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TYPICAL CONFIGURATIONS FOR ONE-DIMENSIONAL RANDOM FIELD KAC MODEL[1]

BY MARZIO CASSANDRO, ENZA ORLANDI AND PIERRE PICCO

Universitá di Roma, Universitá di Roma Tre and CPT-CNRS

In this paper we study the typical profiles of a random field Kac model. We give upper and lower bounds of the space scale where the profiles are constant. The results hold almost surely with respect to the realizations of the random field. The analysis is based on a block–spin construction, deviation techniques for the local empirical order parameters and concentration inequalities for the realizations of the random magnetic field. For the upper bound, we exhibit a scale related to the law of the iterated logarithm, where the random field makes an almost sure fluctuation that obliges the system to break its rigidity. For the lower bound, we prove that on a smaller scale the fluctuations are not strong enough to allow this transition.

1. Introduction. In this paper we consider a one-dimensional spin system with a ferromagnetic two-body Kac potential and a stochastic external magnetic field. Problems where a stochastic contribution is added to the energy of the system naturally arise in condensed matter physics where the presence of impurities causes the microscopic structure to vary from point to point. A lot of work has been dedicated to the subject of spin system with random magnetic field; let us mention [1–6], [9], [11], [13], [15], [16], [17], [20], [24], [28].

The Kac potentials are functions $J_\gamma(r)$ which depend on the scaling parameter γ as $J_\gamma(r) = \gamma J(\gamma r)$. The equilibrium statistical mechanics of these systems in the absence of stochastic external field are well known. In the limit $\gamma \downarrow 0$, it is possible to explicitly derive the thermodynamic potentials, prove the existence of a critical temperature and give a very natural and transparent explanation of the phenomenon of spontaneous magnetization in ferromagnetic systems [18], [21]. It is also possible to analyze the limit Gibbs states, but since direct interaction between any two given spins vanishes when $\gamma \downarrow 0$, in order to get nontrivial limit distributions, it is useful to introduce the so-called block–spins, which are the space average of spins over regions whose size diverges as $\gamma \downarrow 0$ and which describe the configurations of the system in terms of these magnetization profiles. In the one-dimensional case, an analysis, [10] for Ising spin and [7] for more general spin, allows us

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to get a satisfactory description of the typical profiles. The results can be summarized in the following way. The empirical spin average in blocks of size δ/γ, for any positive δ, converges as $\gamma \downarrow 0$, to one of the two thermodynamic magnetizations, uniformly in intervals of size $1/\gamma^p$, for any given $p \geq 1$. Furthermore, the intervals where the magnetization is approximately constant have lengths of the order $\exp((\Delta f)/\gamma)$ where Δf is the activation energy of the corresponding Curie–Weiss model.

In this paper we add a stochastic magnetic field and study how the previous picture is modified. This is a particular case of the general problem of stochastic perturbation of random systems. Random walk in random environment is another famous example [29], [30]. The general theory of such systems is far from being complete; therefore it is important to have examples that can be rigorously treated, where the behavior of the perturbed system is radically different from the unperturbed one. The first step in the analysis of such systems is to find the right scale where new phenomena occur. The rigorous analysis is in general delicate even if the heuristic arguments are simple.

In our case, if we consider the system in a volume of order $1/\gamma$ and let $\gamma \downarrow 0$, the model is equivalent to the random field Curie–Weiss model [1], [3], [4], [6], [20], [24], [28]. It is possible to define a critical temperature and, if the variance of the magnetic fields is small enough, only two distinct magnetization profiles occur, the relative weight of each one being a random variable. When we take first the infinite volume limit and then the limit $\gamma \downarrow 0$, new phenomena occur that depend on the scale we are considering. If we consider what happens in a large interval, say, centered at the origin and of length $\gamma^{-2}[(\log 1/\gamma)^p]$ for some $p > 1$, we start seeing new effects of the random magnetic field. The profiles that were approximately constant on a scale $\exp([(1-\varepsilon)\Delta f]/\gamma)$ and made a transition between the two equilibria on a scale $\exp([(1+\varepsilon)\Delta f]/\gamma)$ when the random magnetic field was switched off, now make a transition on a scale at most $\gamma^{-2}[(\log 1/\gamma)(\log\log 1/\gamma)^2]$ and are constant on a scale at least $l(\gamma) = \gamma^{-2}[\log\log 1/\gamma]^{-1}$. To be a little more precise, for almost all the realizations of the random magnetic fields, for all but a finite number of indices n, if $\gamma = 2^{-n}$, up to a translation of *at most* $l(\gamma)$, we meet a constant profile which is constant on an interval which is *at least* $l(\gamma)$. Note that for a *given* interval of scale $l(\gamma)$, say, centered at the origin, the system can be approximately constant around one of the two equilibria or make just one transition between the two equilibria. That is, there is at most one transition in such a fixed interval. Let us note that in a recent paper [8], the Kac–Hopfield model was considered and it was proved that the system made at most one transition in an interval of scale $\gamma^{-2}[\log 1/\gamma]^{-1}$ which is smaller than $l(\gamma)$. Here it is possible to get results on a scale $l(\gamma)$ mainly because the system we consider is simpler and this allows us to make more accurate estimates. Moreover, to get the scale $\gamma^{-2}[(\log 1/\gamma)(\log\log 1/\gamma)^2]$, a very special representation of the system is used. It is possible to get similar results for the Kac–Hopfield model in the regime where the number of patterns is bounded by $(\log 1/\gamma)/\log 2$. This is just a tedious modification of what is done in this work and no new ideas are needed.

The plan of the paper is the following. In Section 2 we introduce notation and state the main results. In Section 3 we perform the block–spin representation, giving an explicit representation of the random part. A large deviation principle in the strong form, that is, with estimates of the subexponential terms, for a hypergeometric random variable is given there. In Section 4 we prove the upper bound and in Section 5 we prove the lower bound for the typical length of profiles.

2. The model and the main results. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Let $h \equiv \{h_i\}_{i \in \mathbb{Z}}$ be a family of independent, identically distributed Bernoulli random variables defined on this space; that is $\mathbb{P}[h_i = +1] = \mathbb{P}[h_i = -1] = 1/2$. We denote by σ a function $\mathbb{Z} \to \{-1, +1\}$ and call σ_i, $i \in \mathbb{Z}$ the spin at site i. $\mathscr{S}$ is the space of such functions, equipped with the product topology. Given $\Lambda \subset \mathbb{Z}$, we denote by σ_Λ a function $\Lambda \to \{-1, +1\}$ and the space of such functions is denoted by $\mathscr{S}_\Lambda$. We choose a Kac potential of the form $J_\gamma(i-j) \equiv \gamma J(\gamma|i-j|)$, $\gamma > 0$, where $J(x) = \mathbb{1}_{|x| \le 1/2}$. Note that more general ferromagnetic potentials could be used without changing the behavior of the model. The relevant conditions are (1) $J(x) \ge 0$ (i.e., ferromagnetism), (2) $J(x) = J(-x)$ (symmetry), (3) fast decay at infinity; that could be short range or exponential $J(x) = \exp(-2|x|)$ as in the original Kac model. We assume that $\int J(x)\, dx = 1$.

The Hamiltonian in a finite volume $\Lambda \subset \mathbb{Z}$ with free boundary conditions is the random variable

$$H_\gamma(\sigma_\Lambda)[\omega] = -\tfrac{1}{2} \sum_{(i,j) \in \Lambda \times \Lambda} J_\gamma(i-j)\sigma_i \sigma_j - \theta \sum_{i \in \Lambda} h_i[\omega]\sigma_i, \tag{2.1}$$

where θ is a strictly positive parameter. The interaction between the spins in Λ and those outside Λ will be denoted by

$$W_\gamma(\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i \in \Lambda} \sum_{j \in \Lambda^c} J_\gamma(i-j)\sigma_i \sigma_j. \tag{2.2}$$

We will usually drop the ω dependence for all quantities we consider.

The *Gibbs measure* at inverse temperature $\beta > 0$ in the finite region Λ with free boundary conditions is the probability measure-valued random variable $\mu_{\beta,\theta,\gamma,\Lambda}$ on $\{-1, +1\}^\Lambda$ defined by

$$\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \exp\{-\beta H_\gamma(\sigma_\Lambda)\}. \tag{2.3}$$

Here $Z_{\beta,\theta,\gamma,\Lambda}$ is the partition function, that is, the normalization factor to make $\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda)$ into a probability measure on $\mathscr{S}_\Lambda$.

If $\tilde\sigma$ is a spin configuration in $\mathscr{S}$, the Gibbs measure with boundary condition $\tilde\sigma$ is the probability measure-valued random variable $\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ on

$\{-1, +1\}^{\Lambda}$ defined by

$$\mu^{\tilde{\sigma}_{\Lambda^C}}_{\beta, \theta, \gamma, \Lambda}(\sigma_\Lambda) = \frac{1}{Z^{\tilde{\sigma}_{\Lambda^c}}_{\beta, \theta, \gamma, \Lambda}} \exp\Big\{-\beta\big(H_\gamma(\sigma_\Lambda) + W_\gamma(\sigma_\Lambda, \tilde{\sigma}_{\Lambda^c})\big)\Big\}. \tag{2.4}$$

Here $Z^{\tilde{\sigma}_{\Lambda^c}}_{\beta, \theta, \gamma, \Lambda}$, the partition function in the volume Λ with the boundary condition $\tilde{\sigma}$, is the normalization factor to make $\mu^{\tilde{\sigma}_{\Lambda^C}}_{\beta, \theta, \gamma, \Lambda}$ into a probability measure on $\mathscr{S}_\Lambda$.

Given a realization of h, $\forall \gamma > 0$, the infinite volume Gibbs measure $\mu_{\beta, \theta, \gamma}$ is obtained as the unique weak-limit of $\mu_{\beta, \theta, \gamma, \Lambda}$ along a family of volumes $\Lambda_L = [-L, +L]$, $L \in \mathbb{N}$. It is also the unique weak-limit of $\mu^{\tilde{\sigma}_{\Lambda^C}}_{\beta, \theta, \gamma, \Lambda}$ for any $\tilde{\sigma}$ that could depend on h. Note that different realizations of h give different infinite volume Gibbs measures.

The *free energy* in the volume Λ, with free boundary conditions, is defined by

$$F_\Lambda(\beta, \theta, \gamma) = -\frac{1}{\beta|\Lambda|} \log Z_{\beta, \theta, \gamma, \Lambda}. \tag{2.5}$$

The infinite volume limit $F(\beta, \theta, \gamma)$ of the free energy with free boundary conditions, for fixed γ, exists $\mathbb{P}$-almost surely by standard subadditive argument; see [34], [19]. Being measurable with respect to the tail σ-algebra of $\mathscr{F}$, $F(\beta, \theta, \gamma)$ is a nonrandom quantity and it is equal to the limit of the average of $F_\Lambda(\beta, \theta, \gamma)$ with respect to $\mathbb{P}$.

Given a volume $\Delta \subset \mathbb{Z}$, we define the sample magnetization in Δ by

$$\tilde{m}_\Delta(\sigma) = \frac{1}{|\Delta|} \sum_{i \in \Delta} \sigma_i. \tag{2.6}$$

A relevant order parameter of this system is the limit, when $\Delta \uparrow \mathbb{Z}$, of the infinite volume Gibbs average of $\tilde{m}_\Delta$. Note that $\tilde{m}_\Delta$ can be written as $\tilde{m}_\Delta(\sigma) = \hat{m}_\Delta(+, \sigma) + \hat{m}_\Delta(-, \sigma)$ where

$$\hat{m}_\Delta(\pm, \sigma) = \frac{1}{|\Delta|} \sum_{i \in \Delta} \sigma_i \left(\frac{1 \pm h_i}{2}\right) \tag{2.7}$$

is the local sample magnetization on the random subset of Δ where the magnetic field is positive (resp. negative).

Given $\varepsilon > 0$ and $(m_1, m_2) \in [-1, +1]^2$, we define the constrained partition function,

$$\begin{aligned} &\hat{Z}_{\beta, \theta, \gamma, \Lambda}(m_1, m_2, \varepsilon) \\ &\quad = \frac{1}{2^{|\Lambda|}} \sum_{\sigma_\Lambda \in \mathscr{S}_\Lambda} \exp\big(-\beta H_\gamma(\sigma_\Lambda)\big) \mathbb{1}_{\{|\hat{m}_\Lambda(+, \sigma) - m_1| \le \varepsilon\}} \mathbb{1}_{\{|\hat{m}_\Lambda(-, \sigma) - m_2| \le \varepsilon\}} \end{aligned} \tag{2.8}$$

and the constrained finite volume free energy

$$\hat{F}_\Lambda(\beta, \theta, \gamma, m_1, m_2, \varepsilon) = -\frac{1}{\beta|\Lambda|} \log \hat{Z}_{\beta, \theta, \gamma, \Lambda}(m_1, m_2, \varepsilon). \tag{2.9}$$

Using as before standard subadditive arguments [34], [19], $\mathbb{P}$-almost surely, $\lim_{\varepsilon \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}} \hat{F}_\Lambda(\beta, \theta, \gamma, m_1, m_2, \varepsilon) = \hat{F}(\beta, \theta, \gamma, m_1, m_2)$ exists and it is nonrandom. Moreover, it follows from general arguments (see [33]) that it is a convex function of (m_1, m_2) and $F(\beta, \theta, \gamma) = \inf_{m_1, m_2} \hat{F}(\beta, \theta, \gamma, m_1, m_2)$.

We want to give a precise description of the typical configurations in term of profiles of local magnetizations in a given scale. This leads naturally to the notion of block–spin transformations that will be defined later. Similar analysis was done in the one-dimensional ferromagnetic Kac model without external magnetic field in [10], [7].

We will not use $\hat{m}_\Delta(\pm, \sigma)$ to transform our system into a block–spin system. We will use an equivalent set of two local averages. The main reason is that the cardinality of the subset of Δ where h is positive is a random number with mean $\Delta/2$. The random fluctuations of this cardinality govern the stochastic fluctuations of the system. We use another representation of the system in term of a priori less physical quantities. They are the empirical magnetizations over random sets with *fixed* length equal to $\Delta/2$. However, the local magnetization in a block is just one-half the sum of these two empirical magnetizations. This allows us to extract from the random terms a volume term $\Delta/2$ which is deterministic. Moreover, with this choice some important quantities, such as the logarithm of (4.8), are symmetric random variables.

The effect of the block–spin transformation is to transform our microscopic system on $\mathbb{Z}$ into a *macroscopic* system on $\mathbb{R}$. Since the interaction length is γ^{-1}, we consider the system in a macroscopic scale where the interaction length becomes 1. The volumes we consider will always be expressed in this macroscopic scale; that is, a macroscopic volume $V \subset \mathbb{R}$ corresponds to a microscopic volume $\Lambda = \Lambda(V) = \gamma^{-1}V \cap \mathbb{Z}$. Now, given $0 < \delta^* < 1$, we partition $\mathbb{R}$ into blocks of length δ^*. This will induce a partition of $\mathbb{Z}$ into blocks of length $\delta^*\gamma^{-1}$. We assume for convenience that $\gamma = 2^{-n}$ for some integer n and δ^* is a function of n such that $\delta^*\gamma^{-1}$ is an integer.

We denote by $\mathscr{A}(x)$ a block of length δ^* centered at x. This corresponds in a microscopic scale to a block of length $\delta^*\gamma^{-1}$, $A(x) \equiv \{i \in \mathbb{Z}, \gamma^{-1}\delta^*(x - 1/2) \le i < \gamma^{-1}\delta^*(x + 1/2)\}$. We denote by $a^-(x) = \inf\{i\colon i \in A(x)\}$ and $a^+(x) = \sup\{i\colon i \in A(x)\}$.

Given a realization of h: $h[\omega] \equiv (h_i[\omega])_{i \in \mathbb{Z}}$, let us call $A^+(x) = \{i \in A(x), h_i[\omega] = +1\}$ and $A^-(x) = \{i \in A(x), h_i[\omega] = -1\}$. We denote by $\lambda(x) \equiv \operatorname{sgn}(|A^+(x)| - (2\gamma)^{-1}\delta^*)$, where sgn is the sign function, with the convention that sgn(0) = 0. Note that if $\delta^*\gamma^{-1}$ is odd, $\lambda(x)$ is a Bernoulli symmetric random variable. However, for convenience we assume $\delta^*\gamma^{-1}$ even. In this case, the distribution of $\lambda(x)$ has the following mass at zero:

$$\mathbb{P}[\lambda(x) = 0] = 2^{-\delta^*\gamma^{-1}} \binom{\delta^*\gamma^{-1}}{\delta^*\gamma^{-1}/2}. \tag{2.10}$$

We define, for a given realization of h such that $\lambda(x) = \pm 1$,

$$l^\lambda(x) \equiv l^{\lambda(x)}(x) = \inf\left\{ l \ge a^-(x)\colon \sum_{j = a^-(x)}^{l} \mathbb{1}_{\{A^{\lambda(x)}(x)\}}(j) \ge \delta^*\gamma^{-1}/2 \right\}. \tag{2.11}$$

We denote the corresponding subset $B^{\lambda}(x)=\{i\in A^{\lambda(x)}(x), i\le l^{\lambda}(x)\}$ and $B^{-\lambda}(x)=A(x)\setminus B^{\lambda}(x)$. If $\lambda(x)=0$, we take $B^{+}(x)=A^{+}(x)$ and $B^{-}(x)=A^{-}(x)$. Let us call $A^{\lambda}(x)\setminus B^{\lambda}(x)\equiv D^{\lambda}(x)$. Note that with this construction, since we have assumed $\delta^*\gamma^{-1}$ even, we have always $|B^{+}(x)|=|B^{-}(x)|=\delta^*\gamma^{-1}/2$.

