\alpha\rangle}\bigr| &= |tg+sf|^\alpha + |tg-sf|^\alpha - 2|tg|^\alpha \\ &\le 2K^{(0)}_{\alpha,\nu}|sf|^\nu|tg|^{\alpha-\nu}\end{aligned}$$

when $|sf| < |tg|$, while

$$\left.\begin{array}{l}\bigl|2(tg)^{\langle\alpha\rangle} - (tg+sf)^{\langle\alpha\rangle} - (tg-sf)^{\langle\alpha\rangle}\bigr| \\ \bigl|2|tg|^\alpha - |tg+sf|^\alpha - |tg-sf|^\alpha\bigr|\end{array}\right\} \le 2(1+2^\alpha)|sf|^\alpha \le 2(1+2^\alpha)|sf|^\nu|tg|^{\alpha-\nu}$$

when $|sf| \ge |tg|$. It follows that (4.2) holds with $K^{(2)}_{\alpha,\nu} = \max\{2K^{(0)}_{\alpha,\nu}, 2(1+2^\alpha)\}$.

We will also need the elementary inequality

$$\begin{aligned}&\bigl|2\cos(x) - \cos(y) - \cos(z)\bigr| \\ &\quad = 4\left|\cos^2\left(\frac{y-z}{4}\right)\sin\left(\frac{2x-y-z}{4}\right)\sin\left(\frac{2x+y+z}{4}\right)\right. \\ &\qquad \left. - \sin^2\left(\frac{y-z}{4}\right)\cos\left(\frac{2x-y-z}{4}\right)\cos\left(\frac{2x+y+z}{4}\right)\right| \\ &\quad \le |2x-y-z| + \tfrac14|y-z|^2 \quad \text{for } x, y, z \in \mathbb{R},\end{aligned} \tag{4.3}$$

and its corollary $|\cos(x) - \cos(y)| \le |x-y|$, as well as the facts that

$$\begin{aligned}&\bigl|e^{-x} - e^{-y}\bigr| \le e^{-(x\wedge y)}|x-y|, \\ &\bigl|2e^{-x} - e^{-y} - e^{-z}\bigr| \\ &\qquad \le \exp(-(x\wedge y\wedge z))\bigl(|2x-y-z| + 2|x-y|^2 + 2|x-z|^2\bigr).\end{aligned} \tag{4.4}$$

Combining these inequalities with (4.1), (4.2) and using symmetry, we get

$$\begin{aligned}&\pm\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} |t|^\rho \bigl[\cos\bigl(ty - \tau_\alpha\langle tg\rangle_\alpha\bigr)\exp(-\|tg\|_\alpha^\alpha) \\ &\qquad\qquad - \cos\bigl(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\bigr)\exp\bigl(-\|sf+tg\|_\alpha^\alpha\bigr)\bigr]\frac{ds\,dt}{s^{1+\varrho}} \\ &\quad = \pm\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} |t|^\rho \bigl[\cos\bigl(ty - \tau_\alpha\langle tg\rangle_\alpha\bigr) - \cos\bigl(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\bigr)\bigr] \\ &\qquad\quad \times \bigl(\exp(-\|sf+tg\|_\alpha^\alpha) - \exp(-\|tg\|_\alpha^\alpha)\bigr)\frac{ds\,dt}{s^{1+\varrho}} \\ &\quad \pm\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} |t|^\rho \cos\bigl(ty - \tau_\alpha\langle tg\rangle_\alpha\bigr) \\ &\qquad \times \bigl(2\exp(-\|tg\|_\alpha^\alpha) - \exp(-\|tg+sf\|_\alpha^\alpha) - \exp(-\|tg-sf\|_\alpha^\alpha)\bigr)\frac{ds\,dt}{s^{1+\varrho}}\end{aligned}$$

$$
\begin{aligned}
&\pm \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} |t|^{\rho}\big[2\cos\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)-\cos\big(ty-\tau_\alpha\langle tg+sf\rangle_\alpha\big)\\
&\qquad\qquad -\cos\big(ty-\tau_\alpha\langle tg-sf\rangle_\alpha\big)\big]\exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}\\
&\le \tau_\alpha \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} |t|^{\rho}\big|\langle tg\rangle_\alpha-\langle sf+tg\rangle_\alpha\big|\big|\|tg\|_\alpha^\alpha-\|sf+tg\|_\alpha^\alpha\big|\\
&\quad\times\exp\big(-(\|sf+tg\|_\alpha^\alpha\wedge\|tg\|_\alpha^\alpha)\big)\frac{ds\,dt}{s^{1+\varrho}}\\
&\quad+\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} |t|^{\rho}\Big(\big|2\|tg\|_\alpha^\alpha-\|tg+sf\|_\alpha^\alpha-\|tg-sf\|_\alpha^\alpha\big|\\
&\qquad\qquad +2\big|\|tg\|_\alpha^\alpha-\|tg+sf\|_\alpha^\alpha\big|^2+2\big|\|tg\|_\alpha^\alpha-\|tg-sf\|_\alpha^\alpha\big|^2\Big)\\
&\quad\times\exp\big(-\big(\|tg+sf\|_\alpha^\alpha\wedge\|tg-sf\|_\alpha^\alpha\wedge\|tg\|_\alpha^\alpha\big)\big)\frac{ds\,dt}{s^{1+\varrho}}\\
&\quad+\tau_\alpha\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} |t|^{\rho}\Big(\big|2\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha-\langle tg-sf\rangle_\alpha\big|\\
&\qquad\qquad +\frac{1}{4}\tau_\alpha\big|\langle tg-sf\rangle_\alpha-\langle tg+sf\rangle_\alpha\big|^2\Big)\exp\big(-\|tg\|_\alpha^\alpha\big)\frac{ds\,dt}{s^{1+\varrho}} \qquad (4.5)\\
&\le\Big(\frac{1}{2}\tau_\alpha+2\Big)^2\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} t^{\rho}\big(K_\alpha^{(1)}\big)^2\big(\langle|f|,|g|\rangle_{1,\alpha}st^{\alpha-1}+\|f\|_\alpha^\alpha s^\alpha\big)^2\\
&\quad\times\exp(\|f\|_\alpha^\alpha-\|tg/2\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}\\
&\quad+(\tau_\alpha+1)\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} t^{\rho}K_{\alpha,\nu}^{(2)}\langle|f|,|g|\rangle_{\nu,\alpha}s^{\nu}t^{\alpha-\nu}\exp\big(\|f\|_\alpha^\alpha-\|tg/2\|_\alpha^\alpha\big)\frac{ds\,dt}{s^{1+\varrho}}\\
&=\Big(\frac{1}{2}\tau_\alpha+2\Big)^2(K_\alpha^{(1)})^2\frac{2^{\rho+2\alpha-1}\Gamma((\rho+2\alpha-1)/\alpha)}{\alpha(2-\varrho)}\frac{\langle|f|,|g|\rangle_{1,\alpha}^2}{\|g\|_\alpha^{\rho+2\alpha-1}}\exp(\|f\|_\alpha^\alpha)\\
&\quad+\Big(\frac{1}{2}\tau_\alpha+2\Big)^2(K_\alpha^{(1)})^2\frac{2^{\rho+\alpha+1}\Gamma((\rho+\alpha)/\alpha)}{\alpha(\alpha+1-\varrho)}\frac{\langle|f|,|g|\rangle_{1,\alpha}\|f\|_\alpha^\alpha}{\|g\|_\alpha^{\rho+\alpha}}\exp(\|f\|_\alpha^\alpha)\\
&\quad+\Big(\frac{1}{2}\tau_\alpha+2\Big)^2(K_\alpha^{(1)})^2\frac{2^{\rho+1}\Gamma((\rho+1)/\alpha)}{\alpha(2\alpha-\varrho)}\frac{\|f\|_\alpha^{2\alpha}}{\|g\|_\alpha^{\rho+1}}\exp(\|f\|_\alpha^\alpha)\\
&\quad+(\tau_\alpha+1)K_{\alpha,\nu}^{(2)}\frac{2^{\rho+\alpha+1-\nu}\Gamma((\rho+\alpha+1-\nu)/\alpha)}{\alpha(\nu-\varrho)}\frac{\langle|f|,|g|\rangle_{\nu,\alpha}}{\|g\|_\alpha^{\rho+\alpha+1-\nu}}\exp(\|f\|_\alpha^\alpha)\\
&\le\frac{K_{\alpha,\nu,\rho}^{(3)}\langle|f|,|g|\rangle_{\nu,\alpha}}{(\nu-\varrho)\|g\|_\alpha^{\rho+\alpha+1-\nu}}\exp(2\|f\|_\alpha^\alpha)\quad\text{for } \rho\ge 0, \text{for some constant } K_{\alpha,\nu,\rho}^{(3)}>0
\end{aligned}
$$

(that depends on α, ν and ρ only). Here we used the inequalities $\langle |f|, |g|\rangle_{1,\alpha} \le \|f\|_\alpha \|g\|_\alpha^{\alpha-1}$ and $\|f\|_\alpha^\nu \le \langle |f|, |g|\rangle_{\nu,\alpha} \|g\|_\alpha^{\nu-\alpha}$, together with the elementary fact that $\|f\|_\alpha^\kappa \le e^{\|f\|_\alpha^\alpha}$ for $\kappa \in [0, \alpha\mathrm{e}]$, to obtain the last inequality.

Let X be a random variable with Fourier transform ϕ_X. Using the inequality

$$\int_0^1 t^{-(1+\varrho)}(1-\cos(t))\,dt \ge \int_0^1 t^{-(1+\varrho)}\tfrac{1}{4}t^2\,dt = \frac{1}{4(2-\varrho)} \quad \text{for } \varrho \in [0,2),$$

in a calculation inspired by Ramachandran and Rao (1968) and Ramachandran (1969) [cf. ST (1994), Theorem 5.1.2], we get

$$\begin{aligned} \frac{1}{4(2-\varrho)}\mathbf{E}\{|X|^\varrho I_{\{|X|>1\}}\} &\le \int_{|x|>1}\int_{t=0}^{t=|x|} |x|^\varrho \frac{1-\cos(t)}{t^{1+\varrho}}\,dt\,dF(x) \\ &\le \int_{x\in\mathbb{R}}\int_{s=0}^{s=1} \frac{1-\cos(s|x|)}{s^{1+\varrho}}\,ds\,dF(x) \qquad (4.6) \\ &= \int_0^1 \Re\big(1-\phi_X(s)\big)\frac{ds}{s^{1+\varrho}}. \end{aligned}$$

Adapting this estimate to the context (1.2) and (2.4), (4.5) now shows that

$$\begin{aligned} &\frac{2\pi f_Y(y)}{4(2-\varrho)}\mathbf{E}\{|X|^\varrho I_{\{|X|>1\}} \mid Y=y\} \\ &\quad \le \int_0^1 \Re\bigg(\int_{\mathbb{R}} \exp(-ity)\big[\phi_{X,Y}(0,t)-\phi_{X,Y}(s,t)\big]\,dt\bigg)\frac{ds}{s^{1+\varrho}} \\ &\quad = \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \Big(\cos\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\exp(-\|tg\|_\alpha^\alpha) \qquad (4.7) \\ &\qquad\qquad - \cos\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\exp(-\|sf+tg\|_\alpha^\alpha)\Big)\frac{ds\,dt}{s^{1+\varrho}} \\ &\quad \le \frac{K^{(3)}_{\alpha,\nu,0}\langle |f|,|g|\rangle_{\nu,\alpha}}{(\nu-\varrho)\|g\|_\alpha^{\alpha+1-\nu}}\exp(2\|f\|_\alpha^\alpha). \end{aligned}$$

PROOF OF THEOREM 2 WHEN $\alpha < 1$ AND $\varrho < \alpha+1$. When $\nu \ge \alpha+1$, the fact that $\varrho < \alpha+1$, together with Hölder's inequality show that $\langle |f|,|g|\rangle_{\hat\nu,\alpha} < \infty$ for some $\hat\nu \in (\varrho, \alpha+1)$. Therefore we can without loss assume that $\nu < \alpha+1$.