We define, for $\lambda=\pm 1$,

$$m^{\delta^*}(\lambda,x,\sigma)=\frac{2\gamma}{\delta^*}\sum_{i\in B^{\lambda}(x)}\sigma_i. \tag{2.12}$$

Notice that we have still $(\gamma/\delta^*)\sum_{i\in A(x)}\sigma_i=\frac{1}{2}(m^{\delta^*}(+,x,\sigma)+m^{\delta^*}(-,x,\sigma))$ but now,

$$\begin{aligned}&\frac{\gamma}{\delta^*}\sum_{i\in A(x)}h_i\sigma_i\\&\quad=\frac{1}{2}\big(m^{\delta^*}(+,x,\sigma)-m^{\delta^*}(-,x,\sigma)\big)+\lambda(x)\frac{2\gamma}{\delta^*}\sum_{i\in D^{\lambda}(x)}\sigma_i.\end{aligned} \tag{2.13}$$

Given a microscopic volume Λ, we denote by

$$\mathcal{M}_{\delta^*}(\Lambda)\equiv\prod_{x\in\mathcal{C}_{\delta^*}(\Lambda)}\left[-1,-1+\frac{4\gamma}{\delta^*},-1+\frac{8\gamma}{\delta^*},\dots,1-\frac{4\gamma}{\delta^*},1\right]^2, \tag{2.14}$$

where $\mathcal{C}_{\delta^*}(\Lambda)$ is the set of the centers of the blocks of length $\delta^*\gamma^{-1}$ that we get making a partition of Λ into such blocks. Namely, $\mathcal{M}_{\delta^*}(\Lambda)$ is the set of possible configurations of the pair $m^{\delta^*}(x,\sigma)=(m^{\delta^*}(+,x,\sigma),m^{\delta^*}(-,x,\sigma))$ for $x\in\mathcal{C}_{\delta^*}(\Lambda)$. We denote by

$$m^{\delta^*}(\Lambda)\equiv\big(m^{\delta^*}(x)\big)_{x\in\mathcal{C}_{\delta^*}(\Lambda)}\equiv\big(m_1^{\delta^*}(x),m_2^{\delta^*}(x)\big)_{x\in\mathcal{C}_{\delta^*}(\Lambda)} \tag{2.15}$$

an element of $\mathcal{M}_{\delta^*}(\Lambda)$.

We call a block–spin transformation the random map,

$$\begin{aligned}&\{-1,+1\}^{\Lambda}\to\mathcal{M}_{\delta^*}(\Lambda),\\&\qquad\sigma_{\Lambda}\to\big((m^{\delta^*}(+,x,\sigma),m^{\delta^*}(-,x,\sigma))\big)_{x\in\mathcal{C}_{\delta^*}(\Lambda)}.\end{aligned} \tag{2.16}$$

By abuse of notation, we denote by $\mu_{\beta,\theta,\gamma,\Lambda}$ the probability measure induced by the Gibbs measure through this map. The infinite volume limit $\lim_{\Lambda\uparrow\mathbb{Z}}\mu_{\beta,\theta,\gamma,\Lambda}$ will be denoted $\mu_{\beta,\theta,\gamma}$.

If $\lim_{\gamma\downarrow 0}\delta^*(\gamma)=0$, the induced Gibbs measure $\mu_{\beta,\theta,\gamma}$ will have a support in the subset $\mathcal{T}$ of $L^{\infty}(\mathbb{R},dx)\times L^{\infty}(\mathbb{R},dx)$ of all measurable functions $(m_1(x),m_2(x))$, $x\in\mathbb{R}$ such that $\max(|m_1(x)|,|m_2(x)|)\le 1$. $\mathcal{T}$ is a compact convex set with respect to the weak L^2-loc topology.

We want to study the block–spin profiles which are typical with respect to the Gibbs measure $\mu_{\beta,\theta,\gamma}$ when $\gamma\downarrow 0$. However, since the Gibbs measure is a random variable defined on Ω, we have also to specify in what $\mathbb{P}$-probabilistic sense this is true. In this paper we consider results that are true $\mathbb{P}$-almost surely.

These typical configurations will have a spatial structure that will critically depend on the values of the parameters β, θ and on the length scale we are considering. As in all Kac models, the local behavior is related to the one of the corresponding Curie–Weiss model. In our case it is the random field Curie–Weiss model (RFCW). This model is well studied (see [1], [3], [4], [20], [24], [28]) for various distributions of the random field $h[\omega]$. The Bernoulli and the Gaussian distributions are the most commonly used. Note that even if parameters similar to the $\hat{m}(\pm, x)$ were already introduced in [28], in all the previously mentioned references, the results were given for the measure induced by the Gibbs measure through magnetization.

Since our approach is slightly different, let us state some results for the RFCW model in term of the parameters $m(\pm, x)$.

The random field Curie–Weiss model. This is the case where we assume $\Lambda = 1/\gamma \equiv N$, so that the thermodynamic limit and the limit $\gamma \downarrow 0$ are not independent. The Hamiltonian of the random field Curie–Weiss model is given by

$$H(\sigma_N)[\omega'] = -\frac{1}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - \theta\sum_{i=1}^{N} h_i[\omega']\sigma_i, \tag{2.17}$$

where θ is a strictly positive parameter.

The partition function is $Z_N(\beta, \theta) = \sum_{\sigma \in \mathscr{S}_N} \exp(-\beta H(\sigma_N))$ and the finite volume free energy is $f_N(\beta, \theta) = -(1/\beta N)\log Z_N(\beta, \theta)$. We make the partition of $\{1, \dots, N\}$ into two random blocks of equal length $N/2$ exactly as we did between (2.11) and (2.12). Considering the empirical pair of magnetization over the previous blocks, we denote by $Z_N(\beta, \theta, m_1, m_2, \varepsilon)$ the constrained partition function defined in a similar way to (2.8) and by $f_N(\beta, \theta, m_1, m_2, \varepsilon) = -(1/\beta N)\log Z_N(\beta, \theta, m_1, m_2, \varepsilon)$ the associated free energy.

It is easy to check that $\mathbb{P}$-almost surely, uniformly with respect to $(m_1, m_2) \in [-1, +1]^2$, we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\lim_{N\uparrow\infty} f_N(\beta, \theta, m_1, m_2, \varepsilon) &= \frac{-(m_1+m_2)^2}{8} - \frac{\theta}{2}(m_1 - m_2) - \frac{1}{2\beta}(I(m_1) + I(m_2)) \\ &\equiv f_{\beta,\theta}(m_1, m_2);\end{aligned} \tag{2.18}$$

here $I(m) = -((1+m)/2)\log((1+m)/2) - ((1-m)/2)\log((1-m)/2)$. The function $f_{\beta,\theta}(m_1, m_2)$ is called the canonical free energy. Moreover, it can be checked that, $\mathbb{P}$-almost surely,

$$\lim_{N\uparrow\infty} f_N(\beta, \theta) = f(\beta, \theta) = \inf_{(m_1, m_2)\in[-1,+1]^2} f_{\beta,\theta}(m_1, m_2). \tag{2.19}$$

Our first result relates the free energy of the random field Kac model to the one of the random field Curie–Weiss model.

THEOREM 2.1. *For all positive* β, *for all positive* θ, $\mathbb{P}$-*almost surely we have*

$$\lim_{\gamma\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}} F_\Lambda(\beta,\theta,\gamma)=f(\beta,\theta). \tag{2.20}$$

The proof of this result, being straightforward although lengthy, will not be given here. It is a consequence of the block–spin representation made in Section 3 and modification of classical arguments that can be found, for example, in [33].

To state our next results, we need some results on the RFCW model. The critical points of $f_{\beta,\theta}(m_1,m_2)$ are the two-dimensional vectors $m=(m_1,m_2)$ solutions of the system of equations,

$$\begin{aligned} m_1&=\tanh\left(\beta\frac{(m_1+m_2)}{2}+\beta\theta\right),\\ m_2&=\tanh\left(\beta\frac{(m_1+m_2)}{2}-\beta\theta\right). \end{aligned} \tag{2.21}$$

We assume throughout this paper that $\beta>1$ and $\beta\theta$ satisfies

$$\tanh\beta\theta\le\min\left(1/\sqrt{3}\,,(1-\beta^{-1})^{1/2}\right). \tag{2.22}$$

This implies that the system (2.21) has only three solutions, two of them being absolute minima and one the local maximum of $f_{\beta,\theta}(m_1,m_2)$. This can be proved easily by considering

$$m=\tfrac{1}{2}\tanh\beta(m+\theta)+\tfrac{1}{2}\tanh\beta(m-\theta). \tag{2.23}$$

The previous condition implies that the derivative at the origin of the function on the right-hand side of (2.23) is bigger than 1 and the function is concave on the positive real, convex on the negative real number. Moreover, if $\tilde{m}_\beta$ is the largest positive solution of (2.23), then the two absolute minima of $f_{\beta,\theta}(m_1,m_2)$ are of the form $m_\beta=(m_{\beta,1},m_{\beta,2})$ and $Tm_\beta=(-m_{\beta,2},-m_{\beta,1})$ where $m_{\beta,1}=\tanh\beta(\tilde{m}_\beta+\theta)$ and $m_{\beta,2}=\tanh\beta(\tilde{m}_\beta-\theta)$.

It is easy to see that the function $f_{\beta,\theta}(m_1,m_2)$ is quadratic around its minima. Moreover, there exists a constant $c(\beta,\theta)$ such that for all $m=(m_1,m_2)$,

$$f_{\beta,\theta}(m)-f_{\beta,\theta}(m_\beta)\ge c(\beta,\theta)\min\left(\|m-m_\beta\|_2^2,\|m-Tm_\beta\|_2^2\right). \tag{2.24}$$

Here $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$.

Our second result is the analogue of the Lebowitz–Penrose theorem [21, 25]. It relates the canonical free energy of the random field Kac model to the convex envelope of the canonical free energy of the random field Curie–Weiss model. Recall that the convex envelope of a function f is the largest convex function that is smaller than f. It will be denoted by $\mathrm{Conv}(f)$.

THEOREM 2.2. *For all positive* β, *for all positive* θ, $\mathbb{P}$-*almost surely, uniformly with respect to* $(m_1,m_2)\in[-1,+1]$, *we have*

$$\lim_{\varepsilon\downarrow 0}\lim_{\gamma\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}}\hat{F}_\Lambda(\beta,\theta,\gamma,m_1,m_2,\varepsilon)=\mathrm{Conv}(f_{\beta,\theta})(m_1,m_2). \tag{2.25}$$

The proof of this theorem will not be given here. It is a consequence of the block–spin representation of Section 3 and can be done following step-by-step the usual proof of the Lebowitz and Penrose theorem; see [33].

To describe the asymptotic properties of the support of the measure $\mu_{\beta,\theta,\gamma}$, we need to introduce another scale. To avoid possible confusion, we emphasize that we do not make a block–spin transformation on this scale. Given $\delta > \delta^*$ and assuming that $\delta = k\delta^*$ for some positive integer $k \geq 2$, for $l \in \mathbb{Z}$, we denote by $C_\delta(l)$ the set of centers of those blocks of length δ that are in the macroscopic interval $[l - \frac{1}{2}, l + \frac{1}{2}[$ and given $r \in C_\delta(l)$ we denote by $C_{\delta^*/\delta}(r)$ the set of centers of those blocks of length δ^* that are in the interval of length δ indexed by r. We define the notion of being near an equilibrium with tolerance ζ. We impose that $0 < \zeta \leq m_{\beta,2}$ to separate the two equilibria and define for $l \in \mathbb{Z}$, the random variable

$$\eta^{\delta,\zeta}(l) = \begin{cases} 1, & \text{if } \forall_{u \in C_\delta(l)} \dfrac{\delta^*}{\delta} \sum_{x \in \mathscr{C}_{\delta^*/\delta}(u)} \|m^{\delta^*}(x) - m_\beta\|_1 \leq \zeta, \\ -1, & \text{if } \forall_{u \in C_\delta(l)} \dfrac{\delta^*}{\delta} \sum_{x \in \mathscr{C}_{\delta^*/\delta}(u)} \|m^{\delta^*}(x) - Tm_\beta\|_1 \leq \zeta, \\ 0, & \text{otherwise,} \end{cases} \tag{2.26}$$

where $\|\cdot\|_1$ is the l^1 norm in $\mathbb{R}^2$. In analogy with [10], we expect that when $\gamma \downarrow 0$, the typical profiles will be described by runs of $\eta^{\delta,\zeta} = 1$ followed by runs of $\eta^{\delta,\zeta} = -1$. It was proved in [10] that, for the ferromagnetic Kac model, the profiles make runs of $\eta^{\delta,\zeta} = 1$ on a scale which is of order $\exp(\Delta f/\gamma)$ where Δf is the activation energy of the Curie–Weiss model, that is, the difference between the value of the canonical free energy at its saddle point and at its minima. Roughly speaking, this means that on a scale $\exp(((1+\varepsilon)\Delta f/\gamma))$ the profiles are nonconstant if $\varepsilon > 0$ and are constant if $\varepsilon < 0$.

As we will see, the presence of the random magnetic field makes the profiles nonconstant on a much smaller scale. To be more precise, given $\tau \in \{-1, +1\}$, $l_1 \in \mathbb{Z}$, $l_2 \in \mathbb{Z}$ with $l_1 < l_2$, we define

$$\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau) = \{m^{\delta^*} : \eta^{\delta,\zeta}(l) = \tau, \forall\, l;\ l_1 \leq l \leq l_2\} \tag{2.27}$$

and

$$\mathscr{R}^{\delta,\zeta}(l_1, l_2) = \mathscr{R}^{\delta,\zeta}(l_1, l_2, +) \cup \mathscr{R}^{\delta,\zeta}(l_1, l_2, -), \tag{2.28}$$

that is, the set of profiles that between l_1 and l_2 are near the equilibrium m_β, Tm_β, respectively, for $\tau = \pm 1$, with tolerance ζ.

Given positive constants $\hat{c}$, $\tilde{c}$, $p > 1$, L_1, we denote by $N_\gamma = [(\tilde{c}/\hat{c})(\log(1/\gamma))^p(\log\log(1/\gamma))]$, where $[x]$ is the integer part of x, by $l_{\hat{c}}(\gamma) = \hat{c}/\gamma \log\log(1/\gamma)$ and by

$$\mathscr{R}^{\delta,\zeta}\left(L_1, \hat{c}, \frac{\tilde{c}(\log(1/\gamma))^p}{\gamma}\right) \equiv \bigcup_{k=-N_\gamma}^{N_\gamma} \mathscr{R}^{\delta,\zeta}(kl_{\hat{c}}(\gamma), L_1 + kl_{\hat{c}}(\gamma)). \tag{2.29}$$

That is, the set of profiles that in an interval of length $2(\tilde{c}(\log(1/\gamma))^p/\gamma)$, centered at the origin, have at least one interval of length L_1 that it is rigid. We have the following result.

THEOREM 2.3. *Given $\tilde{c} > 0$, $\beta > 1$, $p > 1$, $\rho > 0$, $c_0 > 0$, $\beta\theta$ small enough, for all $x > 0$, $\delta > \delta^* = c_0\gamma \log\log(1/\gamma)$, there exist an absolute constant $c > 0$ and a positive constant $\hat{c} = \hat{c}(\beta, \theta, x)$ such that if $\gamma = 2^{-n}$, $\mathbb{P}$-almost surely, for all but a finite number of indices n, if*

$$L_1 \geq \frac{(\log(1/\gamma))(\log\log(1/\gamma))^{2+\rho}}{\gamma}\left[\frac{c(x,\rho,\gamma)}{(\beta\theta)^2(m_{\beta,1}+m_{\beta,2})^2}\right], \tag{2.30}$$

where

$$c(x,\rho,\gamma) = \frac{2(4+x)^2}{1+\left(2+\frac{3\rho}{4}\right)\frac{\log\log\log(1/\gamma)}{\log\log(1/\gamma)}},$$

then

$$\mu_{\beta,\theta,\gamma}\left[\mathscr{R}^{\delta,\zeta}\left(L_1, \hat{c}, \frac{\tilde{c}(\log(1/\gamma))^p}{\gamma}\right)\right] \leq \exp(-\beta x\gamma^{-1}), \tag{2.31}$$

provided that for some function $g_2(1/\zeta)$, with $\lim_{\zeta\downarrow 0} g_2(1/\zeta) = \infty$, slowly varying at infinity,

$$\lim_{\zeta\downarrow 0}\sqrt{\zeta}\, g_2(1/\zeta) = 0$$

and

$$\zeta g_2(1/\zeta) < \beta\theta(m_{\beta,1}+m_{\beta,2})^2(c/(p+2)).$$

To make the previous theorem meaningful, we need a result in the opposite direction, that is, to prove that the system is rigid with the *same* tolerance ζ on a scale smaller than L_1. As we will see later, this will give a constraint from below on ζ. We introduce two different tolerance parameters that we call ζ_4 and ζ_1 and the corresponding δ_4 and δ_1. The parameter ζ_4 plays the role of ζ in the previous theorem.