The inequality (4.2) holds also for $\alpha < 1$, while instead of (4.1) we now have

$$\left.\begin{matrix} \big|\langle tg\rangle_\alpha - \langle tg+sf\rangle_\alpha\big| \\ \big|\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha\big| \end{matrix}\right\} \le K^{(4)}_{\alpha,\nu}\min\big\{\langle |f|,|g|\rangle_{\tilde\nu,\alpha}s^{\tilde\nu}t^{\alpha-\tilde\nu}, \|f\|_\alpha^\alpha s^\alpha\big\}$$

for some constant $K^{(4)}_{\alpha,\nu} > 0$, where $\tilde{\nu} = \nu \wedge 1$. Combining appropriate parts of the sequences of estimates (4.5) and (4.7) we therefore readily obtain

$$
\begin{aligned}
&\frac{2\pi f_Y(y)}{4(2-\varrho)}\mathbf{E}\{|X|^{\varrho} I_{\{|X|>1\}} \mid Y=y\} \\
&\quad \le \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \Big(\cos\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\exp(-\|tg\|_\alpha^\alpha) \\
&\qquad\qquad - \cos\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\exp\big(-\|sf+tg\|_\alpha^\alpha\big)\Big)\frac{ds\,dt}{s^{1+\varrho}} \\
&\quad \le \left(\frac{1}{2}|\tau_\alpha|+2\right)^2 \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} (K^{(4)}_{\alpha,\nu})^2 \langle |f|,|g|\rangle_{\tilde{\nu},\alpha} s^{\tilde{\nu}} t^{\alpha-\tilde{\nu}} \|f\|_\alpha^\alpha s^\alpha \\
&\qquad\qquad \times \exp(\|f\|_\alpha^\alpha - \|tg/2\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}} \\
&\qquad + (|\tau_\alpha|+1)\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} K^{(2)}_{\alpha,\nu}\langle |f|,|g|\rangle_{\nu,\alpha} s^{\nu} t^{\alpha-\nu}\exp(\|f\|_\alpha^\alpha - \|tg/2\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}} \\
&\quad = \big(\tau_\alpha^2 + 2|\tau_\alpha| + 4\big)\big(K^{(4)}_{\alpha,\nu}\big)^2 \frac{2^{\alpha+1-\tilde{\nu}}\Gamma((\alpha+1-\tilde{\nu})/\alpha)}{\alpha(\tilde{\nu}+\alpha-\varrho)} \frac{\langle |f|,|g|\rangle_{\tilde{\nu},\alpha}\|f\|_\alpha^\alpha}{\|g\|_\alpha^{\alpha+1-\tilde{\nu}}} \\
&\qquad \times \exp(\|f\|_\alpha^\alpha) \\
&\qquad + (|\tau_\alpha|+1)K^{(2)}_{\alpha,\nu}\frac{2^{\alpha+1-\nu}\Gamma((\alpha+1-\nu)/\alpha)}{\alpha(\nu-\varrho)}\frac{\langle |f|,|g|\rangle_{\nu,\alpha}}{\|g\|_\alpha^{\alpha+1-\nu}}\exp(\|f\|_\alpha^\alpha) \\
&\quad \le \frac{K^{(5)}_{\alpha,\nu}\langle |f|,|g|\rangle_{\nu,\alpha}}{(\nu-\varrho)\|g\|_\alpha^{\alpha+1-\nu}}\exp(2\|f\|_\alpha^\alpha) \quad \text{for some constant } K^{(5)}_{\alpha,\nu} > 0.
\end{aligned}
$$

Here we used the estimates $\langle |f|,|g|\rangle_{1,\alpha} \le \langle |f|,|g|\rangle_{\nu,\alpha}^{1/\nu}\|g\|_\alpha^{\alpha(\nu-1)/\nu}$ and $\|f\|_\alpha^{\nu-1} \le \langle |f|,|g|\rangle_{\nu,\alpha}^{1-1/\nu}\|g\|_\alpha^{\nu-1-\alpha(\nu-1)/\nu}$ together with the fact that $\|f\|_\alpha^{\alpha+1-\nu} \le \exp(\|f\|_\alpha^\alpha)$ when $\tilde{\nu}=1$, and the fact that $\|f\|_\alpha^\alpha \le \exp(\|f\|_\alpha^\alpha)$ only when $\tilde{\nu}=\nu$. □

THEOREM 3. *Consider the α-stable random variable (X,Y) given by* (1.2) *where $\alpha \in (1,2)$ and $\|g\|_\alpha > 0$. If $\varrho \in [0,\alpha)$ we have*

$$
\mathbf{E}\{|X|^{\varrho} I_{\{|X|>\lambda\}} \mid |Y=y\} \le \frac{K_\alpha}{\alpha-\varrho}\,\frac{\|f\|_\alpha^\alpha \lambda^{\varrho-\alpha}}{\max\{|y|,\|g\|_\alpha\}f_Y(y)}\exp(2\|f\|_\alpha^\alpha\lambda^{-\alpha})
$$

for $\lambda > 0$ and $y \in \mathbb{R}$, where $K_\alpha > 0$ is a constant that depends on α only.

PROOF. Taking $\nu = \alpha$ in (4.5) we obtain

$$\pm \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} |t|^{\rho}\Big[\cos\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\exp(-\|tg\|_\alpha^\alpha) - \cos\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\exp(-\|sf+tg\|_\alpha^\alpha)\Big]\frac{ds\,dt}{s^{1+\varrho}} \le \frac{K^{(3)}_{\alpha,\alpha,\rho}\|f\|_\alpha^\alpha}{(\alpha-\varrho)\|g\|_\alpha^{\rho+1}}\exp(2\|f\|_\alpha^\alpha) \quad \text{for } \rho\ge 0. \tag{4.8}$$

Replacing the use of inequality (4.3) with that of

$$\begin{aligned} &\big|2\sin(x)-\sin(y)-\sin(z)\big| \\ &\quad = 4\bigg|\cos^2\Big(\frac{y-z}{4}\Big)\sin\Big(\frac{2x-y-z}{4}\Big)\cos\Big(\frac{2x+y+z}{4}\Big) \\ &\qquad + \sin^2\Big(\frac{y-z}{4}\Big)\cos\Big(\frac{2x-y-z}{4}\Big)\sin\Big(\frac{2x+y+z}{4}\Big)\bigg| \\ &\quad \le \big|2x-y-z\big| + \frac{1}{4}|y-z|^2 \quad \text{for } x, y, z\in\mathbb{R}, \end{aligned} \tag{4.9}$$

the sequence of estimates (4.5) further readily carry over to prove the bound

$$\pm \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} t^{\langle\rho\rangle}\Big[\sin\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\exp(-\|tg\|_\alpha^\alpha) - \sin\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\exp(-\|sf+tg\|_\alpha^\alpha)\Big]\frac{ds\,dt}{s^{1+\varrho}} \le \frac{K^{(3)}_{\alpha,\alpha,\rho}\|f\|_\alpha^\alpha}{(\alpha-\varrho)\|g\|_\alpha^{\rho+1}}\exp(2\|f\|_\alpha^\alpha) \quad \text{for } \rho\ge 0. \tag{4.10}$$

Using the elementary inequality $\big||x|^{\alpha-1}-|y|^{\alpha-1}\big| \le |x-y|^{\alpha-1}$ we obtain

$$\begin{aligned} \big||tg|^{\alpha-1}-|tg+sf|^{\alpha-1}\big| &\le \big|(tg)^{\langle\alpha-1\rangle}-(tg+sf)^{\langle\alpha-1\rangle}\big| \le (1+2^{\alpha-1})|sf|^{\alpha-1} \\ &\le 2^\alpha|sf|^{\alpha-1} \end{aligned}$$

(by treating the cases $|tg|>|sf|$ and $|tg|\le|sf|$ separately). This in turn gives

$$\left.\begin{array}{l} \big|\langle g, tg\rangle_{1,\alpha}-\langle g, tg+sf\rangle_{1,\alpha}\big| \\ \big|\langle g, |tg|\rangle_{1,\alpha}-\langle g, |tg+sf|\rangle_{1,\alpha}\big| \end{array}\right\} \le 2^\alpha\langle|g|,|f|\rangle_{1,\alpha}|s|^{\alpha-1} \quad \text{for } s, t\in\mathbb{R}. \tag{4.11}$$

Now let $T = T(\alpha)$ denote the unique solution $t = T$ in $(1,\infty)$ to the equation $(t+1)^{\alpha-1}+(t-1)^{\alpha-1}=2$. Clearly we have

$$2|tg|^{\alpha-1}-|tg+sf|^{\alpha-1}-|tg-sf|^{\alpha-1} \begin{array}{ll} \ge 0 & \text{when } |sf|/|tg|\le T \\ \le 0 & \text{when } |sf|/|tg|\ge T. \end{array}$$

By integration by parts it therefore follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^+} \big|2\langle g, |tg|\rangle_{1,\alpha} - \langle g, |tg+sf|\rangle_{1,\alpha} - \langle g, |tg-sf|\rangle_{1,\alpha}\big| \exp(-\|tg\|_\alpha^\alpha)\, dt \\
&\quad \le \int_{\substack{t\in\mathbb{R}^+\\ x\in\mathbb{R}}} \big|2|tg(x)|^{\alpha-1} - |tg(x)+sf(x)|^{\alpha-1} - |tg(x)-sf(x)|^{\alpha-1}\big| \\
&\qquad \times \frac{|g(x)|}{\exp(\|tg\|_\alpha^\alpha)}\, dx\, dt \\
&\quad = \int_{\substack{x\in\mathbb{R}\\ 0<t\le sf(x)/Tg(x)}} \big(|tg(x)+sf(x)|^{\alpha-1} + |tg(x)-sf(x)|^{\alpha-1} - 2|tg(x)|^{\alpha-1}\big) \\
&\qquad \times \frac{|g(x)|}{\exp(\|tg\|_\alpha^\alpha)}\, dx\, dt \\
&\qquad + \int_{\substack{x\in\mathbb{R}\\ t>sf(x)/Tg(x)}} \\
&\qquad\quad \times \big(2|tg(x)|^{\alpha-1} - |tg(x)+sf(x)|^{\alpha-1} - |tg(x)-sf(x)|^{\alpha-1}\big) \\
&\qquad\quad \times \frac{|g(x)|}{\exp(\|tg\|_\alpha^\alpha)}\, dx\, dt \\
&\quad = \int_{\substack{x\in\mathbb{R}\\ 0<t\le sf(x)/Tg(x)}} \|g\|_\alpha^\alpha t^{\alpha-1} \\
&\qquad \times \big([tg(x)+sf(x)]^{\langle\alpha\rangle} + [tg(x)-sf(x)]^{\langle\alpha\rangle} - 2[tg(x)]^{\langle\alpha\rangle}\big) \\
&\qquad \times \frac{\operatorname{sign}(g(x))}{\exp(\|tg\|_\alpha^\alpha)}\, dx\, dt \\
&\qquad + \int_{\substack{x\in\mathbb{R}\\ t>sf(x)/Tg(x)}} \|g\|_\alpha^\alpha t^{\alpha-1} \\
&\qquad\quad \times \big(2[tg(x)]^{\langle\alpha\rangle} - [tg(x)+sf(x)]^{\langle\alpha\rangle} - [tg(x)-sf(x)]^{\langle\alpha\rangle}\big) \\
&\qquad\quad \times \frac{\operatorname{sign}(g(x))}{\exp(\|tg\|_\alpha^\alpha)}\, dx\, dt \\
&\qquad + \int_{\mathbb{R}} \bigg[\big([tg(x)+sf(x)]^{\langle\alpha\rangle} + [tg(x)-sf(x)]^{\langle\alpha\rangle} - 2[tg(x)]^{\langle\alpha\rangle}\big) \\
&\qquad\qquad\qquad\qquad \times \frac{\operatorname{sign}(g(x))}{\alpha\exp(\|tg\|_\alpha^\alpha)}\bigg]_{t=0}^{t=sf(x)/Tg(x)}\, dx
\end{aligned}
$$

$$+\int_{\mathbb{R}}\bigg[\big(2[tg(x)]^{\langle\alpha\rangle}-[tg(x)+sf(x)]^{\langle\alpha\rangle}-[tg(x)-sf(x)]^{\langle\alpha\rangle}\big) \times \frac{\operatorname{sign}(g(x))}{\alpha\exp(\|tg\|_\alpha^\alpha)}\bigg]_{t=sf(x)/Tg(x)}^{t=\infty}dx$$

$$\le\int_{\mathbb{R}^+}\|g\|_\alpha^\alpha t^{\alpha-1}\big|2\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha-\langle tg-sf\rangle_\alpha\big|\exp(-\|tg\|_\alpha^\alpha)\,dt$$
$$+\int_{\mathbb{R}}\frac{2[(T+1)^\alpha-(T-1)^\alpha-2]}{\alpha T^\alpha}[sf(x)]^{\langle\alpha\rangle}\operatorname{sign}(g(x)) \times\exp(-\|g\|_\alpha^\alpha|sf(x)/Tg(x)|^\alpha)\,dx.$$

By inspection of (4.5) we thus readily conclude that

$$\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}}\Big|2\langle g,|tg|\rangle_{1,\alpha}-\langle g,|tg+sf|\rangle_{1,\alpha}-\langle g,|tg-sf|\rangle_{1,\alpha}\Big| \times\exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}$$
$$\le\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}}\|g\|_\alpha^\alpha t^{\alpha-1}\Big|2\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha-\langle tg-sf\rangle_\alpha\Big| \times\exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}$$
$$+\int_0^1\frac{2|(T+1)^\alpha-(T-1)^\alpha-2|}{\alpha T^\alpha}\|f\|_\alpha^\alpha s^\alpha\frac{ds}{s^{1+\varrho}}$$
$$\le\frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho}\|f\|_\alpha^\alpha\exp(2\|f\|_\alpha^\alpha)+\frac{2|(T+1)^\alpha-(T-1)^\alpha-2|}{\alpha(\alpha-\varrho)T^\alpha}\|f\|_\alpha^\alpha. \tag{4.12}$$

Using the elementary inequality $(1+x)^{\langle\alpha-1\rangle}+(1-x)^{\langle\alpha-1\rangle}\le 2$ we similarly obtain

$$\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}}\Big|2\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}-\langle g,tg-sf\rangle_{1,\alpha}\Big| \times\exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}$$
$$=\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}}\Big(2\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}-\langle g,tg-sf\rangle_{1,\alpha}\Big) \times\exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}} \tag{4.13}$$

$$= \|g\|_\alpha^\alpha \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} t^{\alpha-1}\Big(2\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha - \|tg-sf\|_\alpha^\alpha\Big)$$

$$\times \exp(-\|tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}$$

$$+ \frac{1}{\alpha}\int_0^1 \Big[\big(2\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha - \|tg-sf\|_\alpha^\alpha\big)\exp(-\|tg\|_\alpha^\alpha)\Big]_{t=0}^{t=\infty}\frac{ds}{s^{1+\varrho}}$$

$$\le \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho}\|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) + \frac{2}{\alpha(\alpha-\varrho)}\|f\|_\alpha^\alpha.$$

Integrating (4.7) by parts, the estimates (4.1), (4.3), (4.4) and (4.8)–(4.13) yield

$$\left|\frac{2\pi y f_Y(y)}{4\alpha(2-\varrho)}\mathbf{E}\{|X|^\varrho I_{\{|X|>1\}} \mid |Y=y\}\right|$$

$$\le \left|\frac{y}{\alpha}\int_0^1 \Re\bigg(\int_{\mathbb{R}} \exp(-ity)\big[\phi_{X,Y}(0,t) - \phi_{X,Y}(s,t)\big]\,dt\bigg)\frac{ds}{s^{1+\varrho}}\right|$$

$$= \left|\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \Im\bigg(\frac{1}{\alpha}\exp(-ity)\frac{\partial}{\partial t}\big[\phi_{X,Y}(0,t) - \phi_{X,Y}(s,t)\big]\bigg)\,dt\frac{ds}{s^{1+\varrho}}\right|$$

$$= \bigg|\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}}\Big(\tau_\alpha\cos(ty-\tau_\alpha\langle tg\rangle_\alpha)\langle g,|tg|\rangle_{1,\alpha}\exp(-\|tg\|_\alpha^\alpha)$$

$$-\tau_\alpha\cos(ty-\tau_\alpha\langle sf+tg\rangle_\alpha)\langle g,|sf+tg|\rangle_{1,\alpha}$$

$$\times\exp(-\|sf+tg\|_\alpha^\alpha)$$

$$+\sin\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\langle g,tg\rangle_{1,\alpha}\exp(-\|tg\|_\alpha^\alpha)$$

$$-\sin\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\langle g,sf+tg\rangle_{1,\alpha}$$

$$\times\exp(-\|sf+tg\|_\alpha^\alpha)\Big)\frac{ds\,dt}{s^{1+\varrho}}\bigg|$$

$$= \bigg|\tau_\alpha\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}}\langle g,|tg|\rangle_{1,\alpha}\big(\cos(ty-\tau_\alpha\langle tg\rangle_\alpha)\exp(-\|tg\|_\alpha^\alpha)$$

$$-\cos\big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\big)\exp(-\|sf+tg\|_\alpha^\alpha)\big)\frac{dsdt}{s^{1+\varrho}}$$

$$-\tau_\alpha\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}}\big(\cos(ty-\tau_\alpha\langle tg\rangle_\alpha) - \cos(ty-\tau_\alpha\langle sf+tg\rangle_\alpha)\big)$$

$$\times\big(\langle g,|tg|\rangle_{1,\alpha} - \langle g,|sf+tg|\rangle_{1,\alpha}\big)\exp(-\|sf+tg\|_\alpha^\alpha)\frac{ds\,dt}{s^{1+\varrho}}$$

$$+\tau_\alpha\int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}}\cos\big(ty-\tau_\alpha\langle tg\rangle_\alpha\big)\big(\langle g,|tg|\rangle_{1,\alpha} - \langle g,|sf+tg|\rangle_{1,\alpha}\big)$$

$$
\begin{aligned}
&\times \big(\exp(-\|sf+tg\|_\alpha^\alpha) - \exp(-\|tg\|_\alpha^\alpha)\big) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \tau_\alpha \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} \big(2\langle g, |tg|\rangle_{1,\alpha} - \langle g, |tg+sf|\rangle_{1,\alpha} - \langle g, |tg-sf|\rangle_{1,\alpha}\big) \\
&\qquad \times \cos\big(ty - \tau_\alpha\langle tg\rangle_\alpha\big) \exp(-\|tg\|_\alpha^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \langle g, tg\rangle_{1,\alpha} \big(\sin\big(ty - \tau_\alpha\langle tg\rangle_\alpha\big) \exp(-\|tg\|_\alpha^\alpha) \\
&\qquad - \sin\big(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\big) \exp(-\|sf+tg\|_\alpha^\alpha)\big) \frac{ds\,dt}{s^{1+\varrho}} \\
&- \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \Big(\sin\big(ty - \tau_\alpha\langle tg\rangle_\alpha\big) - \sin\big(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\big)\Big) \\
&\qquad \times \big(\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha}\big) \exp(-\|sf+tg\|_\alpha^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \sin\big(ty - \tau_\alpha\langle tg\rangle_\alpha\big) \big(\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha}\big) \\
&\times \big(\exp(-\|sf+tg\|_\alpha^\alpha) - \exp(-\|tg\|_\alpha^\alpha)\big) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}^+}} \big(2\langle g, tg\rangle_{1,\alpha} - \langle g, tg+sf\rangle_{1,\alpha} - \langle g, tg-sf\rangle_{1,\alpha}\big) \\
&\qquad\qquad\qquad \times \sin\big(ty - \tau_\alpha\langle tg\rangle_\alpha\big) \exp(-\|tg\|_\alpha^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \bigg| \\
\le\ & \tau_\alpha \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) \\
&+ \tau_\alpha^2 \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \big|\langle tg\rangle_\alpha - \langle sf+tg\rangle_\alpha\big| \, \big|\langle g, |tg|\rangle_{1,\alpha} - \langle g, |sf+tg|\rangle_{1,\alpha}\big| \\
&\times \exp(-\|sf+tg\|_\alpha^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \tau_\alpha \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \big|\langle g, |tg|\rangle_{1,\alpha} - \langle g, |sf+tg|\rangle_{1,\alpha}\big| \, \big|\|sf+tg\|_\alpha^\alpha - \|tg\|_\alpha^\alpha\big| \\
&\times \exp\big(-(\|sf+tg\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)\big) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \tau_\alpha \bigg[\frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) + \frac{2|(T+1)^\alpha - (T-1)^\alpha - 2|}{\alpha(\alpha-\varrho)T^\alpha} \|f\|_\alpha^\alpha\bigg] \\
&+ \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tau_\alpha \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \big|\langle tg\rangle_\alpha - \langle sf+tg\rangle_\alpha\big| \, \big|\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha}\big| \\
&\qquad \times \exp(-\|sf+tg\|_\alpha^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} \big|\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha}\big| \, \big|\|sf+tg\|_\alpha^\alpha - \|tg\|_\alpha^\alpha\big| \\
&\qquad \times \exp\big(-(\|sf+tg\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)\big) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) + \frac{2}{\alpha(\alpha-\varrho)} \|f\|_\alpha^\alpha \\
\le{}& (\tau_\alpha+1)^2 \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} K^{(1)}_\alpha \big(\langle |f|, |g|\rangle_{1,\alpha} s|t|^{\alpha-1} + \|f\|_\alpha^\alpha s^\alpha\big) 2^\alpha \langle |g|, |f|\rangle_{1,\alpha} s^{\alpha-1} \\
&\qquad \times \exp(\|f\|_\alpha^\alpha - \|g\|_\alpha^\alpha |\tfrac{1}{2} t|^\alpha) \frac{ds\,dt}{s^{1+\varrho}} \\
&+ 2(\tau_\alpha+1) \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) \\
&+ \tau_\alpha \frac{2|(T+1)^\alpha - (T-1)^\alpha - 2|}{\alpha(\alpha-\varrho)T^\alpha} \|f\|_\alpha^\alpha \\
&+ \frac{2}{\alpha(\alpha-\varrho)} \|f\|_\alpha^\alpha \\
={}& (\tau_\alpha+1)^2 K^{(1)}_\alpha 2^\alpha \langle |g|, |f|\rangle_{1,\alpha} \\
&\times \left(\frac{2^{\alpha+1} \langle |f|, |g|\rangle_{1,\alpha}}{\alpha(\alpha-\varrho)\|g\|_\alpha^\alpha} + \frac{4\Gamma(1/\alpha)\|f\|_\alpha^\alpha}{\alpha(2\alpha-1-\varrho)\|g\|_\alpha} \right) \exp(\|f\|_\alpha^\alpha) \\
&+ 2\bigg[(\tau_\alpha+1) \frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} \exp(2\|f\|_\alpha^\alpha) + \tau_\alpha \frac{|(T+1)^\alpha - (T-1)^\alpha - 2|}{\alpha(\alpha-\varrho)T^\alpha} \\
&\qquad\qquad + \frac{1}{\alpha(\alpha-\varrho)} \bigg] \|f\|_\alpha^\alpha \\
\le{}& (\tau_\alpha+1)^2 K^{(1)}_\alpha 2^\alpha \|g\|_\alpha \|f\|_\alpha^{\alpha-1} \left(\frac{2^{\alpha+1}\|f\|_\alpha \|g\|_\alpha^{\alpha-1}}{\alpha(\alpha-\varrho)\|g\|_\alpha^\alpha} + \frac{4\Gamma(1/\alpha)\|f\|_\alpha^\alpha}{\alpha(\alpha-\varrho)\|g\|_\alpha} \right) \\
&\times \exp(\|f\|_\alpha^\alpha) \\
&+ 2(\tau_\alpha+1) \left[\frac{K^{(3)}_{\alpha,\alpha,\alpha-1}}{\alpha-\varrho} + \frac{|(T+1)^\alpha - (T-1)^\alpha - 2|}{\alpha(\alpha-\varrho)T^\alpha} + \frac{1}{\alpha(\alpha-\varrho)} \right] \\
&\times \|f\|_\alpha^\alpha \exp(2\|f\|_\alpha^\alpha) \\
\le{}& K^{(6)}_\alpha \frac{\|f\|_\alpha^\alpha}{\alpha-\varrho} \exp(2\|f\|_\alpha^\alpha) \quad \text{for some constant } K^{(6)}_\alpha > 0 \\
&\qquad\qquad\qquad\qquad\qquad \text{(depending on } \alpha \text{ only),}
\end{aligned}
$$