Given $l_1 \in \mathbb{Z}$, $l_2 \in \mathbb{Z}$ with $l_1 < l_2$, $\delta_4 > 0$, $\zeta_4 > 0$, $\delta_1 > 0$, $\zeta_1 > 0$, $R_1 \in \mathbb{R}$, $x \in [l_1 + 2R_1 + 1, l_2 - 3R_1 - 1]$ and $\tau \in \{-1, +1\}$, we define a front starting at the equilibrium τ at the point x by

$$\begin{aligned}
&\mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1, l_2, \tau, x)\\
&= \{m^{\delta^*}: \forall\, l \in [l_1 + R_1, x], \eta^{\delta_4,\zeta_4}(l) = \tau = \eta^{\delta_1,\zeta_1}(x),\\
&\qquad \forall\, l \in [x + R_1 + 1, l_2 - R_1[, \eta^{\delta_4,\zeta_4}(l) = -\tau = \eta^{\delta_1,\zeta_1}(x + R_1)\}
\end{aligned} \tag{2.32}$$

and the set of fronts in all possible starting points,

$$\mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1, l_2, \tau) = \bigcup_{l_1+2R_1+1\leq x\leq l_2-3R_1-1} \mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1, l_2, \tau, x).$$

Let us note that we do not specify the configurations in a block of length R_1 at the beginning and at the end of the interval $[l_1, l_2]$. Moreover, we specify the front by a starting point x and by a final point $x + R_1$ where the other equilibrium is reached with a tolerance ζ_1. We do not specify what happens in the interval of length R_1 in between. This length $R_1 = R_1(\zeta_1, \delta_1)$ is the longest interval where the system can stay out of equilibrium with a tolerance ζ_1, a fact that will be proved in Corollary 5.2.

We denote the set of fronts that occur within $[l_1, l_2]$ by

$$\mathscr{V}^{\delta_1, \zeta_1, \delta_4, \zeta_4}(l_1, l_2) = \bigcup_{\tau \in \{+1, -1\}} \mathscr{V}^{\delta_1, \zeta_1, \delta_4, \zeta_4}(l_1, l_2, \tau). \tag{2.33}$$

Moreover, to shorten notation we set [see (2.28)]

$$\mathscr{R}^{\delta_4, \zeta_4}(l_1, l_2, R_1) \equiv \mathscr{R}^{\delta_4, \zeta_4}(l_1 + 2R_1, l_2 - 2R_1). \tag{2.34}$$

Let us note that on this set, since we have not specified what happens in the first two blocks of length R_1, we could have a configuration that looks like a front with a transition that occurs in these two first blocks and stays rigid after. These events are not in the set defined in (2.32).

We have the following result.

THEOREM 2.4. *Given $\beta > 1$, $\rho > 0$ and $\hat{c}$, there exists an ε_0 such that if $\beta\theta \le \varepsilon_0$, we can find $\gamma_0 > 0$, $c_0 > 0$ and constants $c_i = c_i(\beta, \theta)$ for $i = 1, 2, 3$, such that for all $\gamma \le \gamma_0$, for all $\zeta_4 > \zeta_1 > 0$, $\delta_4 > \delta_1 > \delta^* = c_0 \gamma \log\log(1/\gamma)$ that satisfy*

$$\delta_4 \zeta_4^3 \ge c_1 \left(\sqrt{\frac{1}{\log\log(1/\gamma)}} \vee \zeta_1 \right) \tag{2.35}$$

for $R_1 = c_2(\delta_1 \zeta_1^3)^{-1}$, for any interval $I = [l_1, l_2]$ such that $4R_1 \le |l_1 - l_2| \le (\hat{c}/\gamma \log\log(1/\gamma))$, there exists $\Omega_1 = \Omega_1(\beta, \theta, l_1, l_2, \gamma)$ such that $\mathbb{P}[\Omega_1] \ge 1 - \exp(-(\log\log(1/\gamma))(1 + 2\rho))$ and on Ω_1,

$$\mu_{\beta, \theta, \gamma}\left(\mathscr{R}^{\delta_4, \zeta_4}(l_1, l_2, R_1) \cup \mathscr{V}^{\delta_1, \zeta_1, \delta_4, \zeta_4}(l_1, l_2)\right) \ge 1 - \exp\left(-\frac{c_3 \delta_4 \zeta_4^3}{\gamma}\right) \tag{2.36}$$

in particular, if $\gamma = 2^{-n}$, $\mathbb{P}$-almost surely, (2.36) occurs for all but a finite number of indices n.

Roughly speaking, inside an interval of length $\hat{c}(\gamma \log\log(1/\gamma))^{-1}$ centered, say, at the origin, the typical profiles are rigid with a tolerance ζ_4 around one of the two equilibria or make only one transition between the two equilibria. Note also that we have allowed a fuzzy region of length $2R_1$ around the extremes of the intervals considered and also a region R_1 around the front. However, using Corollary 5.4, it can be proved that in a fuzzy zone there is at most one transition from one equilibrium to the other. Note that $R_1 = c_2(\delta_1 \zeta_1^3)^{-1}$, the length of the fuzzy zones, is very small with respect to $\hat{c}(\gamma \log\log(1/\gamma))^{-1}$. As will be proved in Section 5, this R_1 corresponds to the longest runs of $\eta^{\delta_1, \zeta_1} = 0$ which is typical with respect to the Gibbs measure.

Note that it is possible to take ζ_4 in Theorem 2.4 and ζ in Theorem 2.3 equal.

Theorem 2.4 suggests that a good notion of rigidity is not to fix all of those intervals where the profiles are at equilibrium with a given tolerance but to allow those intervals of rigidity to have a fuzzy zone of length $2R_1$ at the extremes. To describe the typical profiles, we combine the results of the two previous theorems. We can expect to give an upper and lower bound on the distance between two fronts for the typical profiles in an interval of length, say, $\gamma^{-1}(\log 1/\gamma)^p$ for some $p > 1$. Namely, this is the scale where we know from Theorem 2.3 that such fronts exist. This corresponds to give an upper and lower bound on the number of transitions from one equilibrium to the other in such an interval. To be more precise, we need some more definitions. Given an interval $\mathcal{I} = [-j_1, j_1]$, centered at the origin, and positives integers k and L, we define, for $l_1, l_2 \in \mathbb{Z}$, $\tau \in \{-1, +1\}$, $\mathcal{R}^{\delta_4, \zeta_4}(l_1, l_2, R_1, \tau) \equiv \mathcal{R}^{\delta_4, \zeta_4}(l_1 + 2R_1, l_2 - 2R_1, \tau)$, $\mathcal{T}^{\delta_4, \zeta_4}(L, 1, \tau, \mathcal{I}) \equiv \mathcal{R}^{\delta_4, \zeta_4}(-j_1, j_1, R_1, \tau)$ and

$$\mathcal{T}^{\delta_4, \zeta_4}(L, k, \tau, \mathcal{I}) = \bigcup_{l_1 = -j_1}^{j_1} \bigcup_{\substack{l_2 > l_1 \\ l_2 - l_1 > L}}^{j_1} \cdots \bigcup_{\substack{l_k > l_{k-1} \\ l_k - l_{k-1} > L}}^{j_1} \bigcap_{k_1 = 1}^{k} \mathcal{R}^{\delta_4, \zeta_4}\Big(l_{k_1}, l_{k_1+1}, R_1, (-1)^{k_1+1}\tau\Big); \tag{2.37}$$

that is, the profiles in $\mathcal{T}^{\delta_4, \zeta_4}(L, k, \tau, \mathcal{I})$ change exactly $k - 1$ times equilibrium, starting from τ somewhere within $[-j_1, -j_1 + 2R_1]$ and remaining in a given equilibrium for a length at least L. We define also

$$\mathcal{T}^{\delta_4, \zeta_4}(L, \le k, \tau, \mathcal{I}) = \bigcup_{k_2 = 1}^{k} \mathcal{T}^{\delta_4, \zeta_4}(L, k_2, \tau, \mathcal{I}) \tag{2.38}$$

and $\mathcal{T}^{\delta_4, \zeta_4}(L, \le k, \mathcal{I}) = \mathcal{T}^{\delta_4, \zeta_4}(L, \le k, +, \mathcal{I}) \cup \mathcal{T}^{\delta_4, \zeta_4}(L, \le k, -, \mathcal{I})$. The profiles in $\mathcal{T}^{\delta_4, \zeta_4}(L, \le k, \mathcal{I})$ change equilibrium at most $k - 1$ times, starting from one equilibrium somewhere within $[-j_1, -j_1 + 2R_1]$ and remaining in a given equilibrium for a length at least L.

THEOREM 2.5. *Given $\beta > 1$ and $\rho > 0$, there exists $\varepsilon_0 > 0$ such that for all $\beta\theta \le \varepsilon_0$, we can find $p = p(\beta\theta) > 1$, $\bar{\zeta}_4(\beta\theta) > 0$, $\gamma_0 > 0$, $c_0 > 0$, $\hat{c} > 0$ and constants $c_i = c_i(\beta, \theta)$ for $i = 1, 2, 3$, such that for all $\gamma \le \gamma_0$, for all $\bar{\zeta}_4(\beta\theta) \ge \zeta_4 > \zeta_1 > 0$, $\delta_4 > \delta_1 > \delta^* = c_0 \gamma \log\log 1/\gamma$ that satisfy* (2.35), *L_1 that satisfies* (2.30) *and for $R_1 = c_2(\delta_1 \zeta_1^3)^{-1}$, for all given interval $\mathcal{I}$ of length $\tilde{c}(\log(1/\gamma))^p \gamma^{-1}$, for some positive constant $\tilde{c}$, if $\gamma = 2^{-n}$, $\mathbb{P}$-almost surely, for all but a finite number of indices n,*

$$\begin{aligned} \mu_{\beta, \theta, \gamma}\Bigg(&\mathcal{T}^{\delta_4, \zeta_4}\Bigg(l_{\hat{c}}(\gamma), \le \frac{\tilde{c}}{\hat{c}}\left(\log \frac{1}{\gamma}\right)^p \log\log \frac{1}{\gamma}, \mathcal{I}\Bigg) \setminus \\ &\qquad \mathcal{T}^{\delta_4, \zeta_4}\Bigg(L_1, \le \frac{\tilde{c}(\log(1/\gamma))^{p-1}}{\hat{c}(\log\log(1/\gamma))^{2+\rho}}, \mathcal{I}\Bigg)\Bigg) \\ &\ge 1 - \exp\left(-\frac{c_3 \delta_4 \zeta_4^3}{\gamma}\right). \end{aligned} \tag{2.39}$$

Our estimates give the scaling relation $p(\beta\theta) = \varepsilon_0^2/(\beta\theta)^2$. Following a typical profile starting from the left end of the interval $\mathscr{I}$, we meet at least one transition, within a scale $L_1 \approx (1/\gamma)[(\log 1/\gamma)(\log\log 1/\gamma)^{2+\rho}]$, then after this transition, we are near an equilibrium on a scale which is at least $(1/\gamma)[\log\log 1/\gamma]^{-1}$ and at most L_1, then we meet another transition within a scale L_1 and so on. This implies that the number of oscillations between the equilibria in the interval $\mathscr{I}$ is bounded from above by $(\log 1/\gamma)^p \log\log 1/\gamma$ and from below by $(\log 1/\gamma)^{p-1}(\log\log 1/\gamma)^{-(2+\rho)}$.

3. Analysis of the block–spin representation. In this section we perform the block–spin transformation on the scale δ^* mentioned in the previous section and we make a rather precise analysis of the stochastic contribution in order to prove our theorems.

Given a macroscopic interval $I \equiv [i^-, i^+[\subset \mathbb{R}$ with $i^\pm \in \mathbb{Z}$, we denote by $\mathscr{C}_{\delta^*}(I)$ the set of centers of blocks of length δ^* that we get making a partition of I into such blocks. Note that we are making a little abuse of notation since a similar quantity was defined for a microscopic interval [see after (2.14)] and there the partition was done into blocks of length $\delta^*(\gamma)^{-1}$. However, we consider the two sets equivalent. In particular, we identify $\mathscr{M}_{\delta^*}(I)$ with $\mathscr{M}_{\delta^*}(\gamma^{-1}I)$. Let us denote by $\Sigma_I^{\delta^*}$ the σ-algebra of $\mathscr{S}$ generated by the variables $(m^{\delta^*}(x,\sigma))_{x\in\mathscr{C}_{\delta^*}(I)}$ where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$ For such an interval I we denote by $\partial^+ I \equiv \{x\in\mathbb{R},\ i^+ \le x < i^+ + 1\}$ and by $\partial^- I \equiv \{x \in \mathbb{R},\ i^- - 1 \le x < i^-\}$ the two macroscopic intervals of length 1 that are on the right and on the left of I. We call $\partial I = \partial^+ I \cup \partial^- I$.

If F^{δ^*} is a $\Sigma_I^{\delta^*}$-measurable bounded function, we define the conditional expectation of F^{δ^*}, given the σ-algebra $\Sigma_{\partial I}^{\delta^*}$, as the real $\Sigma_{\partial I}^{\delta^*}$-measurable function that associates to $m^{\delta^*}(\partial I) \equiv \{m^{\delta^*}(x),\ x\in\mathscr{C}_{\delta^*}(\partial I)\}$ the value

$$
\begin{aligned}
&\mu_{\beta,\theta,\gamma}\big(F^{\delta^*} \mid \Sigma_{\partial I}^{\delta^*}\big)\big(m^{\delta^*}(\partial I)\big)\\
&\quad = \frac{1}{Z_{\beta,\gamma,\theta,I}\big(m^{\delta^*}(\partial I)\big)} \sum_{\sigma_{\gamma^{-1}I}\in\mathscr{S}_{\gamma^{-1}I}} F^{\delta^*}(\sigma_{\gamma^{-1}I})\\
&\qquad \times \exp\Big(-\beta\Big[H(\sigma_{\gamma^{-1}I}) + W\big(\sigma_{\gamma^{-1}I} \mid m^{\delta^*}(\partial I)\big)\Big]\Big),
\end{aligned}
\tag{3.1}
$$

where

$$
W\big(\sigma_I \mid m^{\delta^*}(\partial I)\big) \equiv \frac{\delta^*}{\gamma} \sum_{i\in\gamma^{-1}I} \sum_{x\in\mathscr{C}_{\delta^*}(\partial I)} J_\gamma(i - \delta^*\gamma^{-1}x)\sigma_i \tilde{m}^{\delta^*}(x)
\tag{3.2}
$$

with $\tilde{m}^{\delta^*}(x) = (m_1^{\delta^*}(x) + m_2^{\delta^*}(x))/2$ and $Z_{\beta,\gamma,\theta,I}(m^{\delta^*}(\partial I))$ is the normalization factor that gives $\mu_{\beta,\theta,\gamma}(1 \mid \Sigma_{\partial I}^{\delta^*}) = 1$.

Given $(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*})$ in $\mathscr{M}_{\delta^*}(I\cup\partial^+ I\cup\partial^- I)$ let us denote by

$$
E\big(m_I^{\delta^*}\big) \equiv -\frac{\delta^*}{2} \sum_{(x,y)\in\mathscr{C}_{\delta^*}^2(I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y)
\tag{3.3}
$$

and

$$E\big(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}\big) \equiv -\delta^* \sum_{x \in \mathscr{C}_{\delta^*}(I)} \sum_{y \in \mathscr{C}_{\delta^*}(\partial^\pm I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y). \tag{3.4}$$

On the set $M^{\delta^*}(m^{\delta^*}(I)) \equiv \{\sigma \in \gamma^{-1}I\colon m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\ \forall\, x \in \mathscr{C}_{\delta^*}(I)\}$, we have

$$\sup_{\sigma_{\gamma^{-1}I} \in M^{\delta^*}(m^{\delta^*}(I))} \left| H(\sigma_{\gamma^{-1}I}) + \theta \sum_{i \in \gamma^{-1}I} h_i \sigma_i - \frac{1}{\gamma} E\big(m_I^{\delta^*}\big)\right| \le \sigma^* \gamma^{-1}|I|; \tag{3.5}$$

here $|I|$ is the length of the macroscopic interval I. Moreover, we have also

$$\sup_{\sigma_{\gamma^{-1}I} \in M^{\delta^*}(m^{\delta^*}(I))} \left| W\big(\sigma_I \mid m^{\delta^*}(\partial^\pm I)\big) - \frac{1}{\gamma} E\big(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}\big)\right| \le \delta^* \gamma^{-1}. \tag{3.6}$$

The two estimates (3.5) and (3.6) follow from the fact that $|\mathbb{1}_{\{\gamma|i-j| \le 1/2\}} - \mathbb{1}_{\{\delta^*|x-y| \le 1/2\}}| \le 3\,\mathbb{1}_{\{-\delta^* + 1/2 \le \delta^*|x-y| \le \delta^* + 1/2\}}$ and an easy computation. Therefore, using (2.13), we can write

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\big(F^{\delta^*} \mid \Sigma_{\partial I}\big)\big(m^{\delta^*}(\partial I)\big)\\
&\quad= \frac{\exp\big((\pm\delta^*\gamma^{-1}|I|)\big)}{Z_{\beta,\theta,\gamma,I}\big(m^{\delta^*}(\partial I)\big)}\\
&\qquad\times \sum_{m^{\delta^*}(I) \in \mathscr{M}_{\delta^*}(I)} F^{\delta^*}(m^{\delta^*}) \exp\Bigg(-\frac{\beta}{\gamma}\Bigg(E\big(m_I^{\delta^*}\big) + E\big(m_I^{\delta^*}, m_{\partial I}^{\delta^*}\big)\\
&\qquad\qquad - \frac{\theta\delta^*}{2} \sum_{x \in \mathscr{C}_{\delta^*}(I)} \big(m_1^{\delta^*}(x) - m_2^{\delta^*}(x)\big)\Bigg)\Bigg)\\
&\qquad\times \sum_{\sigma_{\gamma^{-1}I}} \mathbb{1}_{\{m^{\delta^*}(x,\sigma) = m^{\delta^*}(x),\, \forall\, x \in \mathscr{C}_{\delta^*}(I)\}} \prod_{x \in \mathscr{C}_{\delta^*}(I)} \exp\Bigg(2\beta\theta\lambda(x) \sum_{i \in D^{\lambda}(x)} \sigma_i\Bigg),
\end{aligned} \tag{3.7}$$

where this equality has to be interpreted as an upper bound for $\pm = 1$ and a lower bound for $\pm = -1$ and the first sum is over $m^{\delta^*}(x)_{x \in \mathscr{C}_{\delta^*}(I)} \in \mathscr{M}_{\delta^*}(I)$.