where we used that $\|f\|_\alpha^{\alpha-1} \le \exp(\|f\|_\alpha^\alpha)$ in the last step. The theorem follows from combining this bound with the bound obtained taking $\nu = \alpha$ in Theorem 2. □

COROLLARY 2. *Consider the α-stable random variable (X, Y) given by* (1.2) *where $\alpha \in (0,1) \cup (1,2)$ and $\|g\|_\alpha > 0$.*

(i) *Suppose that $\varrho \in [0, \nu \wedge (\alpha+1))$ for some $\nu \in [\alpha, 2)$ with $\langle |f|, |g| \rangle_{\nu,\alpha} < \infty$. Then we have*

$$\mathbf{E}\{|X|^\varrho | Y = y\} \le \frac{K_{\alpha,\nu} \langle |f|, |g| \rangle_{\nu,\alpha}^{\varrho/\nu}}{[(\nu - \varrho)\|g\|_\alpha^{\alpha+1-\nu} f_Y(y)]^{\varrho/\nu}} \quad \textit{for } y \in \operatorname{int}(\operatorname{supp}(Y)).$$

(ii) *Suppose that $\alpha \in (1,2)$ and $\varrho \in (0, \alpha)$. Then we have*

$$\mathbf{E}\{|X|^\varrho | Y = y\} \le \frac{K_\alpha \|f\|_\alpha^\varrho}{\big[(\alpha - \varrho)\max\{|y|, \|g\|_\alpha\} f_Y(y)\big]^{\varrho/\alpha}} \quad \textit{for } y \in \mathbb{R}.$$

PROOF OF (i). Theorem 2 shows that

$$\mathbf{E}\{|X|^\varrho | Y = y\} \le \lambda^\varrho + \frac{K_{\alpha,\nu} \langle |f|, |g| \rangle_{\nu,\alpha} \|g\|_\alpha^{\nu-\alpha-1}}{(\nu - \varrho)\lambda^{\nu-\varrho} f_Y(y)} \exp(2\|f\|_\alpha^\alpha \lambda^{-\alpha}) \quad \text{for } \lambda > 0.$$

Taking $\lambda = [\langle |f|, |g| \rangle_{\nu,\alpha} / ((\nu - \varrho)\|g\|_\alpha^{\alpha+1-\nu} f_Y(y))]^{1/\nu}$ this becomes

$$\begin{aligned} \mathbf{E}\{|X|^\varrho | Y = y\} &\le \left(\frac{\langle |f|, |g| \rangle_{\nu,\alpha}}{(\nu - \varrho)\|g\|_\alpha^{\alpha+1-\nu} f_Y(y)}\right)^{\varrho/\nu} \\ &\quad \times \left[1 + K_{\alpha,\nu} \exp\left\{\frac{2\big[(\nu - \varrho)\|g\|_\alpha f_Y(y)\big]^{\alpha/\nu} \|f\|_\alpha^\alpha}{\langle |f|, |g| \rangle_{\nu,\alpha}^{\alpha/\nu} \|g\|_\alpha^{(\nu-\alpha)\alpha/\nu}}\right\}\right]. \end{aligned} \tag{4.14}$$

Now note the fact that (3.4) implies (albeit not immediately so)

$$f_{S_\alpha(\sigma,\beta)}(x) \le D_\alpha \sigma^\alpha (|x| + \sigma)^{-(\alpha+1)} \quad \text{for } x \in \mathbb{R}, \tag{4.15}$$

for some constant $D_\alpha > 0$. In particular we have $\|g\|_\alpha f_Y(y) \le D_\alpha$. Using the elementary inequality $\|f\|_\alpha^\alpha \le \langle |f|, |g| \rangle_{\nu,\alpha}^{\alpha/\nu} \|g\|_\alpha^{(\nu-\alpha)\alpha/\nu}$ we therefore conclude that the bracket on the right-hand side of (4.14) is bounded by a constant that depends on α and ν only.

PROOF OF (ii). By Theorem 3 we have

$$\mathbf{E}\{|X|^\varrho | Y = y\} \le \lambda^\varrho + \frac{K_\alpha}{\alpha - \varrho} \frac{\|f\|_\alpha^\alpha \lambda^{\varrho-\alpha}}{\max\{|y|, \|g\|_\alpha\} f_Y(y)} \exp(2\|f\|_\alpha^\alpha \lambda^{-\alpha}) \quad \text{for } \lambda > 0.$$

Taking $\lambda = \|f\|_\alpha / [(\alpha - \varrho)\max\{|y|, \|g\|_\alpha\} f_Y(y)]^{1/\alpha}$, it follows that

$$\mathbf{E}\{|X|^\varrho | Y = y\} \le \frac{\|f\|_\alpha^\varrho \big[1 + K_\alpha \exp(2(\alpha - \varrho)\max\{|y|, \|g\|_\alpha\} f_Y(y))\big]}{\big[(\alpha - \varrho)\max\{|y|, \|g\|_\alpha\} f_Y(y)\big]^{\varrho/\alpha}}.$$

Here the bracket in the numerator on the right-hand side is bounded by a constant that depends on α only, since $\max\{|y|, \|g\|_\alpha\} f_Y(y) \le D_\alpha$ by (4.15). □

COROLLARY 3. *Consider the α-stable random variable (X_u, Y) given by* (3.8) *where $\alpha \in (1,2)$ and $\langle g^- \rangle_\alpha > 0$. If $\varrho \in (0,\alpha)$ we have*

$$\limsup_{u\to\infty} \mathbf{E}\{|X_u/u|^\varrho | Y = u\} \leq \frac{K_\alpha \limsup_{u\to\infty} \|f_u\|_\alpha^\varrho}{[(\alpha-\varrho)\langle g^-\rangle_\alpha]^{\varrho/\alpha}}.$$

For the proof, take $y = u$ in Corollary 2(ii) and use (3.4).

5. Asymptotic behavior of conditional probabilities and moments. In Theorem 4 we prove weak convergence of $(X/u \mid Y = u)$ as $u \to \infty$ when $\alpha > 1$ by approximating X with a random variable $\hat{X}$ such that $\sigma_{X-\hat{X}}$ is small, and such that the limit $\lim_{u\to\infty} \mathbf{P}\{\hat{X}/u > \lambda \mid Y = u\}$ can be calculated. The remainder $X - \hat{X}$ is controlled via Corollary 3 and the Markov inequality.

Clearly, one expects the proof of convergence of $(X/u \mid Y = u)$ to be easier in the case $\alpha < 1$, than in the (usually more interesting) case when $\alpha > 1$. However, our bound on conditional moments in Corollary 3 is only valid when $\alpha > 1$. Thus our proof of Theorem 4 (which builds on Corollary 3) can only be adapted to the case $\alpha < 1$ if (a suitable version of) Corollary 3 is proved for that case.

Besides Corollary 3, the important mechanism in the proof of Theorem 4 is subexponentiality: the essential contribution to a large value for a sum of subexponential random variables comes from a single variable; compare (5.4) below. [See for example Samorodnitsky (1988) and Rosiński and Samorodnitsky (1993) for earlier examples on the use of subexponentiality in asymptotic analysis of α-stable phenomena.]

THEOREM 4. *Consider the α-stable random vector (X, Y) in $\mathbb{R}^{n+1}$ given by* (2.1) *where $\alpha \in (1,2)$ and $\langle g^- \rangle_\alpha > 0$. Then we have* (*with obvious notation*)

$$\big(X/u \mid Y = u\big) \to_d Z \quad \textit{where } \mathbf{P}\{Z \leq z\} = \big\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)\leq z\}}\big\rangle \big/ \langle g^-\rangle_\alpha.$$

To explain how Theorem 4 relates to subexponentiality, we approximate the functions g^- and $f_i I_{G^-}$ [in (2.1)] by simple functions $\hat{g}^- = \sum_{j=1}^k g_j I_{E_j}$ and $\hat{f}_i^- = \sum_{j=1}^k f_i^{(j)} I_{E_j}$, where $\{E_j\}_{j=1}^k$ are disjoint sets in $G^- = \{x \in \mathbb{R}\colon g(x) < 0\}$ such that $\|\hat{g}^- - g^-\|_\alpha$ and $\|\hat{f}_i^- - f_i I_{G^-}\|_\alpha$ are "small." Then we have

$$Y_- = \int_{\mathbb{R}} g^- \, d\xi \approx \hat{Y}_- = \int_{\mathbb{R}} \hat{g}^- \, d\xi$$