Note that the random terms appear only in the last product $\prod_{x \in \mathscr{C}_{\delta^*}(I)}$ and that the last sum in (3.7) factors into a product over the intervals of length $\delta^*\gamma^{-1}$ indexed by $\mathscr{C}_{\delta^*}(I)$.

For all $x \in \mathscr{C}_{\delta^*}(I)$, we introduce on $\{-1,+1\}^{\delta^*\gamma^{-1}} = \mathscr{S}_{\delta^*\gamma^{-1}}$ the measure denoted the *canonical* measure in physics literature,

$$\mathbb{E}^{\delta^*}_{x, m^{\delta^*}(x)}(\varphi) = \frac{\sum_{\sigma \in \mathscr{S}_{\delta^*\gamma^{-1}}} \varphi(\sigma)\, \mathbb{1}_{\{m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\}}}{\sum_{\sigma \in \mathscr{S}_{\delta^*\gamma^{-1}}} \mathbb{1}_{\{m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\}}}. \tag{3.8}$$

The denominator in (3.8) is

$$\binom{|B^+|}{\frac{1+m_1^{\delta^*}(x)}{2}|B^+|}\binom{|B^-|}{\frac{1+m_2^{\delta^*}(x)}{2}|B^-|}, \tag{3.9}$$

where $|B^{\pm}| = |B| = \delta^*(2\gamma)^{-1}$. We set

$$\begin{aligned}\hat{\mathscr{F}}\big(m_I^{\delta^*}, m_{\partial I}^{\delta^*}\big) &= E\big(m_I^{\delta^*}\big) + E\big(m_I^{\delta^*}, m_{\partial I}^{\delta^*}\big)\\ &\quad - \frac{\theta\delta^*}{2}\sum_{x\in\mathscr{C}_{\delta^*}(I)}\big(m_I^{\delta^*}(x) - m_2^{\delta^*}(x)\big)\\ &\quad - \delta^*\sum_{x\in\mathscr{C}_{\delta^*}(I)}\frac{\gamma}{\beta\delta^*}\log\binom{|B^+|}{\frac{1+m_1^{\delta^*}(x)}{2}|B^+|}\binom{|B^-|}{\frac{1+m_2^{\delta^*}(x)}{2}|B^-|}\end{aligned} \tag{3.10}$$

We introduce the moment generating function

$$L_{x,m^{\delta^*}(x)}^{\delta^*}\big(\lambda(x)\beta\theta, D^{\lambda}(x)\big) \equiv \mathbb{E}_{x,m^{\delta^*}(x)}^{\delta^*}\Bigg(\exp\Bigg(2\beta\theta\lambda(x)\sum_{i\in D^{\lambda}(x)}\sigma_i\Bigg)\Bigg) \tag{3.11}$$

and the cumulant generating function

$$\mathscr{G}_{x,m^{\delta^*}(x)}\big(\lambda(x)\big) \equiv -\log L_{x,m^{\delta^*}(x)}^{\delta^*}\big(\lambda(x)\beta\theta, D^{\lambda}(x)\big); \tag{3.12}$$

then (3.7) becomes

$$\begin{aligned}&\mu_{\beta,\theta,\gamma}\big(F^{\delta^*} \mid \Sigma_{\partial I}\big)\big(m^{\delta^*}(\partial I)\big)\\ &\quad = \frac{\exp\big((\pm\delta^*\gamma^{-1}|I|)\big)}{Z_{\beta,\theta,\gamma,I}\big(m^{\delta^*}(\partial I)\big)}\sum_{m^{\delta^*}(I)\in\mathscr{M}_{\delta^*}(I)}F^{\delta^*}(m^{\delta^*})\\ &\qquad\times\exp\bigg(-\frac{1}{\gamma}\Big\{\beta\hat{\mathscr{F}}\big(m_1^{\delta^*}, m_{\partial I}^{\delta^*}\big) + \gamma\mathscr{G}\big(m_I^{\delta^*}\big)\Big\}\bigg),\end{aligned} \tag{3.13}$$

where

$$\mathscr{G}\big(m_I^{\delta^*}\big) \equiv \sum_{x\in\mathscr{C}_{\delta^*}(I)}\mathscr{G}_{x,m^{\delta^*}(x)}\big(\lambda(x)\big); \tag{3.14}$$

that is, up to the error terms $\exp((\pm c\delta^*\gamma^{-1}|I|))$, we have been able to describe our system in terms of the block–spin variables, giving a rather explicit form to the deterministic and the stochastic part.

Note that the stochastic dependence is given only by the fluctuations of the magnetic fields on each block, $\lambda(x) = \operatorname{sgn}(\sum_{i\in A(x)} h_i)$ and by $|D^{\lambda}(x)| = (\lambda(x)/2)\sum_{i\in A(x)} h_i$.

Coming back to (3.11), if $\lambda(x) = +1$, then $D^{\lambda}(x)$ is a subset of B^- and therefore the sum over the sites in B^+ factors out and it is cancelled by the first combinatorial factor in (3.9) [if $\lambda(x) = -1$, it is the second term in (3.9)].

In particular, this means that if $\lambda(x) = +1$, we have

$$L^{\delta^*}_{x, m^{\delta^*}(x)}(\beta\theta, D^+(x)) = \mathbb{E}^{\delta^*}_{x, m_2^{\delta^*}(x)}\left(\exp\left(2\beta\theta \sum_{i \in D^+(x)} \sigma_i\right)\right), \tag{3.15}$$

which depends only on the second coordinate of $m^{\delta^*}(x)$; while if $\lambda(x) = -1$,

$$L^{\delta^*}_{x, m^{\delta^*}(x)}(\beta\theta, D^-(x)) = \mathbb{E}^{\delta^*}_{x, m_1^{\delta^*}(x)}\left(\exp\left(-2\beta\theta \sum_{i \in D^-(x)} \sigma_i\right)\right), \tag{3.16}$$

which depends only on the first coordinate of $m^{\delta^*}(x)$.

We will need in the next section very precise estimates when γ small, of $\mathscr{G}_{x, m^{\delta^*}(x)}(\lambda(x))$ [see (3.12)], which is the cumulant generating function of a hypergeometric. However, from the beginning we know that $\beta\theta$ is small and to simplify the estimates, we will take $\beta\theta$ as small as we need. In fact, what we need is a precise dependence in terms of the volume of $D(x)$ and the result we need has to be valid for all the possible values of $m^{\delta^*}(x)$, even those ones very close to 1. Moreover, we cannot impose any conditions on the size of D. We use large deviation estimates in the strong form with a good control of the polynomial prefactors. We have to consider all the possible behaviors of the fluctuations of a hypergeometric. It is well known in classical probability that there are three possible regimes, namely a Gaussian one, a binomial and a Poissonian one. Classical results are usually given in terms of convergence in distribution. Since we are interested in controlling the error terms, we need some extra work. We give a short proof of the estimates we need. The statements of them are given in Proposition 3.4, for the Gaussian regime and in Proposition 3.5 for the binomial and Poissonian regimes. Since it could be of independent interest, we set the result in a general form. To do it, we set $m_i^{\delta^*}(x) = m$, $D(x) = D$, $2\lambda(x)\beta\theta = z$. We keep in mind that $m \in \{-1 + 2/B, -1 + 4/B, \dots, 1 - 2/B, 1\}$. Denoting $\mathbb{E}_{\sigma_B}$ the normalized symmetric Bernoulli measure on $\{-1, +1\}^{|B|}$, we want to estimate

$$L_m(z, D, B) = \frac{\mathbb{E}_{\sigma_B}\left[\exp\left(z \sum_{i \in D} \sigma_i\right) \mathbb{1}_{\{m_B(\sigma) = m\}}\right]}{\mathbb{E}_{\sigma_B}\left[\mathbb{1}_{\{m_B(\sigma) = m\}}\right]}, \tag{3.17}$$

where D is a subset of B. With a little abuse of notation we will denote $|B| = B$ and $|D| = D$ when no confusion is possible. Moreover, we set $\alpha = |D|/|B|$. There are, roughly speaking, two regimes to consider, depending on whether or not $|m|$ is bounded away from 1. To be able to separate these two possible cases, we introduce a real function $g(x)$ such that $\lim_{x \uparrow \infty} g(x) = \infty$ but $\lim_{x \uparrow \infty} g(x)/x = 0$. Here we will not specify more than this, since the choice of $g(x)$ will be done at the end of the next chapter for reasons that will become clear at that moment. The first case we consider is when $|m| \le 1 - (g(B)/B)$. It is the Gaussian regime. We introduce $\mathbb{E}_\nu$ to be the grand canonical measure with chemical potential ν, defined on $\{-1, +1\}^{|B|}$,

$$\mathbb{E}_{\nu}(\varphi) = \frac{\mathbb{E}_{\sigma_B}\left[\varphi(\sigma)\exp\left(\nu \sum_{i\in B} \sigma_i\right)\right]}{\mathbb{E}_{\sigma_B}\left[\exp\left(\nu \sum_{i\in B} \sigma_i\right)\right]}. \tag{3.18}$$

Note that in classical probability theory and in large deviation theory $(\sigma_i)_{i\in B}$ under the law $\mathbb{E}_\nu$ are called associated random variables; see [12]. Following H. T. Yau [35], we introduce two different chemical potentials and we write the following identity: for all $\nu_1, \nu_2 \in \mathbb{R}$,

$$\begin{aligned} L_m(z, D, B) &= \frac{\mathbb{E}_{\nu_2}\left[\exp\left(z\sum_{i\in D}\sigma_i\right)\mathbb{1}_{\{m_B(\sigma)=m\}}\right]}{\mathbb{E}_{\nu_2}\left[\exp\left(z\sum_{i\in D}\sigma_i\right)\right]} \frac{1}{\mathbb{E}_{\nu_1}\left[\mathbb{1}_{\{m_B(\sigma)=m\}}\right]} \\ &\quad\times \exp(\{m(\nu_1-\nu_2)|B|\}) \frac{(\cosh(\nu_2))^{|B\setminus D|}(\cosh(\nu_2+z))^{|D|}}{(\cosh(\nu_1))^{|B|}}. \end{aligned} \tag{3.19}$$

We choose $\nu_1 \equiv \nu_1(m)$ such that $m = \tanh \nu_1$, in which case the mean value of $m_B(\sigma)$ under $\mathbb{E}_{\nu_1}$ is m. Then $\nu_2 \equiv \nu_2(m, \alpha, z)$ is chosen such that

$$m = \alpha \tanh(\nu_2 + z) + (1-\alpha)\tanh \nu_2, \tag{3.20}$$

in which case

$$m = \frac{\mathbb{E}_{\nu_2}\left[m_B(\sigma)\exp\left(z\sum_{i\in D}\sigma_i\right)\right]}{\mathbb{E}_{\nu_2}\left[\exp\left(z\sum_{i\in D}\sigma_i\right)\right]}.$$

Then writing simply $\{m_B(\sigma) = m\} = \{(|B|)^{-1/2}\sum_{i\in B}(\sigma_i - m) = 0\}$, the two first ratios in (3.19) can be estimated by a *local* central limit theorem (LCLT), exactly as in [35]. Therefore, denoting

$$\Psi_{z,\alpha,m} \equiv \frac{\mathbb{E}_{\nu_2}\left[\exp\left(z\sum_{i\in D}\sigma_i\right)\mathbb{1}_{\{(|B|)^{-1/2}\sum_{i\in B}(\sigma_i-m)=0\}}\right]}{\mathbb{E}_{\nu_2}\left[\exp\left(z\sum_{i\in D}\sigma_i\right)\right]} \tag{3.21}$$

and

$$\begin{aligned} &\exp(zD[m + \hat{\varphi}(m, z, \alpha)]) \\ &\quad\equiv \exp(\{m(\nu_1-\nu_2)|B|\}) \frac{(\cosh(\nu_2))^{|B\setminus D|}(\cosh(\nu_2+z))^{|D|}}{(\cosh(\nu_1))^{|B|}}, \end{aligned} \tag{3.22}$$

we have

$$L_m(z, D, B) = \frac{\Psi_{z,\alpha,m}}{\Psi_{0,0,m}} \exp(zD[m + \hat{\varphi}(m, z, \alpha)]). \tag{3.23}$$

The result in the Gaussian regime is the following.

PROPOSITION 3.1. *There exist an* $\varepsilon > 0$ *and positive constants* c_1, c_2 *such that if* $|z| < \varepsilon$, *for all* $m \in \{-1, -1 + 2/B, -1 + 4/B, \dots, 1 - 2/B, 1\}$ *such that* $|m| < 1$, *then*

$$\log L_m(z, D, B) = z|D|[m + \hat{\varphi}(m, z, \alpha)] + \log \frac{\Psi_{z,\alpha,m}}{\Psi_{0,0,m}}, \tag{3.24}$$

with $\sup_{m:\,|m|<1} |\hat{\varphi}(m, z, \alpha)| \le |z|(1 + c_1|z|)$. *Moreover, for all* $g(n)$ *such that* $\lim_{n\uparrow\infty} g(n) = \infty$ *but* $\lim_{n\uparrow\infty} g(n)/n = 0$, *for all* m *such that* $|m| \le 1 - g(|B|)/|B|$,

$$\left| \frac{\Psi_{z,\alpha,m}}{\Psi_{0,0,m}} - 1 \right| \le c_2 z^2 + \frac{25}{g(|B|)}. \tag{3.25}$$

In the Poissonian and binomial regime we have the following.

PROPOSITION 3.2. *There exist an* $\varepsilon > 0$ *and a positive constant* c_1 *such that if* $0 < |z| < \varepsilon$, *for all* $g(n)$ *such that* $\lim_{n\uparrow\infty} g(n) = \infty$ *but* $\lim_{n\uparrow\infty} g^2(n)/n = 0$, *for all* $m \in \{-1, -1 + 2/B, -1 + 4/B, \dots, 1 - 2/B, 1\}$ *such that* $|m| \ge 1 - g(|B|)/|B|$, *we have*

$$\log L_m(z, D, B) = z|D|[m + \hat{\varphi}_1(m, z, \alpha)] \tag{3.26}$$

with

$$\sup_{m:\,|m|\ge 1 - g(|B|)/|B|} |\hat{\varphi}_1(m, z, \alpha)| \le c_1 \left(\frac{g(|B|)}{|B|}|z| + \frac{g^2(|B|)}{|z|\,|B|} \right). \tag{3.27}$$

The remaining part of this section is devoted to the proofs of the last two propositions and is quite technical. At first reading, this part could be skipped. However, some of the estimates below will be used in a crucial way in the next section.

We start proving Proposition 3.1.

First we give a lower bound for the variance of $m_B(\sigma)$ under $\mathbb{E}_{\nu_2}$.

LEMMA 3.3. *Let* ν_2 *be a solution of* (3.20) *and* σ_z *given by*

$$\sigma_z^2 = \alpha \frac{1}{\cosh^2(\nu_2 + z)} + (1 - \alpha) \frac{1}{\cosh^2(\nu_2)}, \tag{3.28}$$

then for all m such that $|m| < 1$, *for all* $\beta > 1$, *for all z such that* $|z| < \varepsilon$, *for some* $\varepsilon > 0$ *small enough*, *for all* $\alpha \in [0, 1]$,

$$\sigma_z^2 > (1 - m^2)(1 - cz^2) \tag{3.29}$$

for some positive constant c.

PROOF. We have $\sigma_z^2 = 1 - m^2 - \alpha(1-\alpha)(\tanh(\nu_2 + z) - \tanh(\nu_2))^2$. Now calling $\nu_2 - \nu_1 \equiv \Delta$, using m $= \tanh \nu_1$, it is easy to see that

$$\tanh(\nu_2 + z) - \tanh(\nu_2) = \frac{(1 - m^2)(\tanh(z + \Delta) - \tanh(\Delta))}{(1 + m\tanh(z + \Delta))(1 + m\tanh(\Delta))}. \tag{3.30}$$

On the other hand, since $\nu_2 = \nu_2(z)$ and $\nu_2(0) = \nu_1$ [see (3.20)],

$$\frac{d\nu_2}{dz} = \frac{-z\alpha}{\sigma_z^2 \cosh^2(\nu_2 + z)}. \tag{3.31}$$

After an easy computation, we get

$$\begin{aligned} \nu_2 - \nu_1 &= \int_0^z \frac{d\nu_2}{dz}\, dz' \\ &= -\int_0^z \frac{\alpha \cosh^2(\nu_2(z'))}{\alpha\cosh^2(\nu_2(z')) + (1-\alpha)\cosh^2(z + \nu_2(z'))}\, dz' \end{aligned} \tag{3.32}$$

from which it is easy to get

$$|\nu_2 - \nu_1| \le |z|. \tag{3.33}$$

Therefore, we have $|\tanh(z + \Delta) - \tanh(\Delta)| \le [1 - \tanh^2(z)]^{-1}|\tanh(z)|$ and $|1 + m\tanh(z + \Delta)| \ge 1 - |\tanh(2z)|$. Collecting, we get the lemma. □

PROPOSITION 3.4 (LCLT). *There exists an* $\varepsilon > 0$ *such that if* $|z| < \varepsilon$, *for all given m such that* $|m| < 1$, $\alpha = |D|/|B|$ *and* σ_z *given by* (3.28),

$$\Psi_{z,\alpha,m} = \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\left(1 \pm \frac{3}{|B|\sigma_z^2}\right) \tag{3.34}$$

provided $|B|$ *is large enough. Moreover, for all m, such that* $|m| \le 1 - g(|B|)/|B|$ *for some g that satisfies* $\lim_{x\uparrow\infty} g(x) = \infty$ *but* $\lim_{x\uparrow\infty}(g(x)/x) = 0$, *we have*

$$\Psi_{z,\alpha,m} = \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\left(1 \pm \frac{c}{g(|B|)}\right) \tag{3.35}$$

for some positive constant c.