(in the sense of convergence in probability), and

$$\mathbf{P}\big\{X/u \leq z \mid Y = u\big\} \approx \mathbf{P}\big\{X/u \leq z \mid Y_- = u\big\} \approx \mathbf{P}\big\{X/u \leq z \mid \hat{Y}_- = u\big\}$$

for u large, since the tail of $Y - Y_-$ [cf. (3.2)] is much lighter than that of Y_- [cf. (3.4)]. However, by subexponentiality and (3.4) we have asymptotically

$$\begin{aligned}
&\mathbf{P}\{X/u \le z \mid \hat{Y}_- = u\} \\
&\quad\approx \left(\sum_{j=1}^{k} \mathbf{P}\left\{\bigcap_{i=1}^{n}\{X_i/u \le z_i\} \,\middle|\, g_j \int_{E_j} d\xi = u\right\} f_{g_j \int_{E_j} d\xi}(u)\right) \\
&\qquad\bigg/ \left(\sum_{j=1}^{k} f_{g_j \int_{E_j} d\xi}(u)\right) \\
&\quad\approx \left(\sum_{j=1}^{k} \mathbf{P}\left\{\bigcap_{i=1}^{n}\left\{f_i^{(j)} \int_{E_j} d\xi \Big/ u \le z_i\right\} \,\middle|\, g_j \int_{E_j} d\xi = u\right\} f_{g_j \int_{E_j} d\xi}(u)\right) \\
&\qquad\bigg/ \left(\sum_{j=1}^{k} f_{g_j \int_{E_j} d\xi}(u)\right) \\
&\quad\approx \left(\sum_{j=1}^{k} \left(\prod_{i=1}^{n} I_{\{f_i^{(j)}/g_j \le z_i\}}\right) g_j^{\alpha} \int_{E_j} dx\right) \bigg/ \left(\sum_{j=1}^{k} g_j^{\alpha} \int_{E_j} dx\right) \\
&\quad\approx \left\langle |g|^{\alpha} I_{\{x \in G^-:\ f(x)/g(x) \le z\}}\right\rangle \big/ \langle g^- \rangle_{\alpha}.
\end{aligned}$$

PROOF OF THEOREM 4. By considering the vectors $(\pm X_1, \dots, \pm X_n,\ Y)$, convergence for $(X/u \mid Y = u)$ will follow provided that we can prove

$$\lim_{u \to \infty} \mathbf{P}\{X/u > \lambda \mid Y = u\} = \left\langle |g|^{\alpha} I_{\{x \in G^-:\ f(x)/g(x) > \lambda\}}\right\rangle \big/ \langle g^- \rangle_{\alpha} \tag{5.1}$$

for continuity points $\lambda > 0$ (with components $\lambda_1, \dots, \lambda_n > 0$) of the distribution of Z. To that end we define (again using obvious notation)

$$\begin{aligned}
Y_{k,\varepsilon}^{(+)} &\equiv \int_{\mathbb{R}} I_{A_k} g\, d\xi \quad \text{where } A_k \equiv \{x \in G^+\colon (k-1)\varepsilon g(x) < f(x) \le k\varepsilon g(x)\}, \\
Y_{k,\varepsilon}^{(-)} &\equiv \textstyle\int_{\mathbb{R}} I_{B_k} g\, d\xi \quad \text{where } B_k \equiv \{x \in G^-\colon k\varepsilon g(x) < f(x) \le (k-1)\varepsilon g(x)\}
\end{aligned}$$

for $\varepsilon > 0$ and $k \in \mathbb{Z}^n$. Further let $G^0 \equiv \{x \in \mathbb{R}\colon g(x) = 0\}$, $X^{(0)} \equiv \int_{\mathbb{R}} I_{G^0} f\, d\xi$,

$$f^{(l,\varepsilon)} \equiv \sum_{\|k\| \le l} \left(I_{A_k} k\varepsilon g + I_{B_k} k\varepsilon g\right) \quad \text{and} \quad f^{(\varepsilon)} \equiv \sum_{k \in \mathbb{Z}^n} \left(I_{A_k} k\varepsilon g + I_{B_k} k\varepsilon g\right),$$

where $\|k\| = \max\{|k_1|, \dots, |k_n|\}$ for $k \in \mathbb{Z}^n$. Setting $f^{(\pm)} \equiv I_{G^+ \cup G^-} f$ we have $|f_i^{(\varepsilon)} - f_i^{(\pm)}| \le \varepsilon |g|$ for $i = 1, \dots, n$, so that $f_i^{(\varepsilon)} \in \mathbb{L}^{\alpha}(\mathbb{R})$. Writing $X_{k,\varepsilon}^{(+)} \equiv k\varepsilon Y_{k,\varepsilon}^{(+)}$ and $X_{k,\varepsilon}^{(-)} \equiv k\varepsilon Y_{k,\varepsilon}^{(-)}$, we may thus define the α-stable vectors

$$X^{(l,\varepsilon)} \equiv \int_{\mathbb{R}} f^{(l,\varepsilon)}\, d\xi = \sum_{\|k\| \le l} \left(X_{k,\varepsilon}^{(+)} + X_{k,\varepsilon}^{(-)}\right),$$

$$X^{(\varepsilon)} \equiv \int_{\mathbb{R}} f^{(\varepsilon)}\, d\xi = \sum_{k \in \mathbb{Z}^n} \left(X_{k,\varepsilon}^{(+)} + X_{k,\varepsilon}^{(-)}\right).$$

To proceed, we note that by Corollary 3 and since $X^{(0)}$ is independent of Y,

$$\begin{aligned}
&\limsup_{l\to\infty}\limsup_{u\to\infty}\mathbf{E}\big\{|(X-X^{(l,\varepsilon)})_i/u|\mid Y=u\big\}\\
&\quad\le \limsup_{u\to\infty}\mathbf{E}\big\{|(X-X^{(\pm)})_i/u|\mid Y=u\big\}\\
&\qquad+\limsup_{u\to\infty}\mathbf{E}\big\{|(X^{(\pm)}-X^{(\varepsilon)})_i/u|\mid Y=u\big\}\\
&\qquad+\limsup_{l\to\infty}\limsup_{u\to\infty}\mathbf{E}\big\{|(X^{(\varepsilon)}-X^{(l,\,\varepsilon)})_i/u|\mid Y=u\big\}\\
&\quad\le \limsup_{u\to\infty}\mathbf{E}\big\{|X_i^{(0)}/u|\mid Y=u\big\}\\
&\qquad+K_\alpha\|f_i^{(\pm)}-f_i^{(\varepsilon)}\|_\alpha\big/\big[(\alpha-1)\langle g^-\rangle_\alpha\big]^{1/\alpha}\\
&\qquad+\limsup_{l\to\infty}K_\alpha\|f_i^{(\varepsilon)}-f_i^{(l,\,\varepsilon)}\|_\alpha\big/\big[(\alpha-1)\langle g^-\rangle_\alpha\big]^{1/\alpha}\\
&\quad\le 0+K_\alpha\varepsilon\|g\|_\alpha\big/\big[(\alpha-1)\langle g^-\rangle_\alpha\big]^{1/\alpha}+0
\end{aligned}$$

[where $X^{(\pm)}\equiv\int_{\mathbb{R}}f^{(\pm)}\,d\xi$]. For a vector $\delta=(\delta_1,\dots,\delta_n)>0$ we hence have

$$\begin{aligned}
&\limsup_{u\to\infty}\mathbf{P}\big\{X/u>\lambda\mid Y=u\big\}\\
&\quad\le\limsup_{l\to\infty}\limsup_{u\to\infty}\bigg(\mathbf{P}\big\{X^{(l,\,\varepsilon)}/u>\lambda-\delta\big|Y=u\big\}\\
&\qquad\qquad+\sum_{i=1}^{n}\frac{1}{\delta_i}\mathbf{E}\big\{|(X-X^{(l,\,\varepsilon)})_i/u|\mid Y=u\big\}\bigg)\\
&\quad\le\limsup_{l\to\infty}\limsup_{u\to\infty}\mathbf{P}\big\{X^{(l,\,\varepsilon)}/u>\lambda-\delta\mid Y=u\big\}\\
&\qquad+O(\varepsilon)\quad\text{as }\varepsilon\downarrow 0
\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}
&\liminf_{u\to\infty}\mathbf{P}\big\{X/u>\lambda\mid Y=u\big\}\\
&\quad\ge\liminf_{l\to\infty}\liminf_{u\to\infty}\bigg(\mathbf{P}\big\{X^{(l,\,\varepsilon)}/u>\lambda+\delta\mid Y=u\big\}\\
&\qquad\qquad-\sum_{i=1}^{n}\frac{1}{\delta_i}\mathbf{E}\big\{|(X-X^{(l,\,\varepsilon)})_i/u|\mid Y=u\big\}\bigg)\\
&\quad=\liminf_{l\to\infty}\liminf_{u\to\infty}\mathbf{P}\big\{X^{(l,\,\varepsilon)}/u>\lambda+\delta\mid Y=u\big\}\\
&\qquad-O(\varepsilon)\quad\text{as }\varepsilon\downarrow 0.
\end{aligned}\tag{5.3}$$

Now observe that by calculations similar to those that are featured in (3.7),

$$\mathbf{P}\{|X^{(+)}_{k,\varepsilon}/u| > |\lambda| \mid |Y = u\} = \int_{\{x\in\mathbb{R}:\ |k\varepsilon x|>|\lambda|u\}} \frac{f_{Y^{(+)}_{k,\varepsilon}}(x) f_{Y-Y^{(+)}_{k,\varepsilon}}(u-x)}{f_Y(u)}\,dx \to 0$$

as $u \to \infty$ for $k \in \mathbb{Z}^n$ and $\lambda \in \mathbb{R}^n \setminus \{0\}$, while

$$\mathbf{P}\{X^{(-)}_{k,\varepsilon}/u > \lambda \mid Y = u\} = \int_{\{x\in\mathbb{R}:\ k\varepsilon x>\lambda u\}} \frac{f_{Y^{(-)}_{k,\varepsilon}}(x) f_{Y-Y^{(-)}_{k,\varepsilon}}(u-x)}{f_Y(u)}\,dx \to \begin{cases} 0 & \text{for } k\varepsilon - \lambda \not> 0, \\ \dfrac{2\|I_{B_k} g\|_\alpha^\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha} & \text{for } k\varepsilon - \lambda > 0, \end{cases}$$

when $\lambda = (\lambda_1, \ldots, \lambda_n) > 0$. Further, we have

$$\begin{aligned} &\mathbf{P}\Big\{|X^{(\pm)}_{j,\varepsilon}/u| > |\lambda|, |X^{(\pm)}_{k,\varepsilon}/u| > |\gamma| \mid Y = u\Big\} \\ &\quad = \int_{\{(x,y)\in\mathbb{R}^2:\ |j\varepsilon x|>|\lambda|u, |k\varepsilon y|>|\gamma|u\}} \frac{f_{Y^{(\pm)}_{j,\varepsilon}}(x) f_{Y^{(\pm)}_{k,\varepsilon}}(y) f_{Y-Y^{(\pm)}_{j,\varepsilon}-Y^{(\pm)}_{k,\varepsilon}}(u-x-y)}{f_Y(u)}\,dx\,dy \\ &\quad \to 0 \quad \text{for } \lambda, \gamma \neq 0 \text{ and distinct } j, k \in \mathbb{Z}^n. \end{aligned} \tag{5.4}$$