REMARK. The only reason to prove this proposition is to get in the error term the explicit dependence on α through σ_z and the $g(|B|)$ dependence in (3.35). The proof is rather standard and follows the usual strategy to get asymptotic expansions in LCLT. We have been influenced by [35]; see also Renyi's book [26], pages 460–466.

PROOF OF PROPOSITION 3.4. We start with the following simple equation,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,dk = \frac{\sin \pi x}{\pi x} = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0,\ x \in \mathbb{Z}, \end{cases} \tag{3.36}$$

which implies after some algebra,

$$\Psi_{z,\alpha,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(ikm|B|)\Phi(z,\alpha,k)\,dk, \tag{3.37}$$

where

$$\Phi(z,\alpha,k) \equiv \left[\frac{\cosh(z+\nu_2+ik)}{\cosh(z+\nu_2)}\right]^{|D|}\left[\frac{\cosh(\nu_2+ik)}{\cosh(\nu_2)}\right]^{|B\setminus D|}. \tag{3.38}$$

Introducing the variable $e^{x}(2\cosh x)^{-1}$, using $1-y \le e^{-y}, \forall\, y \in \mathbb{R}$ and $1-\cos k \ge k^2/2$, $\forall\, k \in \mathbb{R}$, it is easy to check that for all $(x,k) \in \mathbb{R}^2$,

$$\left|\frac{\cosh(x+ik)}{\cosh(x)}\right| \le \exp\left\{-\frac{k^2}{2\cosh^2(x)}\right\}. \tag{3.39}$$

Then we easily get

$$|\Phi(z,\alpha,k)| \le \exp\left(-|B|\frac{k^2}{2}\sigma_z^2\right). \tag{3.40}$$

If we denote

$$\mathscr{E}_\rho(m) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{1}_{\{\rho<|k|\le\pi\}}\Phi(z,\alpha,k)\exp(ikm|B|)\,dk, \tag{3.41}$$

then, after some standard tail Gaussian estimates, we get, for all $\rho > 0$,

$$|\mathscr{E}_\rho(m)| \le \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\left(\frac{8}{3\sqrt{2\pi}\left(1+\rho\sigma_z\sqrt{|B|}\right)}\exp\left(-\frac{\rho^2}{2}\sigma_z^2|B|\right)\right). \tag{3.42}$$

Equation (3.42) suggests taking $\rho = (\sigma_z\sqrt{|B|})^{-1}f(|B|)$ for some $f(|B|)$ that diverges with $|B|$ but it is such that

$$\lim_{|B|\uparrow\infty}\frac{f(|B|)}{\sqrt{g(|B|)}} = 0 \tag{3.43}$$

and we get

$$|\mathscr{E}_\rho| \le \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\left(\frac{8}{3\sqrt{2\pi}(1+f(|B|))}\exp\left(-\frac{f^2(|B|)}{2}\right)\right). \tag{3.44}$$

It remains to estimate

$$\Psi_{z,\alpha,m}(\rho) \equiv \frac{1}{2\pi}\int_{-\rho}^{\rho}\exp(ikm|B|)\Phi(z,\alpha,k)\,dk. \tag{3.45}$$

Since we restricted the domain of $|k| \le \rho$ and ρ goes to zero when $|B|\uparrow\infty$, using the Taylor formula with an integral rest for the term in k^4, cancelling the linear term in k, we get

$$\Psi_{z,\alpha,m}(\rho) = \frac{1}{2\pi}\int_{-\rho}^{\rho} \exp\left(|B|\left\{-\frac{k^2\sigma_z^2}{2} - \frac{ik^3}{3}\mathscr{R}_\alpha(3) - \frac{k^4}{3}\mathscr{R}_\alpha(4,k)\right\}\right) dk \tag{3.46}$$

with $|\mathscr{R}_\alpha(4, k)| \le (1 + 16\rho e^{\rho})\sigma_z^2$ and $|\mathscr{R}_\alpha(3)| \le \sigma_z^2$. Therefore, if $\lim_{|B|\uparrow\infty}(f^3(|B|)/\sqrt{g(|B|)}) = 0$, f satisfies (3.43) as well and the terms of order k^3 and k^4 in the exponent in (3.46) go to zero.

Therefore, using $|e^{ix} - 1 - ix| \le x^2/2$ for all $x \in \mathbb{R}$ for the term of order three in k and $|e^x - 1| \le |x|e^{|x|}$ for the term of order 4 in (3.46), we get, after Gaussian estimates,

$$\begin{aligned} &\left|\Psi_{z,\alpha,m}(\rho) - \frac{1}{2\pi}\int_{-\rho}^{\rho}\exp\left(\left\{-\frac{k^2\sigma_z^2|B|}{2}\right\}\right)\left[1 - \frac{ik^3|B|}{3}\mathscr{R}_\alpha(3)\right] dk\right| \\ &\quad \le \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\left(\frac{(1 + 32\rho e^{\rho})}{|B|\sigma_z^2}\right). \end{aligned} \tag{3.47}$$

The point is that the term in k^3 in the left-hand side of (3.47) cancel by symmetry. It is now easy to get (3.34) by taking, for example, $f(|B|) = |B|^s$ with s as small as we want.

To get (3.35), we just take $f(|B|) = \sqrt{2\log g(|B|)}$ and we have

$$|\mathscr{E}_\rho| \le \frac{1}{\sqrt{2\pi|B|}\,\sigma_z}\,\frac{8}{3\sqrt{2\pi}\,g(|B|)\left(1 + \sqrt{2\log g(|B|)}\right)} \tag{3.48}$$

and (3.35) is immediate. □

We come back to (3.23) and we estimate the second factor. We have from (3.22),

$$\phi(m,z,\alpha) = |B|\left((\nu_1 - \nu_2)m + \alpha\log\frac{\cosh(\nu_2 + z)}{\cosh(\nu_1)} + (1-\alpha)\log\frac{\cosh(\nu_2)}{\cosh(\nu_1)}\right). \tag{3.49}$$

From (3.17) it is evident that $|\phi(m, \alpha)|/|D|$ is bounded from above by $2\beta\theta$. Therefore there are some important cancellations that occur in (3.49) in order to make it proportional to $|D|$ instead of $|B|$ as it looks at first sight. To achieve this we first prove the lemma.

LEMMA 3.5. *Let* ν_2 *be a solution of* (3.20), ν_1 *a solution of* $m = \tanh\nu_1$ *and* σ_z *given by* (3.28); *then there exists a constant c such that for all m such* *that* $|m| \le 1$, *for all* z *such that* $|z| < \varepsilon$ *for some* $\varepsilon > 0$ *small enough*,

$$|\nu_2 - \nu_1| \le 2(z)^2 \alpha (1 + c|z|) \tag{3.50}$$

for some positive constant c.

PROOF. The proof is easy; starting from (3.31), using the estimate (3.33), we get

$$\tanh(\nu_2 + z) = m + (1 - m^2) p(m, \Delta) \tag{3.51}$$

with $|p(m, \Delta)| \le 2|z|(1 + c|z|)$ for some positive constant c. Therefore, using (3.31) and (3.29), we have

$$\left| \frac{d\nu_2}{d\theta} \right| \le |z| \alpha (1 + c|z|), \tag{3.52}$$

from which we get (3.50). □

With this result, using the Taylor formula with an integral rest, we expand around ν_1 the last two terms in (3.49). Using $m = \tanh(\nu_1)$, we get

$$\begin{aligned} \frac{\phi(m, \alpha)}{|B|} & \\ &= z\alpha m + \alpha(\nu_2 - \nu_1 + z)^2 \int_0^1 \frac{(1 - \xi)}{\cosh^2(\nu_1 + \xi(\nu_2 - \nu_1 + z))}\, d\xi \\ &\quad + (1 - \alpha)(\nu_2 - \nu_1)^2 \int_0^1 \frac{(1 - \xi)}{\cosh^2(\nu_1 + \xi(\nu_2 - \nu_1))}\, d\xi. \end{aligned} \tag{3.53}$$

The only term which is not evidently proportional to α is the last one, but using (3.50) and defining $b(z, \alpha) \equiv (\nu_2 - \nu_1)/z\alpha$ we have $|b(z, \alpha)| \le |z|(1 + c|z|)$.

We denote by

$$\begin{aligned} \hat{\varphi}(m, z, \alpha) & \\ &\equiv (1 + \alpha b(z, \alpha))^2 \int_0^1 \frac{(1 - \xi)}{\cosh^2(\nu_1 + \xi(\nu_2 - \nu_1 + z))}\, d\xi \\ &\quad + z\alpha(1 - \alpha) b^2(z, \alpha) \int_0^1 \frac{(1 - \xi)}{\cosh^2(\nu_1 + \xi(\nu_2 - \nu_1))}\, d\xi \end{aligned} \tag{3.54}$$

and we have $|\hat{\varphi}(m, z, \alpha)| \le \alpha |z|(1 + c|z|)$, for some positive constant c.

At last $\phi(m, z, \alpha) = z|D|[m + \hat{\varphi}(m, z, \alpha)]$. Therefore, (3.17) takes the form

$$L_m(z, D, B) = \frac{\Psi_{z, \alpha, m}}{\Psi_{0, 0, m}} \exp(z|D|[m + \hat{\varphi}(m, z, \alpha)]). \tag{3.55}$$

Collecting what we have done, recalling (3.12) and (3.23), we end the proof of Proposition 3.1. □

Next we prove Proposition 3.2. It is simpler to start directly from the explicit expression of $L_m(z, |D|, B)$ given by (3.17). By symmetry, it is enough to consider the case where $m \geq 1 - g(|B|)/|B|$. To simplify the formulas, it is better to set $m = 1 - 2k/|B|$ and use the variable k instead of m. We set $L_{(1-2k/B)}(z, D, B) \equiv L_k(z, |D|)$. We assume that $1 \leq k \leq g(|B|)$. It is easy to check that

$$L_k(z, D) = e^{z|D|} \binom{B}{k}^{-1} \sum_{l=0}^{k \wedge |D|} e^{-2zl} \binom{B-D}{k-l} \binom{D}{l}. \tag{3.56}$$

The first case to consider is when $k \leq D$. We are in the binomial regime. We use the following standard estimates:

$$\frac{(B-D-k)^{k-l}}{(k-l)!} \leq \binom{B-D}{k-l} \leq \frac{(B-D)^{k-l}}{(k-l)!}. \tag{3.57}$$

On the one hand, using the right part of (3.57) and some easy algebra, we get

$$L_k(z, D) \leq \frac{B^k (B-k)!}{B!} e^{z|D|} \big((1-\alpha) + \alpha e^{-2|z|}\big)^k, \tag{3.58}$$

where, as before, $\alpha = D/B$. Using $(1-x)^{-1} \leq \exp(x(1+x))$ if $0 \leq x \leq 1/2$, we get

$$\frac{B^k (B-k)!}{B!} \leq \left(1 - \frac{(k-1)}{B}\right)^{-k+1} \leq \exp\left(\frac{k^2}{B}\left(1 + \frac{k}{B}\right)\right) \tag{3.59}$$

and this entails

$$L_k(z, D) \leq e^{z|D|} \big((1-\alpha) + \alpha e^{-2|z|}\big)^k \exp\left(\frac{g^2(B)}{B}\left(1 + \frac{g(B)}{B}\right)\right). \tag{3.60}$$

On the other hand, using the left part of (3.57) and calling $\rho_k = k/B$, we get

$$L_k(z, D) \geq \frac{B^k (B-k)!}{B!} e^{z|D|} \big((1-\alpha) + \alpha e^{-2|z|}\big)^k \left(1 - \rho_k \frac{1 + e^{-2|z|}}{1 - \alpha + \alpha e^{-2|z|}}\right)^k. \tag{3.61}$$

Using $1 - x \geq \exp(-x(1+x))$ if $0 \leq |x| \leq 1/2$, the left part of (3.59) and some easy estimates, we get

$$L_k(z, D) \geq e^{zD} \big((1-\alpha) + \alpha e^{-2|z|}\big)^k \exp\left(-\frac{g^2(B)(1 + \exp(2|z|))}{B}\right) \tag{3.62}$$

After some computations, we get

$$e^{zD} \big((1-\alpha) + \alpha e^{-2|z|}\big)^k = \exp\Big(zD\big[m + (1-m)\tilde{f}(z, \alpha)\big]\Big) \tag{3.63}$$

with $|(1-m)\tilde{f}(z, \alpha)| \leq (g(B)/B)|z|e^{|z|}(1 + c|z|)$.

Collecting (3.60), (3.62) and (3.63), we get

$$L_m(z,D) = \exp\big(zD\big[m + \hat{\varphi}_b(m,z,\alpha)\big]\big) \tag{3.64}$$

with

$$\sup_{m:\,|m|\geq 1-g(B)/B} |\hat{\varphi}_b(m,z,\alpha)| \leq c_1\left(\frac{g(B)}{B}|z| + \frac{g^2(B)}{|z|B}\right). \tag{3.65}$$

It remains to consider the case where $D \leq k \leq g(B)$. This is the Poissonian regime. It can be checked that

$$\begin{aligned} L_k(z,D) &\leq \exp(zD)\left(\frac{1-\alpha}{1-\rho_k}\right)^k \sum_{l=0}^{D} \frac{1}{l!}\exp(-2|z|l)\left(\frac{Dk}{B-D}\right)^l \\ &\leq \exp(zD)\exp\left(\frac{\alpha k}{1-\alpha}\exp(-2|z|)\right)\left(\frac{1-\alpha}{1-\rho_k}\right)^k. \end{aligned} \tag{3.66}$$

The last factor in (3.66) is here to make a nice cancellation that will give the correct behavior when $z \downarrow 0$. We have

$$\left(\frac{1-\alpha}{1-\rho_k}\right)^k \leq \exp(-\alpha k)\exp\left(\frac{g^2(B)}{B}\right). \tag{3.67}$$

Therefore after some computations, we get

$$L_m(z,D) \leq \exp\big[zD\big(m + \hat{\varphi}_p(z,m,\alpha)\big)\big] \tag{3.68}$$

with $|\hat{\varphi}_p(z,m,\alpha)| \leq (2g^2(B)/B) + |z|g(B)/B$.

For the lower bound, we have

$$L_k(z,D) \geq e^{zD}\left(\frac{B-D-k}{B}\right)^k \sum_{l=0}^{D} \frac{1}{l!} e^{-2|z|l} \frac{(D-l)^l(k-l)^l}{(B-D-k+l)^l}. \tag{3.69}$$

Keeping the first two terms in the previous sum gives

$$L_k(z,D) \geq e^{zD}\left(\frac{B-D-k}{B}\right)^k \left(1 + e^{-2|z|}\frac{(D-1)(k-1)}{(B-D-k)}\right). \tag{3.70}$$

After some computations, we get

$$L_k(z,D) \geq \exp\big(zD\big[m + \hat{\varphi}_p(z,m,\alpha)\big]\big) \tag{3.71}$$

with $|\hat{\varphi}_p(z,m,\alpha)| \leq c(g^2(B)/B)e^{4|z|}$. □

4. Proof of Theorem 2.3 and some probability estimates. In this section we prove Theorem 2.3. To study the properties of the system, uniformly on an interval V of length $\tilde{c}((\log 1/\gamma)^p/\gamma)$, $p > 1$, we start considering a region $V_1 \subset V$ of scale $L_1 \approx (1/\gamma)(\log 1/\gamma)(\log\log 1/\gamma)^{2+\rho}$, with $\rho > 0$, and divide it in smaller intervals of scale $l(\gamma) = 1/\gamma \log 1/\gamma$. We reduce the proof to the estimate of the upper bound of the ratio of two constrained partition functions over one of these intervals. We then write this ratio as the product of two stochastic contributions and with $\mathbb{P}$-probability 1, we prove the following:

1. There is at *least* one interval of scale $l(\gamma)$ such that the first factor of the stochastic part is smaller than $\exp(-c/\gamma)$, $c > 0$.
2. *For all* the above-mentioned intervals the contribution of the second factor is negligible.
3. This can be done uniformly with respect to the choices of V_1 in V.

The choice of the relative sizes of the intervals involved is suggested by two conflicting conditions: the existence of a large enough fluctuation of the magnetic field, in at least one small interval, for the first factor and the uniform control of the second factor over *all* intervals contained in V. In step 2 we need a deviation inequality for a Lipschitz function of symmetric Bernoulli random variables, but our construction of the stochastic part, in Section 3, does not allow checking the convexity hypothesis assumed in [22] or [32]. Therefore we give a simple proof of such deviation inequality without any convexity hypothesis. See also [23], [36].

We start the proof of Theorem 2.3. Given $\tilde{c} > 0$, $p > 1$, it is enough to prove that

$$\mu_{\beta,\theta,\gamma}\big[\mathscr{R}^{\delta,\zeta}\big(\big[kl_{\hat{c}}(\gamma), L_1 + kl_{\hat{c}}(\gamma)\big], \tau\big)\big] \le \exp\big(-\beta x \gamma^{-1}\big) \tag{4.1}$$

simultaneously for $\tau = 1$ and $\tau = -1$, and for any k such that $|k| \le (\tilde{c}/\hat{c})(\log 1/\gamma)^p \log\log 1/\gamma$, where $\hat{c}$ is a constant to be determined later. We take $I_{12} \equiv [l_1, l_2] \subset [kl_{\hat{c}}(\gamma), L_1 + kl_{\hat{c}}(\gamma)]$ and we start estimating $\mu_{\beta,\theta,\gamma}[\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau)]$, with l_1 and l_2 such that $|l_1 - l_2| = (\hat{c}/\gamma \log\log 1/\gamma) = l_{\hat{c}}(\gamma)$.

The first remark is that if Λ_1 and Λ_2 are two blocks of macroscopic length 1, then $\sup_{\sigma_{\Lambda_1 \cup \Lambda_2}} |W_\gamma(\sigma_{\Lambda_1}, \sigma_{\Lambda_2})| \le \gamma^{-1}$; this follows from $\int J(x)\,dx = 1$. Therefore, cutting all the interactions between $[l_1, l_2]$ and its complementary, we have the estimate

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\big[\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau)\big] \\ &\qquad \le \exp\big(4\beta\gamma^{-1}\big)\mu_{\beta,\theta,\gamma}\big(\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau) \mid \Sigma_{\partial I_{12}}\big)(0). \end{aligned} \tag{4.2}$$

We bound from below the partition function $Z_{\beta,\gamma,\theta,[l_1,l_2]}(0)$ [see (3.1)] by restricting the sum over all the spin configurations in $\mathscr{R}^{\delta,\zeta}(l_1, l_2, -\tau)$. Taking into account that the two normalization factors cancel, we have

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\big[\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau)\big] \\ &\qquad \le \exp\big(4\beta\gamma^{-1}\big)\frac{\mu_{\beta,\theta,\gamma}\big(\mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau) \mid \Sigma_{\partial I_{12}}\big)(0)}{\mu_{\beta,\theta,\gamma}\big(\mathscr{R}^{\delta,\zeta}(l_1, l_2, -\tau) \mid \Sigma_{\partial I_{12}}\big)(0)}. \end{aligned} \tag{4.3}$$

For simplicity, let us denote $\mathscr{R}(\tau) \equiv \mathscr{R}^{\delta,\zeta}(l_1, l_2, \tau)$.

Performing a block spin transformation on the scale $\delta^*\gamma^{-1}$ and using (3.13) we get

$$\mu_{\beta,\theta,\gamma}(\mathscr{R}(\tau)) \le \exp\big(\beta\gamma^{-1}[\delta^*|l_2 - l_1| + 4]\big)\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})}, \tag{4.4}$$

where

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12})\in\mathscr{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathscr{R}(\tau)\}}\exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}},0\big)+\gamma\mathscr{G}\big(m^{\delta^*}_{I_{12}}\big)\Big\}\Big)}{\sum_{m^{\delta^*}(I_{12})\in\mathscr{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathscr{R}(-\tau)\}}\exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}},0\big)+\gamma\mathscr{G}\big(m^{\delta^*}_{I_{12}}\big)\Big\}\Big)}. \tag{4.5}$$

We denote by T, the linear bijection on $\mathscr{M}_{\delta^*}(I_{12})$ defined by

$$T(m_1(x), m_2(x)) = (-m_2(x), -m_1(x)) \qquad \forall\, x \in \mathscr{C}_{\delta^*}(I_{12}), \tag{4.6}$$

then $T\mathscr{R}(\tau) = \mathscr{R}(-\tau)$. Moreover from (3.10), it is immediate to check that $\hat{\mathscr{F}}(Tm^{\delta^*}_{I_{12}},0) = \hat{\mathscr{F}}(m^{\delta^*}_{I_{12}},0)$ by using the symmetry properties of the combinatorial factors. Therefore, performing the change of variables induced by T in the denominator in (4.5), we get

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12})\in\mathscr{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathscr{R}(\tau)\}}\exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}},0\big)+\gamma\mathscr{G}\big(m^{\delta^*}_{I_{12}}\big)\Big\}\Big)}{\sum_{m^{\delta^*}(I_{12})\in\mathscr{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathscr{R}(\tau)\}}\exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}},0\big)+\gamma\mathscr{G}\big(Tm^{\delta^*}_{I_{12}}\big)\Big\}\Big)}. \tag{4.7}$$

By construction we note that changing h_i into $-h_i$ makes the following changes: $\lambda(x) \to -\lambda(x)$, $B^+ \to B^-$ while $|D(x)|$ is left invariant. Therefore we get the following:

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})}(-h) = \frac{Z_{-\tau}(I_{12})}{Z_\tau(I_{12})}(h), \tag{4.8}$$

which implies the nontrivial fact that $\log(Z_\tau(I_{12})/Z_{-\tau}(I_{12}))(h)$ is a symmetric random variable and therefore has mean zero. The next step is to extract what we expect to be the leading term of the stochastic part coming in (4.7). Recalling (3.14), we introduce

$$\Delta\mathscr{G}\big(m^{\delta^*}_{\beta,I_{12}},\tau\big) \equiv \tau\Big[\mathscr{G}\big(m^{\delta^*}_{\beta,I_{12}}\big) - \mathscr{G}\big(Tm^{\delta^*}_{\beta,I_{12}}\big)\Big], \tag{4.9}$$

where $m^{\delta^*}_{\beta,I_{12}}$ is the configuration of $m^{\delta^*}(x) = m^{\delta^*}_\beta\ \forall\, x \in I_{12}$ and $m^{\delta^*}_\beta$ is any point in $[-1, -1+4\gamma(\delta^*)^{-1},\dots,1]^2$ which is among the nearest to m_β defined before (2.24).

We write

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} = \exp\Big(\Delta\mathscr{G}\big(m^{\delta^*}_{\beta, I_{12}}, \tau\big)\Big)\frac{Z_{\tau,0}(I_{12})}{Z_{-\tau,0}(I_{12})}, \tag{4.10}$$

where

$$\frac{Z_{\tau,0}(I_{12})}{Z_{-\tau,0}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12}) \in \mathscr{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathscr{R}(\tau)\}} \exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}}, 0\big) + \gamma\Delta^\tau_0\mathscr{G}\big(m^{\delta^*}_{I_{12}}\big)\Big\}\Big)}{\sum_{m^{\delta^*}(I_{12}) \in \mathscr{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathscr{R}(\tau)\}} \exp\Big(-(1/\gamma)\Big\{\beta\hat{\mathscr{F}}\big(m^{\delta^*}_{I_{12}}, 0\big) + \gamma\Delta^\tau_0\mathscr{G}\big(Tm^{\delta^*}_{I_{12}}\big)\Big\}\Big)} \tag{4.11}$$

and

$$\Delta^\tau_0\mathscr{G}\big(m^{\delta^*}_{I_{12}}\big) \equiv \mathscr{G}\big(T^{(1-\tau)/2} m^{\delta^*}_{I_{12}}\big) - \mathscr{G}\big(T^{(1-\tau)/2} m^{\delta^*}_{\beta, I_{12}}\big) \tag{4.12}$$

with $T^0 = \mathbb{1}$, the identity.

Recall that $m_{\beta,1}$ and $m_{\beta,2}$ which are defined before (2.24) are bounded away from 1. For $\beta > 1$ and $\beta\theta$ small enough, we can use Proposition 3.1 to control $\Delta\mathscr{G}(m^{\delta^*}_{\beta, I_{12}})$. Recall that this term has mean zero. Using (3.24) and the definition of Tm_β given before (2.24), we can write $\Delta\mathscr{G}(m^{\delta^*}_{\beta, I_{12}}, \tau) = -\tau\sum_{x \in \mathscr{C}_{\delta^*}(I_{12})} X(x)$ with

$$\begin{aligned} X(x) \equiv {} & -2\beta\theta\lambda(x)|D(x)|\big[m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + \Xi(x, \beta\theta, \alpha)\big] \\ & - \lambda(x)\log\frac{\Psi_{\beta\theta,\,\alpha(x),\,m^{\delta^*}_{\beta,2}}\Psi_{0,0,m^{\delta^*}_{\beta,1}}}{\Psi_{\beta\theta,\,\alpha(x),\,m^{\delta^*}_{\beta,1}}\Psi_{0,0,m^{\delta^*}_{\beta,2}}} \end{aligned} \tag{4.13}$$

and

$$\Xi(x, \beta\theta, \alpha) \equiv \Big[\hat{\varphi}\big(m^{\delta^*}_{\beta,1}, \lambda(x)\beta\theta, \alpha\big) - \hat{\varphi}\big(m^{\delta^*}_{\beta,2}, \lambda(x)\beta\theta, \alpha\big)\Big]. \tag{4.14}$$

The next step is to get a lower bound for the probability of $\{\tau\gamma\sum_{x \in \mathscr{C}_{\delta^*}(I_{12})} X(x) > u\}$. We follow de Acosta [14] and write this sum as a sum over $|\mathscr{C}_{\delta^*}(I_{12})|/N$ blocks, each block having N elements, $1 \le N \le |I_{12}|/\delta^*$.

Calling $V^2(N) = V^2(N(w)) \equiv \sum_{x \in N(w)} \mathbb{E}[X^2(x)]$ for $1 \le w \le |I_{12}|/(\delta^* N)$, we require that N satisfies also

$$\gamma\sum_{x \in \mathscr{C}_{\delta^*}(I_{12})} X(x) = \frac{N}{|\mathscr{C}_{\delta^*}(I_{12})|}\sum_{w=1}^{|\mathscr{C}_{\delta^*}(I_{12})|/N}\frac{1}{V(N)}\sum_{x \in N(w)} X(x). \tag{4.15}$$

Assuming that such N can be found, then we have

$$\Big\{\tau\gamma\sum_{x \in \mathscr{C}_{\delta^*}(I_{12})} X(x) > u\Big\} \supset \bigcap_{w=1}^{|\mathscr{C}_{\delta^*}(I_{12})|/N}\Big\{\frac{\tau}{V(N)}\sum_{x \in N(w)} X(x) > u\Big\}. \tag{4.16}$$

Using the fact that the events in the right-hand side are independent, we apply the central limit theorem to estimate their individual probabilities.

To check that we can find an $1 \le N \le |I_{12}|/\delta^*$ such that (4.15) is true, we use Proposition 3.1. On the one hand we have

$$\mathbb{E}\big[X^2(x)\big] \ge \Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} - c\beta\theta\big)\Big)^2 \frac{\delta^*}{\gamma}\left(1 - c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right) \tag{4.17}$$

and on the other hand we get, if $g(|B|)$ is large enough,

$$\mathbb{E}\big[X^2(x)\big] \le \Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + c\beta\theta\big)\Big)^2 \frac{\delta^*}{\gamma}\left(1 + c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right) \tag{4.18}$$

for some positive constant c. Therefore using (4.15), it is easy to check that N must satisfy

$$\begin{aligned} &\Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} - c\beta\theta\big)\Big)^2\left(1 - c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right)\frac{\gamma|I_{12}|^2}{\delta^*} \\ &\qquad \le N \le \Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + c\beta\theta\big)\Big)^2\left(1 + c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right)\frac{\gamma|I_{12}|^2}{\delta^*}. \end{aligned} \tag{4.19}$$

Therefore, $N \le |I_{12}|/\delta^*$ provided

$$|I_{12}| \le \frac{1}{\gamma}\Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + c\beta\theta\big)\Big)^{-2}\left(1 + c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right)^{-1}. \tag{4.20}$$

Obviously $N \ge 1$ provided

$$|I_{12}| \ge \left(\frac{\delta^*}{\gamma}\right)^{1/2}\left(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} - c\beta\theta\big)\left(1 - c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right)\right)^{-1}. \tag{4.21}$$

Therefore, since $|I_{1,2}| = l_{\hat{c}}(\gamma) = \hat{c}(\gamma \log\log \gamma^{-1})^{-1}$, (4.20) and (4.21) are satisfied if γ is small enough. To continue, using (4.17) and (4.19) we have

$$\begin{aligned} V^2(N) &\ge \Big(\beta\theta\big(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} - c\beta\theta\big)\Big)^4\left(1 - c\beta\theta\sqrt{\frac{\gamma}{\delta^*}}\right)^2 |I_{12}|^2 \\ &\equiv (\beta\theta\alpha(\beta,\theta))^2 |I_{12}|^2. \end{aligned} \tag{4.22}$$

Therefore, since $\lim_{\gamma\downarrow 0}|I_{12}| = \infty$, it is clear that we are in the domain of application of the central limit theorem and we have, for all $\tilde{\varepsilon} > 0$ and $u > 0$,

$$\begin{aligned} &\mathbb{P}\left[\frac{\tau}{V(N)}\sum_{x\in N(w)} X(x) > u\right] \\ &\qquad \ge \mathbb{P}\left[u(1+\tilde{\varepsilon}) \ge \frac{\tau}{V(N)}\sum_{x\in N(w)} X(x) > u\right] \\ &\qquad \ge \exp(-u^2(1+\tilde{\varepsilon})/2). \end{aligned} \tag{4.23}$$

Using the lower bound for N [see (4.19)] and for $V(N)$ [see (4.22)] together with (4.16), we get

$$\mathbb{P}\left[\tau\gamma \sum_{x\in\mathscr{C}_{\delta^*}(I_{12})} X(x) \geq u\right] \geq \exp\left(-\frac{u^2(1+\tilde{\varepsilon})}{2(\beta\theta a(\beta,\theta))^2\gamma|I_{12}|}\right). \tag{4.24}$$

Now to end the proof of Theorem 2.3, first we use (4.24), for M consecutive blocks of length $l_{\hat{c}}(\gamma)$, that we denote by $L(1),\dots,L(M)$. Using independence over disjoint blocks and $1-x\leq e^{-x}$, considering the two cases $\tau=1$ and $\tau=-1$ separately, we get

$$\begin{aligned}&\mathbb{P}\left[\inf_{\tau\in\{-1,+1\}}\sup_{1\leq l\leq M}\tau\gamma\sum_{x\in\mathscr{C}_{\delta^*}(L(l))} X(x)\geq u\right]\\ &\qquad\geq 1-2\exp\left[-M\exp\left(-\frac{u^2(1+\tilde{\varepsilon})}{2(\beta\theta a(\beta,\theta))^2\gamma l_{\hat{c}}(\gamma)}\right)\right].\end{aligned}\tag{4.25}$$

Moreover, it follows from the next proposition [see (4.28)] that for all $\varepsilon>0$, provided $g_2(1/\zeta)$ is a diverging, slowly varying function at infinity, $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta)=0$, then

$$\begin{aligned}&\mathbb{P}\left[\sup_{1\leq l\leq M}\left|\log\frac{Z_{+,0}(L(l))}{Z_{-,0}(L(l))}\right|\leq\frac{\varepsilon}{\gamma}\right]\\ &\qquad\geq 1-2M\exp\left(-\frac{\varepsilon^2}{212\gamma l_{\hat{c}}(\gamma)\beta\theta\zeta g_2(1/\zeta)}\right).\end{aligned}\tag{4.26}$$

Given $\rho>0$ and $x>0$ we make the following choice of parameters:

$$\begin{aligned}
c(x,\rho,\gamma) &= \frac{2(4+x)^2\beta^2}{1+(2+3\rho/4)(\log\log\log 1/\gamma/\log\log 1/\gamma)},\\
\hat{c} &= \left[\frac{(1+\tilde{\varepsilon})}{(\beta\theta a(\beta,\theta))^2}c(x,\rho,\gamma)\right],\\
u &= 2\beta(4+x+c_0\hat{c}),\\
M &= \left(\log\frac{1}{\gamma}\right)\left(\log\log\frac{1}{\gamma}\right)^{3+\rho},\\
\varepsilon &= \frac{(4+x)\beta}{2},\\
\zeta g_2(1/\zeta) &\leq \frac{1}{8\times 212(p+2+\tilde{\varepsilon})}\beta\theta(a(\beta,\theta))^2.
\end{aligned}\tag{4.27}$$

An easy computation shows that the right-hand side of (4.25) is bounded below by $1-\exp(-(\log\log 1/\gamma)^{1+\rho/4})$, and the one of (4.26) by $1-1/(\log 1/\gamma)^{p+2+\tilde{\varepsilon}}$. By (4.3), (4.4) and (4.10) we obtain the estimate (4.1).

Moreover, it is immediate to see that we have also uniformity with respect to the $2(\tilde{c}/\hat{c})(\log 1/\gamma)^p \log\log 1/\gamma$ possible choices of k in (2.29). Using the first Borel–Cantelli lemma and the fact that $\gamma = 2^{-n}$, we get Theorem 2.3. □

PROPOSITION 4.1. *Given $\beta > 1$, $\beta\theta < \varepsilon_0$ for some ε_0, let ζ small enough and $g_2(1/\zeta)$ be a real function such that $\lim_{\zeta\downarrow 0} g_2(1/\zeta) = \infty$, slowly varying at infinity that satisfies $\lim_{\zeta\downarrow 0}\sqrt{\zeta} g_2(1/\zeta) = 0$, then for all $\varepsilon > 0$, for all integers l_1, l_2, if γ is small enough,*

$$\mathbb{P}\left[\left|\log\frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})}\right| \geq \frac{\varepsilon}{\gamma}\right] \leq \exp\left(-\frac{\varepsilon^2}{212\gamma|l_1 - l_2|\beta\theta\zeta g_2(1/\zeta)}\right). \tag{4.28}$$

The proof of this proposition is rather long and technical. We first remark that using the explicit expression (3.11), (3.12) and the fact that $T(m_{\beta,1}, m_{\beta,2}) = (-m_{\beta,2}, -m_{\beta,1})$, we get $\mathbb{E}[\mathscr{G}(m^{\delta^*}_{\beta,I_{12}}) - \mathscr{G}(Tm^{\delta^*}_{\beta,I_{12}})] = 0$; using (4.8), we have also $\mathbb{E}[\log Z_{+,0}(I_{12})/Z_{-,0}(I_{12})] = 0$.

Let us prove the above mentioned deviation inequality.