Given vectors $\lambda,\ \delta > 0$ with $\lambda - 4\delta > 0$, these asymptotic relations yield

$$\begin{aligned} &\limsup_{u\to\infty} \mathbf{P}\{X^{(l,\varepsilon)}/u > \lambda - \delta \mid Y = u\} \\ &\quad \le \limsup_{u\to\infty} \mathbf{P}\Bigg\{\sum_{\|k\|\le l} X^{(-)}_{k,\varepsilon}/u > \lambda - 2\delta \;\Bigg|\; Y = u\Bigg\} \\ &\qquad + \limsup_{u\to\infty} \sum_{i=1}^n \sum_{\|k\|\le l} \mathbf{P}\Bigg\{(X^{(+)}_{k,\varepsilon})_i/u > \frac{\delta_i}{(2l+1)^n} \;\Bigg|\; Y = u\Bigg\} \\ &\quad = \limsup_{u\to\infty} \mathbf{P}\Bigg\{\sum_{\|k\|\le l} X^{(-)}_{k,\varepsilon}/u > \lambda - 2\delta, \\ &\qquad\qquad \bigcup_{\|j\|\le l} \Bigg\{|X^{(-)}_{j,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\Bigg\} \;\Bigg|\; Y = u\Bigg\} \\ &\quad \le \limsup_{u\to\infty} \mathbf{P}\Bigg\{\sum_{\|k\|\le l} X^{(-)}_{k,\varepsilon}/u > \lambda - 2\delta, \\ &\qquad\qquad \bigcup_{\|i\|\le l} \Bigg[\Bigg\{|X^{(-)}_{i,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\Bigg\} \end{aligned}$$

$$\cap \bigcap_{\|j\|\le l,\, j\ne i}\left\{\left|X^{(-)}_{j,\,\varepsilon}/u\right| \le \frac{|\delta|}{(2l+1)^n}\right\}\Bigg]\Bigg|\, Y = u\Bigg\}$$

$$+\limsup_{u\to\infty}\mathbf{P}\Bigg\{\sum_{\|k\|\le l} X^{(-)}_{k,\,\varepsilon}/u > \lambda - 2\delta,\tag{5.5}$$

$$\bigcup_{\|i\|\le l}\Bigg[\left\{|X^{(-)}_{i,\,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\right\}$$

$$\cap \bigcup_{\|j\|\le l,\, j\ne i}\left\{|X^{(-)}_{j,\,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\right\}\Bigg]\Bigg| Y = u\Bigg\}$$

$$\le \limsup_{u\to\infty}\sum_{\|i\|\le l}\mathbf{P}\{X^{(-)}_{i,\,\varepsilon}/u > \lambda - 3\delta \mid Y = u\}$$

$$+\limsup_{u\to\infty}\sum_{\|i\|\le l}\sum_{\substack{\|j\|\le l\\ j\ne i}}\mathbf{P}\Bigg\{|X^{(-)}_{i,\,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n},$$

$$|X^{(-)}_{j,\,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\Bigg| Y = u\Bigg\}$$

$$\le \sum_{\{i\in\mathbb{Z}^n:\ \|i\|\le l,\, i\varepsilon>\lambda-4\delta\}}\frac{2\|I_{B_i}g\|_\alpha^\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}$$

$$\to \left\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)>\lambda-4\delta\}}\right\rangle\Big/\langle g^-\rangle_\alpha \quad \text{as } l\to\infty \text{ and } \varepsilon\downarrow 0$$

(in that order). Similarly, we obtain

$$\liminf_{u\to\infty}\mathbf{P}\{X^{(l,\,\varepsilon)}/u > \lambda+\delta \mid Y = u\}$$

$$\ge \liminf_{u\to\infty}\mathbf{P}\Bigg\{\sum_{\|k\|\le l} X^{(-)}_{k,\,\varepsilon}/u > \lambda + 2\delta\Bigg| Y = u\Bigg\}$$

$$-\limsup_{u\to\infty}\sum_{i=1}^{n}\sum_{\|k\|\le l}\mathbf{P}\Bigg\{(X^{(+)}_{k,\,\varepsilon})_i/u < -\frac{\delta_i}{(2l+1)^n}\Bigg| Y = u\Bigg\}$$

$$\ge \liminf_{u\to\infty}\mathbf{P}\Bigg\{\bigcup_{\|k\|\le l}\{X^{(-)}_{k,\,\varepsilon}/u > \lambda + 3\delta\}\Bigg| Y = u\Bigg\}$$

$$-\limsup_{u\to\infty}\mathbf{P}\Bigg\{\bigcup_{\|k\|\le l}\Bigg[\{X^{(-)}_{k,\,\varepsilon}/u > \lambda + 3\delta\}$$

$$\cap \bigcup_{\substack{\|j\|\le l\\ j\ne k}}\left\{|X^{(-)}_{j,\,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\right\}\Bigg]\Bigg| Y = u\Bigg\}\tag{5.6}$$

$$\geq \liminf_{u\to\infty} \sum_{\|k\|\leq l} \mathbf{P}\big\{X^{(-)}_{k,\varepsilon}/u > \lambda + 3\delta \mid Y = u\big\}$$

$$- \limsup_{u\to\infty} \sum_{\|k\|\leq l} \sum_{\substack{\|j\|\leq l \\ j\neq k}} \mathbf{P}\big\{X^{(-)}_{k,\varepsilon}/u > \lambda + 3\delta,\, X^{(-)}_{j,\varepsilon}/u > \lambda + 3\delta \mid Y = u\big\}$$

$$- \limsup_{u\to\infty} \sum_{\|k\|\leq l} \sum_{\substack{\|j\|\leq l \\ j\neq k}} \mathbf{P}\left\{X^{(-)}_{k,\varepsilon}/u > \lambda + 3\delta,\, |X^{(-)}_{l,\varepsilon}/u| > \frac{|\delta|}{(2l+1)^n}\middle| Y = u\right\}$$

$$\geq \sum_{\{k\in\mathbb{Z}^n:\ \|k\|\leq l,\, k\varepsilon > \lambda+4\delta\}} \frac{2\|I_{B_k} g\|_\alpha^\alpha}{(1+\beta_Y)\|g\|_\alpha^\alpha}$$

$$\to \big\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x) > \lambda+4\delta\}}\big\rangle \big/ \langle g^-\rangle_\alpha \quad \text{as } l\to\infty \text{ and } \varepsilon \downarrow 0.$$

Now (5.1) follows from combining (5.2) and (5.3) with (5.5) and (5.6). □

COROLLARY 4. *Let $\{f_u: \mathbb{R}\to\mathbb{R}^n\}_{u>0}$ be a family of maps with components $f_{u,i} \in \mathbb{L}^\alpha(\mathbb{R})$ for $i = 1,\dots,n$, and consider the $\mathbb{R}^{n+1}$-valued α-stable random vector*

$$(X_u, Y) = \left(\int_{\mathbb{R}} f_u\, d\xi, \int_{\mathbb{R}} g\, d\xi\right) = \left(\int_{\mathbb{R}} f_{u,1}\, d\xi, \dots, \int_{\mathbb{R}} f_{u,n}\, d\xi, \int_{\mathbb{R}} g\, d\xi\right),$$

where $\alpha \in (1,2)$ and $g \in \mathbb{L}^\alpha(\mathbb{R})$ with $\langle g^-\rangle_\alpha > 0$. If $f_{u,i} \to_{\mathbb{L}^\alpha(\mathbb{R})} f_i$ as $u\to\infty$ for $i = 1,\dots,n$, for some map $f: \mathbb{R}\to\mathbb{R}^n$, we have

$$\big(X_u/u \mid Y = u\big) \to_d Z \quad \text{where } \mathbf{P}\{Z \leq z\} = \big\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)\leq z\}}\big\rangle \big/ \langle g^-\rangle_\alpha.$$

PROOF. Writing $X \equiv \int_{\mathbb{R}} f\, d\xi$, Corollary 3 shows that $\mathbf{E}\{|(X_u - X)_i/u| \mid Y = u\} \to 0$ for $i = 1,\dots,n$. Hence the corollary follows from Theorem 4. □

When the conditional law $(X/u \mid Y = u)$ converges weakly, convergence of moments of order $\varrho \in (0,\alpha)$ follows from Corollary 3, while convergence of moments of order $\varrho \in [\alpha, 2)$ follows from Theorem 1 if $\langle f, |g|\rangle_{2,\alpha} < \infty$. Moreover, probabilities and moments conditoned on the event that $Y > u$ also converge:

COROLLARY 5. *Consider the α-stable random variable (X_u, Y) given by (3.8) where $\alpha \in (1,2)$, $\langle g^-\rangle_\alpha > 0$ and $f_u \to_{\mathbb{L}^\alpha(\mathbb{R})} f$ as $u\to\infty$. Suppose that $\varrho \in (0,\alpha)$, or that $\varrho \in [\alpha,2)$ and $\limsup_{u\to\infty}\langle f_u, |g|\rangle_{2,\alpha} < \infty$. Then we have*

$$\lim_{u\to\infty} \mathbf{E}\big\{|X_u/u|^\varrho I_{\{X_u/u>\lambda\}} \mid Y = u\big\} = \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)>\lambda\}}\big\rangle \,\big/\, \langle g^-\rangle_\alpha \tag{5.7}$$

for continuity points $\lambda \in \mathbb{R}$ of the function on the right-hand side. Moreover,

$$\lim_{u\to\infty} \mathbf{E}\{[(X_u/u)^+]^\varrho \mid Y = u\} = \langle (f^{\langle\varrho\rangle} g^{\langle\alpha-\varrho\rangle})^+ I_{G^-}\rangle / \langle g^-\rangle_\alpha,$$

$$\lim_{u\to\infty} \mathbf{E}\{[(X_u/u)^-]^\varrho \mid Y = u\} = \langle (f^{\langle\varrho\rangle} g^{\langle\alpha-\varrho\rangle})^- I_{G^-}\rangle / \langle g^-\rangle_\alpha,$$

$$\lim_{u\to\infty} \mathbf{E}\{|X_u/u|^\varrho \mid Y = u\} = \langle |f|^\varrho |g|^{\alpha-\varrho} I_{G^-}\rangle / \langle g^-\rangle_\alpha,$$

$$\lim_{u\to\infty} \mathbf{E}\{(X_u/u)^{\langle\varrho\rangle} \mid Y = u\} = \langle f^{\langle\varrho\rangle} g^{\langle\alpha-\varrho\rangle} I_{G^-}\rangle / \langle g^-\rangle_\alpha.$$

PROOF. Since $(X_u/u \mid Y = u) \to_d Z$ by Corollary 4, it follows that

$$\big(|X_u/u|^\varrho I_{\{X_u/u>\lambda\}} \big| Y = u\big) \to_d |Z|^\varrho I_{\{Z>\lambda\}}$$

when $\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)=\lambda\}}\rangle = 0$. Further, we obviously have

$$\big\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)=\lambda\}}\big\rangle = \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)=\lambda\}}\big\rangle / |\lambda|^\varrho$$

for $\lambda \neq 0$. Hence continuity points $\lambda \neq 0$ for the right-hand side of (5.7) also are continuity points for $\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)>\lambda\}}\rangle$. The fact that (5.7) holds for continuity points $\lambda \neq 0$ thus follows if the family $\{(|X_u/u|^\varrho I_{\{X_u/u>\lambda\}} | Y = u)\}_{u>0}$ is uniformly integrable. However, by Corollary 3 when $\varrho < \alpha$, and by Corollary 1 when $\varrho \in [\alpha, 2)$ and $\limsup_{u\to\infty} \langle f_u, |g|\rangle_{2,\alpha} < \infty$, we have $\limsup_{u\to\infty} \mathbf{E}\{|X_u/u|^\rho \mid Y = u\} < \infty$ for some $\rho > \varrho$. By elementary considerations this establishes uniform integrability.