LEMMA 4.2. *Let N be a positive integer and F be a real function on $\Omega = \{-1, +1\}^N$ and for all $i \in \{1, \dots, N\}$ let*

$$\|\partial_i F\|_\infty = \sup_{(h,\tilde{h}):\ h_j = \tilde{h}_j, \forall j\neq i} \frac{|F(h) - F(\tilde{h})|}{|h_i - \tilde{h}_i|}. \tag{4.29}$$

If $\mathbb{P}$ is the symmetric Bernoulli measure and $\|\partial(F)\|^2_\infty = \sum_{i=1}^N \|\partial_i(F)\|^2_\infty$ then, for all $t > 0$,

$$\mathbb{P}[F - \mathbb{E}(F) \geq t] \leq \exp\left(-\frac{t^2}{4\|\partial(F)\|^2_\infty}\right) \tag{4.30}$$

and also

$$\mathbb{P}[F - \mathbb{E}(F) \leq -t] \leq \exp\left(-\frac{t^2}{4\|\partial(F)\|^2_\infty}\right). \tag{4.31}$$

PROOF. We prove (4.30); the proof of (4.31) is exactly the same. As usual in this kind of estimate, we start with the exponential Markov inequality. For all $\lambda > 0$, we have

$$\mathbb{P}[F - \mathbb{E}(F) \geq t] \leq \exp(-\lambda t)\mathbb{E}[\exp(\lambda(F - \mathbb{E}(F)))]. \tag{4.32}$$

To estimate the last term, we introduce the family of increasing σ-algebra,

$$\begin{aligned}(\varnothing, \Omega) &= \Sigma_0 \subset \Sigma_1 = \sigma(h_1) \subset \Sigma_2 = \sigma(h_1, h_2) \subset \cdots \subset \Sigma_N \\ &= \sigma(h_1, h_2, \dots, h_N)\end{aligned} \tag{4.33}$$

and the martingale difference sequences, $\forall\, 1 \leq k \leq N$; $\Delta_k(F) = \mathbb{E}[F \mid \Sigma_k] - \mathbb{E}[F \mid \Sigma_{k-1}]$. If we prove that

$$\mathbb{E}\left[\exp\left(\lambda\sum_{k=1}^N \Delta_k(F)\right)\right] \leq \exp(\lambda^2\|\partial(F)\|^2_\infty), \tag{4.34}$$

then (4.30) follows from (4.32) by taking $\lambda = t(2\|\partial(F)\|_\infty^2)^{-1}$. To prove (4.34), we perform the integrations in the left-hand side of (4.34) starting from h_N. The only term that depends on h_N is $\Delta_N(F) = F(h_{<N}, h_N) - \int F(h_{<N}, \tilde{h}_N)\,\mathbb{P}(d\tilde{h}_N)$ where $h_{<N} \equiv (h_1, h_2, \dots, h_{N-1})$. Therefore, using the Jensen inequality, we get

$$\begin{aligned}(4.35)\qquad &\int \exp(\lambda\Delta_N(F))\mathbb{P}(dh_N)\\ &\quad\le \int \exp\Big(\lambda\big[F(h_{<N}, h_N) - F\big(h_{<N}, \tilde{h}_N\big)\big]\Big)\mathbb{P}(dh_N)\mathbb{P}\big(d\tilde{h}_N\big).\end{aligned}$$

For all fixed $h_{<N}$, the term into the exponential is the symmetrized of F with respect to the last variable. Then if we expand the exponential and integrate with respect to the product measure $\mathbb{P}(dh_N)\mathbb{P}(d\tilde{h}_N)$, all the odd terms vanish and we get $\forall\, h_{<N}$,

$$\begin{aligned}&\int \exp(\lambda\Delta_N(F))\mathbb{P}(dh_N)\\ &\quad\le \sum_{n=0}^{\infty} \frac{(\lambda)^{2n}}{(2n)!}\int\big[F(h_{<N}, h_N) - F\big(h_{<N}, \tilde{h}_N\big)\big]^{2n}\mathbb{P}(dh_N)\mathbb{P}\big(d\tilde{h}_N\big)\\ (4.36)\qquad &\quad\le \sum_{n=0}^{\infty} \frac{\big(\lambda\|\partial_N F\|_\infty\big)^{2n}}{(2n)!}\int|h_N - \tilde{h}_N|^{2n}\mathbb{P}(dh_N)\mathbb{P}\big(d\tilde{h}_N\big)\\ &\quad= \sum_{n=0}^{\infty} \frac{\big(\lambda\|\partial_N F\|_\infty\big)^{2n}}{(2n)!}2^{(2n-1)^+} \le \exp\big(\lambda^2\|\partial_N(F)\|_\infty^2\big),\end{aligned}$$

where $(x)^+ = \max(x, 0)$ and we have used $2^{(2n-1)^+}/(2n)! \le 1/n!$.

A little difference for the successive integrations comes from the way to use the Jensen inequality. We perform the next h_{N-1} integration. Since the term $\Delta_{N-1}(F)$ is the only one that comes into play, we use the Jensen inequality as follows:

$$\begin{aligned}&\int \exp(\lambda\Delta_{N-1}(F))\mathbb{P}(dh_{N-1})\\ (4.37)\quad &\le \int \exp\Big(\lambda\int\big[F\big(h_{<N-1}, h_{N-1}, \hat{h}_N\big)\\ &\qquad -F\big(h_{<N-1}, \tilde{h}_{N-1}, \hat{h}_N\big)\big]\mathbb{P}\big(d\hat{h}_N\big)\Big)\mathbb{P}(dh_{N-1})\mathbb{P}\big(d\tilde{h}_{N-1}\big).\end{aligned}$$

Now we can make exactly the same computations, since for fixed $h_{<N-1}$,

$$(4.38)\qquad \int\big[F\big(h_{<N-1}, h_{N-1}, \hat{h}_N\big) - F\big(h_{<N-1}, \tilde{h}_{N-1}, \hat{h}_N\big)\big]\mathbb{P}\big(d\hat{h}_N\big)$$

is a symmetric random variable under $\mathbb{P}(dh_{N-1})\mathbb{P}(d\tilde{h}_{N-1})$ and we can use (4.29) to get

$$\int \exp(\lambda\Delta_{N-1}(F))\mathbb{P}(dh_{N-1}) \leq \exp\big(\lambda^2\partial_{N-1}(F)\big). \tag{4.39}$$

Iterating, we get (4.34). □

It is clear that we have to estimate the corresponding Lipschitzian factors [see (4.29)]

$$\left\| \partial_i \log \frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})} \right\|_\infty \tag{4.40}$$

for all $i \in \gamma^{-1}(I_{12})$. Here there is a difficulty that comes from the fact that definition (2.26) of $\eta^{\delta,\zeta}$ is given in term of a Cesaro average of blocks of length δ^* that are contained in a block δ of l_1 norm. So we cannot assume that *all* the blocks of length δ^* are near an equilibrium; some but certainly not all blocks of length δ^* can have $m_i^{\delta^*}(x)$ very near 1. On the other hand, the correction to the leading behavior of $\Delta_0 \mathscr{G}_{x,m^{\delta^*}}$ is dependent on the values of m^{δ^*} and here we have to estimate a Lipschitz norm which certainly becomes more and more singular as $m_i^{\delta^*}(x)$ approaches 1. To solve this problem, we localize the blocks which are near equilibrium (the good ones) and their complementary (the bad ones). We show that the fraction of the bad blocks can be neglected provided we increase the "tolerance" ζ.

We need to introduce some definitions. Given $i \in \gamma^{-1}I_{12}$, let $x(i)$ be the index of the block of length δ^* that contains the microscopic site i. Let $u(i)$ be the index of the block of length δ that contains $x(i)$; let $\mathscr{C}_{\delta/\delta^*}(u(i)) \equiv \mathscr{C}_{\delta/\delta^*}(i)$ be the set of the centers of blocks of length δ^* that are in the blocks of length δ indexed by $u(i)$. We have to estimate

$$\log \frac{Z_{+,0}(I_{12})(h)}{Z_{+,0}(I_{12})(\tilde{h}_i)} - \log \frac{Z_{-,0}(I_{12})(h)}{Z_{-,0}(I_{12})(\tilde{h}_i)}, \tag{4.41}$$

where the only discrepancy between h and $\tilde{h}_i$ is at site i. To continue we need a simple lemma. Its proof is similar to a Markov inequality.

LEMMA 4.3. *If*

$$\sum_{x \in \mathscr{C}_{\delta/\delta^*}(i)} \|m^{\delta^*}(x) - m_\beta\|_1 \leq \frac{\delta}{\delta^*}\zeta, \tag{4.42}$$

then given $g_1(\zeta)$ *such that* $\lim_{\zeta \downarrow 0} g_1(\zeta) = 0$ *but* $\zeta/g_1(\zeta) < 1$ *if* $\zeta \leq 1$, *we have*

$$\sum_{x \in \mathscr{C}_{\delta/\delta^*}(i)} \mathbb{1}_{\{\|m^{\delta^*}(x) - m_\beta\|_1 \leq g_1(\zeta)\}} \geq \frac{\delta}{\delta^*}\left(1 - \frac{\zeta}{g_1(\zeta)}\right). \tag{4.43}$$

This suggests making a partition of $\mathscr{C}_{\delta/\delta^*}(i)$ into two sets,

$$\mathscr{A}(m^{\delta^*}) \equiv \Big\{ x \in \mathscr{C}_{\delta/\delta^*}(i) \colon \|m^{\delta^*}(x) - m_\beta\|_1 \le g_1(\zeta), \sup\big(|m_1^{\delta^*}(x)|, |m_2^{\delta^*}(x)|\big) \le 1 - \frac{g(|B|)}{|B|} \Big\} \tag{4.44}$$

and $\mathscr{B}(m^{\delta^*}) = \mathscr{C}_{\delta/\delta^*}(i) \setminus \mathscr{A}(m^{\delta^*})$. Let us call $\Delta(m_\beta) = 1 - m_{\beta,1}$, recalling that $m_{\beta,2} \le m_{\beta,1}$. We assume that the parameters ζ, δ, δ^* and the functions $g_1(\zeta)$ and $g(|B|)$ are all chosen in such a way that for the given pair (β, θ) we have

$$g_1(\zeta) + \frac{g(|B|)}{|B|} \le \Delta(m_\beta). \tag{4.45}$$

This will imply that $\sup(|m_1^{\delta^*}(x)|, |m_2^{\delta^*}(x)|) \le g_1(\zeta) + 1 - \Delta(m_b) \le 1 - (g(|B|))/|B|$ and therefore the second condition in the definition of $\mathscr{A}$ is automatically satisfied. Let us note that since the two terms in the left-hand side of (4.45) go to zero, we can assume that (4.45) is satisfied by taking ζ and γ small enough.

Let $l(i)$ be the index of the block of length 1 containing the microscopic site i. For all $m^{\delta^*} \equiv m_{l(i)}^{\delta^*}$ we write

$$\mathbb{1}_{\{\eta^{\delta,\zeta}(l(i))=1\}}(m^{\delta^*}) = \sum_{X \subset \mathscr{C}_{\delta/\delta^*}(i)} \mathbb{1}_{\{\mathscr{A}=X\}}(m^{\delta^*})\, \mathbb{1}_{\{\mathscr{B}=X^c\}}(m^{\delta^*})\, \mathbb{1}_{\{\eta^{\delta,\zeta}(l(i))=1\}}(m^{\delta^*}), \tag{4.46}$$

where the sum is over all the subsets of $\mathscr{C}_{\delta/\delta^*}(i)$ and $X^c \equiv \mathscr{C}_{\delta/\delta^*}(i) \setminus X$. Note that it follows from the previous lemma that $\eta_{\delta,\zeta}(l(i)) = 1$ and $|X| \le (\delta/\delta^*)(1 - (\zeta/g_1(\zeta)))$ are incompatible, therefore we can impose that $|X| \ge (\delta/\delta^*)(1 - (\zeta/g_1(\zeta)))$ in (4.46).

Let us call

$$\mathscr{N}(\zeta) = \sum_{X \subset \mathscr{C}_{\delta/\delta^*}(i)} \mathbb{1}_{\{|X| \ge (\delta/\delta^*)(1-(\zeta/g_1(\zeta)))\}} = \sum_{k=(\delta/\delta^*)(1-(\zeta/g_1(\zeta)))}^{\delta/\delta^*} \binom{\frac{\delta}{\delta^*}}{k}. \tag{4.47}$$

Then (4.41) is also equivalent to

$$\log \frac{Z_{+,0}(I_{12})(h)}{\mathscr{N}(\zeta) Z_{+,0}(I_{12})(\tilde{h}_i)} - \log \frac{Z_{-,0}(I_{12})(h)}{\mathscr{N}(\zeta) Z_{-,0}(I_{12})(\tilde{h}_i)}. \tag{4.48}$$

The two terms are estimated in the same way. We consider the first one. It is easy to see that, with self-explanatory notation,

$$\frac{Z_{+,0}(I_{12})(h)}{\mathcal{N}(\zeta)Z_{+,0}(I_{12})(\tilde{h}_i)} = \frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\Big[\exp\Big(\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}\Big)\Big], \tag{4.49}$$

where $\mathcal{Q}$ is the probability measure

$$\begin{aligned}\mathcal{Q}[\Psi] = &\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} \Psi(m^{\delta^*})\,\mathbb{1}_{\{\mathcal{R}(+)\}}\\ &\qquad\times\exp\bigg(-\Big(\frac{1}{\gamma}\Big)\Big\{\beta\hat{\mathcal{F}}\Big(m^{\delta^*}_{I_{12}},0\Big) + \gamma\Delta_0\mathcal{G}^{\tilde{h}_i}\Big(m^{\delta^*}_{I_{12}}\Big)\Big\}\bigg)\\ &\times\Bigg[\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathcal{R}(+)\}}\\ &\qquad\times\exp\bigg(-\Big(\frac{1}{\gamma}\Big)\Big\{\beta\hat{\mathcal{F}}\Big(m^{\delta^*}_{I_{12}},0\Big) + \gamma\Delta_0\mathcal{G}^{\tilde{h}_i}\Big(m^{\delta^*}_{I_{12}}\Big)\Big\}\bigg)\Bigg]^{-1}.\end{aligned} \tag{4.50}$$

Inserting (4.46) in (4.49), we get

$$\begin{aligned}&\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\Big[\exp\Big(\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}\Big)\Big]\\ &\quad = \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\geq(\delta/\delta^*)(1-(\zeta/g_1(\zeta)))}}\mathcal{Q}\Big[\exp\Big(\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}\Big)\mathbb{1}_{\{\mathcal{A}=X\}}\mathbb{1}_{\{\mathcal{B}=X^c\}}\Big]\end{aligned} \tag{4.51}$$

Note that *if* we have an estimate of the form

$$|\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}| \leq f_1(\zeta)\,\mathbb{1}_{\{i\in\mathcal{A}\}} + f_2(\zeta)\,\mathbb{1}_{\{i\in\mathcal{B}\}}, \tag{4.52}$$

then on the one hand, we get

$$\begin{aligned}&\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\Big[\exp\Big(\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}\Big)\Big]\\ &\quad\leq \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\geq(\delta/\delta^*)(1-(\zeta/g_1(\zeta)))}}\Big[\exp(f_1(\zeta))\,\mathbb{1}_{\{i\in X\}} + \exp(f_2(\zeta))\,\mathbb{1}_{\{i\in X^c\}}\Big]\end{aligned} \tag{4.53}$$

and on the other hand

$$\begin{aligned}&\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\Big[\exp\Big(\Delta_0\mathcal{G}^{h}_{x(i)} - \Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}\Big)\Big]\\ &\quad\geq \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\geq(\delta/\delta^*)(1-(\zeta/g_1(\zeta)))}}\exp(-f_1(\zeta))\,\mathbb{1}_{\{i\in X\}}.\end{aligned} \tag{4.54}$$

It is simple to check that

$$1 - \frac{\zeta}{g_1(\zeta)} \le \frac{1}{\mathscr{N}(\zeta)} \sum_{\substack{X \subset \mathscr{C}_{\delta/\delta^*}(i) \\ |X| \ge (\delta/\delta^*)(1-(\zeta/g_1(\zeta)))}} \mathbb{1}_{\{i \in X\}} \le 1. \tag{4.55}$$

Therefore, coming back to (4.49) and using (4.53) and (4.54), we get

$$\left| \log \frac{Z_{+,0}(I_{12})(h)}{\mathscr{N}(\zeta) Z_{+,0}(I_{12})(\tilde{h}_i)} \right| \le f_1(\zeta) + \frac{\zeta}{g_1(\zeta)} \exp(|f_2(\zeta) - f_1(\zeta)|). \tag{4.56}$$

Therefore, recalling (4.52), even if we have a very poor bound $f_2(\zeta)$ on the set $\mathscr{B}$, (4.56) implies that by choosing $g_1(\zeta)$ in such a way that $(\zeta/g_1(\zeta)) \downarrow 0$, we recover something small coming from the prefactor in the second term in (4.56).

Let us prove something similar to (4.52). There are two cases to consider; the first one is when $\lambda^h = -\lambda^{\tilde{h}_i}$ and the second one is when $\lambda^h = \lambda^{\tilde{h}_i}$. In the first case, it is easy to check that we have $|D^h| = |D^{\tilde{h}_i}| = 1$. In this case, it is simpler to use (3.56) directly, and after an easy computation we get, if $|D(x)| = 1$,

$$\mathscr{G}_{x, m^{\delta^*}}(\lambda(x)) = \log\cosh(2\beta\theta) + \log\left(1 + \lambda(x) m^{\delta^*}(x) \tanh(2\beta\theta)\right) \tag{4.57}$$

from which it is immediate that, if $|D(x)| = 1$ and $\beta\theta$ is small enough,

$$\begin{aligned} &|\Delta_0 \mathscr{G}(m^{\delta^*}(x)) - \Delta_0 \mathscr{G}^{\tilde{h}_i}(m^{\delta^*}(x))| \\ &\quad \le \frac{4\tanh(2\beta\theta)\|m^{\delta^*}(x) - m_\beta\|_1}{1 - m_{\beta,1}\tanh(2\beta\theta)} \\ &\quad \le c(\beta, \theta)\|m^{\delta^*}(x) - m_\beta\|_1 \end{aligned} \tag{4.58}$$

and this estimate is valid for all values of m^{δ^*}.