By application of (5.7), for continuity points $\lambda \neq 0$, we readily obtain

$$\begin{aligned}
&\big\langle (f^{\langle\varrho\rangle} g^{\langle\alpha-\varrho\rangle})^+ I_{G^-}\big\rangle / \langle g^-\rangle_\alpha \\
&\quad = \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)>0\}}\big\rangle / \langle g^-\rangle_\alpha - \limsup_{\varepsilon\downarrow 0} |\varepsilon|^\varrho \\
&\quad \le \liminf_{\varepsilon\downarrow 0} \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)>\varepsilon\}}\big\rangle / \langle g^-\rangle_\alpha \\
&\quad \le \limsup_{\varepsilon\downarrow 0} \liminf_{u\to\infty} \mathbf{E}\{|X_u/u|^\varrho I_{\{X_u/u>\varepsilon\}} \mid Y = u\} \\
&\quad \le \liminf_{u\to\infty} \mathbf{E}\{[(X_u/u)^+]^\varrho \big| Y = u\} \\
&\quad \le \limsup_{u\to\infty} \mathbf{E}\{[(X_u/u)^+]^\varrho \big| Y = u\} \\
&\quad \le \liminf_{\varepsilon\uparrow 0} \limsup_{u\to\infty} \mathbf{E}\{|X_u/u|^\varrho I_{\{X_u/u>\varepsilon\}} \mid Y = u\} \\
&\quad \le \limsup_{\varepsilon\uparrow 0} \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)>\varepsilon\}}\big\rangle / \langle g^-\rangle_\alpha \\
&\quad \le \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\ f(x)/g(x)>0\}}\big\rangle / \langle g^-\rangle_\alpha + \limsup_{\varepsilon\uparrow 0} |\varepsilon|^\varrho.
\end{aligned}$$

This completes the proof of (5.7), and by application of what has already been proved to the variable $-X_u$, the proof of the whole corollary. $\square$

EXAMPLE 1. For a moving average process $X(t) = \int_{x\in\mathbb{R}} g(t+x)\,d\xi(x)$ with $g \in \mathbb{L}^\alpha(\mathbb{R})$ and $\langle g^- \rangle_\alpha > 0$, we have $(X(t_1)/u, \dots, X(t_n)/u \mid X(0) = u) \to_d Z$ where

$$\mathbf{P}\{Z \le z\} = \big\langle I_{\{x\in G^-:\ g(t_1+x)/g(x)\le z_1,\dots,g(t_n+x)/g(x)\le z_n\}} |g|^\alpha \big\rangle \big/ \langle g^- \rangle_\alpha.$$

Under the hypothesis of Corollary 4, (3.4) and Corollary 4 imply that

$$\begin{aligned}\mathbf{P}\big\{X_u/u > z \mid Y > u\big\} &= \int_1^\infty \mathbf{P}\bigg\{\frac{X_u}{yu} > \frac{z}{y}\bigg| Y = yu\bigg\} \frac{uf_Y(yu)\,dy}{\mathbf{P}\{Y > u\}} \\ &\to \int_1^\infty \big\langle |g|^\alpha I_{\{x\in G^-:\ f(x)/g(x)>z/y\}}\big\rangle \frac{\alpha\,dy}{y^{\alpha+1}} \bigg/ \langle g^- \rangle_\alpha.\end{aligned}$$

In the case when $f_u = f$ (so that $X_u = X$) this was shown via a direct argument by Samorodnitsky (1988), Theorem 3.1. We now address convergence of moments.

COROLLARY 6. *Consider the α-stable random variable (X_u, Y) given by (3.8) where $\alpha \in (1, 2)$, $\langle g^- \rangle_\alpha > 0$ and $f_u \to_{\mathbb{L}^\alpha(\mathbb{R})} f$ as $u \to \infty$. For each $\varrho \in (0, \alpha)$, we have*

$$\begin{aligned}&\lim_{u\to\infty} \mathbf{E}\big\{|X_u/u|^\varrho I_{\{X_u/u>\lambda\}}\big| Y > u\big\} \\ &\quad = \int_{y=1}^{y=\infty} \int_{x\in G^-} |f(x)|^\varrho |g(x)|^{\alpha-\varrho} I_{\{f(x)/g(x)>\lambda/y\}} \frac{\alpha\,dx\,dy}{y^{\alpha+1-\varrho}} \\ &\quad = \int_{x\in G^-} \bigg([f(x)^+]^\varrho |g(x)|^{\alpha-\varrho} - \frac{f(x)^{\langle\varrho\rangle} |g(x)|^{\alpha-\varrho}}{\big[1 \vee (\lambda g(x)/f(x))\big]^{\alpha-\varrho}} \bigg) \frac{\alpha\,dx}{\alpha-\varrho} \quad \textit{for } \lambda \in \mathbb{R}.\end{aligned}$$

PROOF. In view of the obvious fact that

$$\begin{aligned}&\mathbf{E}\bigg\{\bigg|\frac{X_u}{u}\bigg|^\varrho I_{\{X_u/u>\lambda\}}\bigg| Y > u\bigg\} \\ &\qquad = \int_1^\infty \mathbf{E}\bigg\{\bigg|\frac{X_u}{uy}\bigg|^\varrho I_{\{X_u/(uy)>\lambda/y\}}\bigg| Y = uy\bigg\} \frac{y^\varrho u f_Y(yu)\,dy}{\mathbf{P}\{Y > u\}},\end{aligned}$$

the corollary follows from (5.7) and a change of the order of integration in the resulting limit if we can establish dominated convergence. However, dominated convergence is a simple consequence of (3.4) and Corollary 3. □

6. Upcrossings of α-stable processes. Choose an interval $I = [a, b]$ where $-\infty < a < b < \infty$ and a class of maps $\{f_t(\cdot) \in \mathbb{L}^\alpha(\mathbb{R}): t \in I\}$ where $\alpha \neq 1$, and consider the α-stable process

$$\eta(t) \equiv \text{separable version of } \int_{-\infty}^\infty f_t(x)\,d\xi(x) \quad \text{for } t \in I. \tag{6.1}$$

Each non-pathological (separable in probability) strictly α-stable process has this representation in law [e.g., ST (1994), Theorem 13.2.1].

We shall assume that there exists a power $\nu \in [\alpha, 2) \cap (1, \alpha + 1)$ such that

$$\lim_{\varepsilon \downarrow 0} \sup_{s,\, t \in I,\, 0<|t-s|\le\varepsilon} \big\langle \big|(t-s)^{-1}[f_t - f_s - (t-s)f'_s]\big|, |f_s| \big\rangle_{\nu,\alpha} = 0 \tag{6.2}$$

for some class of maps $\{f'_t(\cdot) \in \mathbb{L}^\alpha(\mathbb{R})\colon t \in I\}$. Further we require that

$$\sup_{t\in I} \langle |f'_t|, |f_t| \rangle_{\nu,\alpha} < \infty. \tag{6.3}$$

When $\alpha < 1$ we make the additional assumption that

$$\lim_{\varepsilon\downarrow 0} \Big\| \sup_{s,\, t\in I,\, 0<|t-s|\le\varepsilon} |f_t - f_s| \Big\|_\alpha = 0. \tag{6.4}$$

Note that (6.2) and (6.3) imply $\sup_{t\in I} \|f_t\|_\alpha$, $\sup_{t\in I} \|f'_t\|_\alpha < \infty$. By continuity of $I \ni t \mapsto \|f_t\|_\alpha \in \mathbb{R}$ we further have $\inf_{t\in I} \|f_t\|_\alpha > 0$ when $\|f_t\|_\alpha > 0$ for $t \in I$.

When $\alpha > 1$ we may take $\nu = \alpha$, so that (6.2) and (6.3) reduce to

$$\lim_{\varepsilon\downarrow 0} \sup_{s,\, t\in I,\, 0<|t-s|\le\varepsilon} \big\|(t-s)^{-1}[f_t - f_s - (t-s)f'_s]\big\|_\alpha = 0 \quad \text{and} \quad \sup_{t\in I} \|f'_t\|_\alpha < \infty.$$

The process $\eta(t)$ given by (6.1) is stationary when the integrals

$$\int_{-\infty}^{\infty} \bigg| \sum_{i=1}^n \theta_i f_{t_i+h}(x) \bigg|^\alpha dx \quad \text{and} \quad \int_{-\infty}^{\infty} \bigg(\sum_{i=1}^n \theta_i f_{t_i+h}(x) \bigg)^{\langle\alpha\rangle} dx$$

do not depend on h. For a stationary process $\eta(t)$ with $\alpha > 1$, (6.2) and (6.3) thus boil down to

$$\lim_{t\to 0} \big\| t^{-1}[f_t(\cdot) - f_0(\cdot) - t f'_0(\cdot)] \big\|_\alpha = 0 \quad \text{for some } f'_0(\cdot) \in \mathbb{L}^\alpha(\mathbb{R}).$$

The difference between our requirements (6.2) and (6.3) and the differentiability conditions used by Adler and Samorodnitsky (1997) in their study of stationary $S\alpha S$-processes appears to be minusculous. In essence these requirements mean that $\eta(t)$ has a stochastic derivative $\eta'(t)$ such that $\mathbf{E}\{\eta'(t) \mid \eta(t)\}$ exists for all $t \in I$.

Writing $\eta'(t) = \int_{\mathbb{R}} f'_t \, d\xi$, Theorem 5 below states that the expected number of upcrossings of a level u by $\{\eta(t)\}_{t\in I}$ is given by

$$\mu(I; u) = \int_I \mathbf{E}\{\eta'(t)^+ | \eta(t) = u\} f_{\eta(t)}(u)\, dt. \tag{6.5}$$

Rice (1944, 1945) proposed this formula for differentiable processes. Under additional technical conditions, proofs were given by Leadbetter (1966) and Marcus (1977) for stationary and nonstationary processes, respectively, but, although natural and reasonable, even in the stationary case these conditions are so forbidding that they have been verified for very few processes except Gaussian ones. Indeed, when Adler and Samorodnitsky (1997) verify Marcus's conditions for $S\alpha S$-processes (via a ten-page argument), the key ingredient in their proof is that symmetric α-stable processes allow representations as mixtures of centered Gaussian processes.

If $(\eta'(t), \eta(t))$ has a density $f_{\eta'(t),\eta(t)}$, then (2.5) and (6.5) show that

$$\mu(I;u) = \int_I \left[\int_0^\infty x f_{\eta'(t),\eta(t)}(x,u)\,dx \right] dt.$$

Our proof of (6.5) builds on Lemmas 7.2.1 and 7.2.2 in Leadbetter, Lindgren and Rootzén (1983). Albeit these lemmas are stated for stationary processes, only the last paragraph of the proof of Lemma 7.2.2(iii) uses stationarity. All other arguments are valid for processes $\{\eta(t)\}_{t\in I}$ possessing continuous paths a.s. and a continuous univariate marginal distribution function $F_{\eta(t)}$ at each $t \in I$. Further the mesh used when approximating $\eta(t)$ with a step process need not be uniform.

For each family of sequences $\{a = s_0^{(n)} \le s_1^{(n)} \le \cdots \le s_n^{(n)} \le s_{n+1}^{(n)} = b \colon n \in \mathbb{N}\}$ such that $q_k^{(n)} \equiv s_k^{(n)} - s_{k-1}^{(n)}$ satisfy $\lim_{n\to\infty} \sup_{1\le k\le n+1} q_k^{(n)} = 0$, we thus have

$$\mu(I;u) = \lim_{n\to\infty} \sum_{k=1}^{n+1} \mathbf{P}\{\eta(s_{k-1}^{(n)}) < u < \eta(s_k^{(n)})\}. \tag{6.6}$$

THEOREM 5. *Consider the process* $\{\eta(t)\}_{t\in I}$ *given by* (6.1) *where the maps* $\{f_t(\cdot) \in \mathbb{L}^\alpha(\mathbb{R}) \colon t \in I\}$ *satisfy* (6.2) *and* (6.3) *with* $\alpha \in (0,1)\cup(1,2)$ *and* $\|f_t\|_\alpha > 0$ *for* $t \in I$. *If* $\alpha < 1$ *we also assume that* (6.4) *holds and that* $u \in \operatorname{int}(\operatorname{supp}(\eta(t)))$ *for* $t \in I$. *Then the expected number of upcrossings* $\mu(I;u)$ *of the level* u *by* $\{\eta(t)\}_{t\in I}$ *satisfies Rice's formula* (6.5).

PROOF. To be able to use the formula (6.6) we must prove that $\eta(t)$ is continuous: when $\alpha > 1$ we can choose a power $\varrho \in (1,\alpha)$ and use (6.2) and (6.3) to obtain

$$\begin{aligned}\mathbf{E}\{|\eta(t)-\eta(s)|^\varrho\} &\le \sup_{\beta\in[-1,1]} \mathbf{E}\{|S_\alpha(1,\beta)|^\varrho\} \\ &\qquad \times \left[\|f_t - f_s - (t-s)f_s'\|_\alpha + |t-s|\|f_s'\|_\alpha\right]^\varrho \\ &\le \text{constant} \times |t-s|^\varrho \quad \text{for } s,t \in I.\end{aligned}$$

A well-known and classic argument [e.g., Cramér and Leadbetter (1967), Section 4.2] therefore shows that $\eta(t)$ has continuous sample paths a.s.

When $\alpha < 1$ $\eta(t)$ is continuous a.s. if and only if $\|\sup_{t\in I}|f_t|\|_\alpha < \infty$ and

$$\operatorname{length}\left(\{x \in \mathbb{R} \colon f_t(x) \text{ is not uniformly continuous in } t \in I\}\right) = 0$$

[cf. ST (1994), Theorem 10.4.2]: Clearly these requirements are satisfied when (6.4) holds with $f_t \in \mathbb{L}^\alpha(\mathbb{R})$ for $t \in I$.

Now take $\delta \in (0,1)$ and $\Delta \in (1,\infty)$, and let $t_k^{(n)} = a + k(b-a)/n$ for $k = 0,\ldots,n$. Define a mesh $\{s_k^{(n)}\}_{k=0}^{n+1}$ by setting $s_0^{(n)} = a$, choosing $s_k^{(n)} \in (t_{k-1}^{(n)}, t_k^{(n)}]$

so that

$$\int_\delta^\Delta \mathbf{P}\{\eta'(s_k^{(n)}) > (1-\delta)x \mid \eta(s_k^{(n)}) = u + q_k^{(n)}x\} f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx - \delta$$
$$\le \inf_{s\in(t_{k-1}^{(n)},\,t_k^{(n)}]} \int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1-\delta)x \big| \eta(s) = u + (s - s_{k-1}^{(n)})x\}$$
$$\times f_{\eta(s)}(u + (s - s_{k-1}^{(n)})x)\,dx$$

for $k = 1, \dots, n$ [where $q_k^{(n)} = s_k^{(n)} - s_{k-1}^{(n)}$ as above], and setting $s_{n+1}^{(n)} = b$. Also note that for $\varrho \in (1, \nu)$, Corollary 2(i) together with (4.15) shows that

$$\mathbf{P}\{\eta'(s) > (1-\delta)x \big| \eta(s) = u + (s - s_{k-1}^{(n)})x\} f_{\eta(s)}(u + (s - s_{k-1}^{(n)})x)$$
$$\le \frac{K_{\alpha,\nu} D_\alpha^{1-\varrho/\nu} \langle |f_s'|, |f_s| \rangle_{\nu,\alpha}^{\varrho/\nu}}{(\nu-\varrho)^{\varrho/\nu} \|f_s\|_\alpha^{1-(\nu-\alpha)\varrho/\nu} [(1-\delta)x]^\varrho} \quad \text{for } x > 0 \text{ and } s \in I. \tag{6.7}$$

Using (2.3) and dominated convergence [ensured by (6.7)], we therefore obtain

$$\limsup_{n\to\infty} \sum_{k=1}^{n+1} q_k^{(n)} \int_\delta^\Delta \mathbf{P}\{\eta'(s_k^{(n)}) > (1-\delta)x \mid \eta(s_k^{(n)}) = u + q_k^{(n)}x\}$$
$$\times f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx$$
$$\le \limsup_{n\to\infty} \sum_{k=1}^{n} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \Bigg[\int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1-\delta)x \big| \eta(s) = u + (s - s_{k-1}^{(n)})x\}$$
$$\times f_{\eta(s)}(u + (s - s_{k-1}^{(n)})x)\,dx \Bigg] ds$$
$$+ \limsup_{n\to\infty} [2(b-a)/n](\Delta - \delta) \sup\nolimits_{x\in[\delta,\Delta]} f_{\eta(b)}(u + q_{n+1}^{(n)}x) \tag{6.8}$$
$$+ \delta(b-a)$$
$$= \int_a^b \Bigg[\int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1-\delta)x \big| \eta(s) = u\}\,dx \Bigg] f_{\eta(s)}(u)\,ds + 0 + \delta(b-a)$$
$$\to \int_a^b \mathbf{E}\{\eta'(s)^+ \mid \eta(s) = u\} f_{\eta(s)}(u)\,ds \quad \text{as } \delta \downarrow 0 \text{ and } \Delta \uparrow \infty.$$

Define another mesh by setting $s_0^{(n)} = a$, choosing $s_k^{(n)} \in (t_{k-1}^{(n)}, t_k^{(n)}]$ so that

$$\int_\delta^\Delta \mathbf{P}\{\eta'(s_k^{(n)}) > (1+\delta)x \big| \eta(s_k^{(n)}) = u + q_k^{(n)}x\} f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx + \delta$$
$$\ge \sup_{s\in(t_{k-1}^{(n)},\,t_k^{(n)}]} \int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1+\delta)x \mid \eta(s) = u + (s - s_{k-1}^{(n)})x\}$$
$$\times f_{\eta(s)}(u + (s - s_{k-1}^{(n)})x)\,dx,$$

for $k = 1, \dots, n$, and setting $s_{n+1}^{(n)} = b$. By (2.3) and Fatou's lemma, we have

$$
\begin{aligned}
&\liminf_{n\to\infty} \sum_{k=1}^{n+1} q_k^{(n)} \int_\delta^\Delta \mathbf{P}\{\eta'(s_k^{(n)}) > (1+\delta)x \mid \eta(s_k^{(n)}) = u + q_k^{(n)}x\} \\
&\qquad\qquad \times f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx \\
&\quad \geq \liminf_{n\to\infty} \sum_{k=1}^{n} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \biggl[\int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1+\delta)x \mid \eta(s) = u + (s - s_{k-1}^{(n)})x\} \\
&\qquad\qquad \times f_{\eta(s)}(u + (s - s_{k-1}^{(n)})x)\,dx\biggr]\,ds \qquad (6.9)\\
&\qquad - \delta(b-a) \\
&\quad = \int_a^b \biggl[\int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1+\delta)x \mid \eta(s) = u\}\,dx\biggr] f_{\eta(s)}(u)\,ds - \delta(b-a) \\
&\quad \to \int_a^b \mathbf{E}\{\eta'(s)^+ | \eta(s) = u\} f_{\eta(s)}(u)\,ds \quad \text{as } \delta \downarrow 0 \text{ and } \Delta \uparrow \infty.
\end{aligned}
$$

By application of (6.6) together with an obvious modification of (6.7) we get

$$
\begin{aligned}
&\mu([a,b];u) \\
&\quad = \lim_{n\to\infty} \sum_{k=1}^{n+1} \int_0^\infty q_k^{(n)} \mathbf{P}\biggl\{\frac{\eta(s_k^{(n)}) - \eta(s_{k-1}^{(n)})}{q_k^{(n)}} > x \biggm| \eta(s_k^{(n)}) = u + q_k^{(n)}x\biggr\} \\
&\qquad\qquad \times f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx \\
&\quad \leq \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_0^\delta q_k^{(n)} D_\alpha \|f_{s_k^{(n)}}\|_\alpha^{-1}\,dx \\
&\qquad + \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\delta^\Delta q_k^{(n)} \mathbf{P}\{\eta'(s_k^{(n)}) > (1-\delta)x \mid \eta(s_k^{(n)}) = u + q_k^{(n)}x\} \\
&\qquad\qquad \times f_{\eta(s_k^{(n)})}(u + q_k^{(n)}x)\,dx \\
&\qquad + \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\delta^\Delta q_k^{(n)} \frac{K_{\alpha,\nu}\langle |(q_k^{(n)})^{-1}[f_{s_k^{(n)}} - f_{s_{k-1}^{(n)}} - q_k^{(n)} f'_{s_k^{(n)}}]|, |f_{s_k^{(n)}}|\rangle_{\nu,\alpha}^{\varrho/\nu}}{D_\alpha^{\varrho/\nu - 1}(\nu-\varrho)^{\varrho/\nu} \|f_{s_k^{(n)}}\|_\alpha^{1-(\nu-\alpha)\varrho/\nu} (\delta x)^\varrho}\,dx \\
&\qquad + \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\Delta^\infty q_k^{(n)} \frac{K_{\alpha,\nu}\langle |(q_k^{(n)})^{-1}[f_{s_k^{(n)}} - f_{s_{k-1}^{(n)}}]|, |f_{s_k^{(n)}}|\rangle_{\nu,\alpha}^{\varrho/\nu}}{D_\alpha^{\varrho/\nu - 1}(\nu-\varrho)^{\varrho/\nu} \|f_{s_k^{(n)}}\|_\alpha^{1-(\nu-\alpha)\varrho/\nu} x^\varrho}\,dx.
\end{aligned}
$$

Sending $\delta \downarrow 0$ and $\Delta \uparrow \infty$, and invoking (6.2), (6.3) and (6.8), it follows that

$$
\mu([a,b];u) \leq \int_a^b \mathbf{E}\{\eta'(s)^+ \mid \eta(s) = u\} f_{\eta(s)}(u)\,ds.
$$

In a similar but somewhat simpler way we obtain

$$
\begin{aligned}
&\mu([a,b];u)\\
&\quad\geq \liminf_{n\to\infty}\sum_{k=1}^{n}\int_{\delta}^{\Delta} q_k^{(n)}\mathbf{P}\bigl\{\eta'(s_k^{(n)})>(1+\delta)x\big|\eta(s_k^{(n)})=u+q_k^{(n)}x\bigr\}\\
&\qquad\qquad\times f_{\eta(s_k^{(n)})}\bigl(u+q_k^{(n)}\bigr)\,dx\\
&\qquad-\limsup_{n\to\infty}\sum_{k=1}^{n+1}\int_{\delta}^{\Delta} q_k^{(n)}\frac{K_{\alpha,\nu}\langle\bigl|(q_k^{(n)})^{-1}\bigl[f_{s_k^{(n)}}-f_{s_{k-1}^{(n)}}-q_k^{(n)}f'_{s_k^{(n)}}\bigr]\bigr|,|f_{s_k^{(n)}}|\rangle_{\nu,\alpha}^{\varrho/\nu}}{D_\alpha^{\varrho/\nu-1}(\nu-\varrho)^{\varrho/\nu}\|f_{s_k^{(n)}}\|_\alpha^{1-(\nu-\alpha)\varrho/\nu}(\delta x)^{\varrho}}\,dx.
\end{aligned}
$$

Sending $\delta\downarrow 0$ and $\Delta\uparrow\infty$, and using (6.2), (6.3) and (6.9), we therefore conclude

$$\mu([a,b];u)\geq\int_a^b \mathbf{E}\bigl\{\eta'(s)^+\mid\eta(s)=u\bigr\}f_{\eta(s)}(u)\,ds. \qquad \square$$

In the particular case when $\eta(t)$ is stationary, (6.6) readily yields

$$\mu(I;u)=\text{length}(I)\lim_{s\downarrow 0}s^{-1}\mathbf{P}\{\eta(-s)<u<\eta(0)\}. \tag{6.10}$$

Taking off from (6.10) rather than (6.6), and using the Markov inequality and Corollary 2(i), the proof of (6.5) reduces to just a few lines of elementary calculations.

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