In the second case, it is a rather long task to make all the estimates. We have the following.

PROPOSITION 4.4. *There exists an $\varepsilon > 0$ and an absolute constant c such that if $\beta\theta \le \varepsilon$, for all $g(n)$ such that $\lim_{n\uparrow\infty} g(n) = \infty$ but $\lim_{n\uparrow\infty} g(n)/n = 0$,*

$$\begin{aligned} &|\Delta_0 \mathscr{G}^h[m^{\delta^*}(x(i))] - \Delta_0 \mathscr{G}^{\tilde{h}_i}[m^{\delta^*}(x(i))]| \\ &\quad \le 2\beta\theta\left(1 + 16\beta\theta + \frac{|B|}{g^2(|B|)}\right)\|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 \\ &\qquad + \frac{c}{g(|B|)\sqrt{\log g(|B|)}} \end{aligned} \tag{4.59}$$

on the set $\{|m^{\delta^*}(x(i))| \le 1 - (g(|B|))/|B|\}$, while

$$\begin{aligned} &|\Delta_0 \mathscr{G}^h[m^{\delta^*}(x(i))] - \Delta_0 \mathscr{G}^{\tilde{h}_i}[m^{\delta^*}(x(i))]| \\ &\quad \le 2\beta\theta\|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 + c\left(\frac{g^2(|B|)}{|B|}\right) \end{aligned} \tag{4.60}$$

on the set $\{|m^{\delta^*}(x(i))| \geq 1 - (g(|B|))/|B|\}$.

PROOF. Formula (4.60) is immediate from Proposition 3.2. To prove (4.59), remembering (3.24), we have to study three terms. The first one is the simplest,

$$\begin{aligned}\Delta_0^1 \mathscr{G}\big[m^{\delta^*}(x(i))\big] \\ \equiv 2\beta\theta\big(\lambda^h|D^h| - \lambda^{\tilde h_i}|D^{\tilde h_i}|\big)\big[m^{\delta^*}_{\iota(x(i))}(x(i)) - m^{\delta^*}_{\beta,\iota(x(i))}(x(i))\big]\end{aligned} \tag{4.61}$$

and using

$$|\lambda^h|D^h| - \lambda^{\tilde h_i}|D^{\tilde h_i}|| = 1, \tag{4.62}$$

we get

$$|\Delta_0^1 \mathscr{G}\big[m^{\delta^*}(x(i))\big]| \leq 2\beta\theta\|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1. \tag{4.63}$$

The next one corresponds to $\hat\varphi$ and we start from (3.49) and cancel from it the previously estimated term. That is, we consider

$$\begin{aligned}&\Delta_0^2 \mathscr{G}\big[m^{\delta^*}(x(i))\big] \\ &\quad\equiv \phi\Big(m^{\delta^*}_{\iota(x(i))}(x(i)), \lambda^h(x)\beta\theta, \alpha^h\Big) - \phi\Big(m^{\delta^*}_{\iota(x(i))}(x(i)), h^{\tilde h_i}(x)\beta\theta, \alpha^{\tilde h_i}\Big) \\ &\quad\quad - \Big(\phi\Big(m^{\delta^*}_{\beta,\iota(x(i))}(x(i)), \lambda^h(x)\beta\theta, \alpha^h\Big) \\ &\quad\quad\quad - \phi\Big(m^{\delta^*}_{\beta,\iota(x(i))}(x(i)), \lambda^{\tilde h_i}(x)\beta\theta, \alpha^{\tilde h_i}\Big)\Big) \\ &\quad\quad - \Delta_0^1 \mathscr{G}\big[m^{\delta^*}(x(i))\big].\end{aligned} \tag{4.64}$$

A simple way to estimate this term is to compute the double integral of its second derivative with respect to α and m.

After easy estimates we get

$$|\Delta_0^2 \mathscr{G}\big[m^{\delta^*}(x(i))\big]| \leq (32\beta^2\theta^2)\|m^{\delta^*} - m_\beta\|_1. \tag{4.65}$$

It remains to consider the last term in (3.24). We use that $\Psi_{0,0,m}$ does not depend on α and define

$$\begin{aligned}&\Delta_0^3 \mathscr{G}\big(m^{\delta^*}(x(i))\big) \\ &\quad\equiv \log \Psi_{\lambda^h(x(i))\beta\theta,\, \alpha^h,\, m^{\delta^*}_{\iota(x(i))}(x(i))} - \log \Psi_{\lambda^{\tilde h_i}(x(i))\beta\theta,\, \alpha^{\tilde h_i},\, m^{\delta^*}_{\iota(x(i))}(x(i))} \\ &\quad\quad - \Big(\log \Psi_{\lambda^h(x(i))\beta\theta,\, \alpha^h,\, m^{\delta^*}_{\beta,\iota(x(i))}(x(i))} - \log \Psi_{\lambda^{\tilde h_i}(x(i))\beta\theta,\, \alpha^{\tilde h_i},\, m^{\delta^*}_{\beta,\iota(x(i))}(x(i))}\Big).\end{aligned} \tag{4.66}$$

The estimates are done in two different ways depending on whether the blocks we consider belong to $\mathscr{B}$ or to $\mathscr{A}$. In the first case, recalling (4.56), we do not need a sharp estimate. We use simply (3.25), bounding the difference in (4.66) by a sum of four terms, and we get immediately

$$|\Delta_0^3 \mathscr{G}\big(m^{\delta^*}(x(i))\big)| \leq c(\beta\theta)^2 + \frac{200}{g(|B|)} \tag{4.67}$$

for some positive constant c.

In the second case, as becomes clear in a moment, we need to use the fact that $\|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 \leq g_1(\zeta)$ and this makes the computations more involved. $\square$

LEMMA 4.5. *There exists an* $\varepsilon > 0$ *and an absolute constant c such that if* $\beta\theta \le \varepsilon$, *for all* $g(n)$ *such that* $\lim_{n\uparrow\infty} g(n) = \infty$ *but* $\lim_{n\uparrow\infty} g(n)/n = 0$ *for all m such that* $|m| \le 1 - (g(|B|))/|B|$,

$$\begin{aligned}|\Delta_0^3 \mathscr{G}(m^{\delta^*}(x(i)))| &\le \|m^{\delta^*}_{\iota(x(i))} - m^{\delta^*}_{\beta,\,\iota(x(i))}\|_1 \frac{c\beta\theta|B|}{g^2(|B|)}\left(1 + \frac{c}{\sqrt{g(|B|)}}\right)\\ &\quad + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}.\end{aligned} \tag{4.68}$$

PROOF. We use first (3.41) and (3.45) to write

$$\log \Psi_{\lambda\beta\theta,\,\alpha,\,m} = \log \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho) + \log\left(1 + \frac{\mathscr{E}_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}{\Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}\right) \tag{4.69}$$

with $\rho = (\sigma_{\lambda\beta\theta}\sqrt{|B|})^{-1}\sqrt{2\log g(|B|)}$ and we use (3.44), setting $f(|B|) = \sqrt{2\log g(|B|)}$ together with (3.34) to control the last term. This leads to

$$\left|\frac{\mathscr{E}_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}{\Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}\right| \le \frac{5}{3\pi g(|B|)\left(1 + \sqrt{2\log g(|B|)}\right)}. \tag{4.70}$$

Therefore the four terms of this type in (4.66) will give a contribution which corresponds to the last term in (4.68). For the remaining terms, we proceed as before, starting with

$$\begin{aligned}&\Delta_0^3 \mathscr{G}(m^{\delta^*}(x(i)), \rho)\\ &\quad = \int_{\alpha^{\bar h_i}}^{\alpha^h} \int_{m^{\delta^*}_{\beta,\,\iota(x(i))}}^{m^{\delta^*}_{\iota(x(i))}(x(i))} \frac{\partial^2 \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}{\partial\alpha\partial m} \frac{1}{\Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}\, d\alpha\, dm\\ &\qquad - \int_{\alpha^{\bar h_i}}^{\alpha^h} \int_{m^{\delta^*}_{\beta,\,\iota(x(i))}}^{m^{\delta^*}_{\iota(x(i))}(x(i))} \frac{\partial \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}{\partial\alpha} \frac{\partial \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}{\partial m} \frac{1}{\Psi^2_{\lambda\beta\theta,\,\alpha,\,m}(\rho)}\, d\alpha\, dm.\end{aligned} \tag{4.71}$$

We estimate separately the last two lines of (4.71). We start from (3.5). We perform the derivative of the integral with respect to m. This gives a term proportional to $|B|$, which is bad. Using the Taylor formula with an integral rest, we expand in k up to order 1 the term in the integrand that comes from deriving $\Phi(\lambda\beta\theta, \alpha, k)$. Then making computations similar to the ones that we did in (3.47), being aware of the cancellation of the previous linear term in k, we get the leading term of order $|B|k^2$. Performing the Gaussian integral, we get

$$|\partial_m \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)| \le \frac{(1 + c\rho e^{\rho})e^{\rho_4}}{\sqrt{2\pi|B|}\,\sigma_{\lambda\beta\theta}} \frac{2}{\sigma^2_{\lambda\beta\theta}}\left(1 + \frac{4}{\sqrt{g(|B|)}}\right). \tag{4.72}$$

Let us note that in the denominator the term $\sqrt{2\pi|B|}\,\sigma_{\lambda\beta\theta}$ will be cancelled out by the corresponding term in $\Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)$ [see (3.35)] when estimating the ratios in (4.71).

For the derivative with respect to α, we proceed in a similar way. It can be checked that the linear term in k is not present here and the result is

$$|\partial_\alpha \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)| \le \frac{(1+c\varepsilon e^{\rho})8\beta\theta e^{\rho_4}}{\sqrt{2\pi|B|}\,\sigma_{\lambda\beta\theta}} \frac{c}{\sigma^2_{\lambda\beta\theta}} \left(1 + \frac{1}{\sqrt{g(|B|)}}\right). \tag{4.73}$$

For the second order derivative, we get a term proportional to $|B|$ and another to $|B|^2$, this last one being really dangerous. The one proportional to $|B|$ is treated as previously. For the one proportional to $|B|^2$, we expand up to the fourth order in k all the integrand except the exponential terms. By making explicit computations, similar to the one we did in (3.47), all the terms of order strictly less than 4 in k give a zero contribution. The result is

$$|\partial_m \partial_\alpha \Psi_{\lambda\beta\theta,\,\alpha,\,m}(\rho)| \le \frac{\beta\theta}{\sqrt{2\pi|B|}\,\sigma_{\lambda\beta\theta}} \frac{c}{\sigma^4_{\lambda\beta\theta}} \left(1 + \frac{c}{\sqrt{g(|B|)}}\right) \tag{4.74}$$

for some positive constant c. Recalling (4.71), and using (4.72), (4.73), (4.74) together with (3.35) we get, for some positive constant c,

$$\begin{aligned} &\Delta^3_0 \mathscr{G}\big(m^{\delta^*}(x(i)), \rho\big) \\ &\quad \le |\alpha^h - \alpha^{\tilde h_i}|\, \|m^{\delta^*}_{\iota(x(i))} - m^{\delta^*}_{\beta,\,\iota(x(i))}\|_1 \frac{c\beta\theta}{\sigma^4_{\lambda\beta\theta}} \left(1 + \frac{c}{\sqrt{g(|B|)}}\right). \end{aligned} \tag{4.75}$$

Using now the fact that $|\alpha^h - \alpha^{\tilde h_i}| \le |B|^{-1}$ and that $\sigma^2_{\lambda\beta\theta} \ge cg(|B|)|B|^{-1}$, we have

$$\frac{1}{|B|\sigma^4_{\lambda\beta\theta}} \le \frac{|B|}{g^2(|B|)}; \tag{4.76}$$

therefore,

$$\Delta^3_0 \mathscr{G}\big(m^{\delta^*}(x(i)), \rho\big) \le \|m^{\delta^*}_{\iota(x(i))} - m^{\delta^*}_{\beta,\,\iota(x(i))}\|_1 \frac{c\beta\theta|B|}{g^2(|B|)} \left(1 + \frac{c}{\sqrt{g(|B|)}}\right) \tag{4.77}$$

and this ends the proof of Lemma (4.5). □

With Proposition 4.4, we get easily an estimate like (4.53) with

$$\begin{aligned} &f_1(\zeta) \le \|h - \tilde h_i\| \\ &\quad \times \left[2\beta\theta g_1(\zeta)\left(1 + 16\beta\theta + \frac{|B|}{g^2(|B|)}\right) + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}\right] \end{aligned} \tag{4.78}$$

and recalling (4.67),

$$f_2(\zeta) \le \|h - \tilde{h}_i\| \Bigg[8\beta\theta \left(1 + 17\beta\theta + \frac{200}{g(|B|)} + c\frac{g^2(|B|)}{|B|} \right) + \frac{c}{g(|B|)\sqrt{\log g(|B|)}} \Bigg] \tag{4.79}$$

for some positive constant c.

The presence of both terms $|B|/g^2(|B|)$ and $g^2(|B|)/|B|$ suggests taking $g(|B|) = \sqrt{|B|/g_2(1/\zeta)}$ for some function $g_2(x)$ that diverges with x but is slowly varying at infinity. Assuming that ζ is such that $1/\sqrt{|B|} \le g_1(\zeta)\sqrt{g_2(1/\zeta)}$ and choosing $g_1(\zeta) = \sqrt{\zeta/2\beta\theta g_2(1/\zeta)}$, recalling (4.56), we get, if g_2 satisfies also $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta) = 0$,

$$\left\| \partial_i \log \frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})} \right\|_\infty \le 8\sqrt{2\beta\theta\zeta g_2(1/\zeta)}\,. \tag{4.80}$$

Then we apply Lemma 4.2 and we end the proof of Proposition 4.1. □

5. Some deviations estimates and proof of Theorems 2.4 and 2.5. In the previous section, we have used the fact that the difference between the stochastic contribution computed on the profiles constantly equal to one minimum and the one computed on the other minimum has mean zero. In this section, we consider profiles that are nonconstant and make arbitrary oscillations so that in general we lose the mean zero property. Roughly speaking, there are basically three kinds of possible oscillations that we expect to be unlikely. The first one is when the system stays out of the equilibria for a too long interval. The second one is when the system jumps from one equilibrium to the other one, stays there for a too short interval and comes back to the first equilibrium. The third one is when the system makes too many oscillations around one equilibrium without reaching the other one. We have to be careful since without "too long," "too short" and "too many," the previous oscillations could by typical of the Gibbs measure.

To prove Theorem 2.4, we first consider the case where such oscillations occur on macroscopic intervals Δ that are not bigger than $\sqrt{\log\log 1/\gamma}$. In this case, our estimates will be true on a subset, say $\hat{\Omega} \subset \Omega$ of $\mathbb{P}$-probability 1, uniformly with respect to all the possible positions of such intervals Δ inside a bigger interval $\mathcal{I}$ centered at the origin, of macroscopic length γ^{-k}, for any given k. A priori we have to consider only the case $|\mathcal{I}| \approx \gamma^{-1}(\log 1/\gamma)^p$, $p > 1$; however, when it is possible, we consider $|\mathcal{I}| = \gamma^{-2}$, that is, γ^{-3} in microscopic units. However, while for the first and third type of oscillations, it will be enough to estimate them in an interval not bigger than $\sqrt{\log\log 1/\gamma}$, since being outside of equilibria or fluctuating around one equilibrium for "too long" is very unlikely and can be detected already in the scale $\sqrt{\log\log 1/\gamma}$, for the second type of oscillations we must be more careful.

Namely, we have to distinguish when being close to one equilibrium is typical and when it is not. This requires analyzing the system over longer intervals and controlling the contribution of the magnetic field and the entropy terms over intervals where the estimates used in the scale $\sqrt{\log\log 1/\gamma}$ will give a too large contribution.

Let Δ_R be a macroscopic interval of length $R \in \mathbb{N}$ and δ_1, ζ_1 be two positive real numbers. Let $\mathscr{O}_0^{\delta_1, \zeta_1}(\Delta_R) \equiv \{\eta^{\delta_1, \zeta_1}(l) = 0, \forall\, l \in \Delta_R \cap \mathbb{Z}\}$, then our first result is the following.

PROPOSITION 5.1. *There exists an absolute positive constant c such that given $\beta > 1$ and $\beta\theta$ that satisfies* (2.22), *there exists a positive constant $c(\beta, \theta)$, such that for all $\delta_1 > \delta^* > 0$, $\zeta_1 > 0$ and $z_1 > 0$, we can find $\Omega_1 = \Omega_1(\gamma, \delta^*, \delta_1, \zeta_1, z_1, \Delta_R) \subset \Omega$ such that on Ω_1,*

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\big(\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R)\big)\\
&\quad \le \exp\Bigg(-\frac{\beta}{\gamma}\Bigg[c(\beta,\theta)\zeta_1^3\delta_1 R - 4 - 2cR\Bigg(\delta^* + \frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}\Bigg)\\
&\qquad\qquad -2R\theta\sqrt{\frac{\gamma}{\delta^*}} - \sqrt{R\gamma}\,4\theta z_1\Bigg]\Bigg)
\end{aligned} \tag{5.1}$$

and $\mathbb{P}[\Omega_1] \ge 1 - \exp(-z_1^2/64)$.

PROOF. By the very same argument that leads to (4.2) we have

$$\mu_{\beta,\theta,\gamma}\big(\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R)\big) \le \exp\left(4\frac{\beta}{\gamma}\right)\mu_{\beta,\theta,\gamma}\big(\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R) \mid \Sigma_{\partial\Delta_R}\big)(0). \tag{5.2}$$

Performing a block spin transformation on the scale δ^*, recalling (3.13), we have

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\big(\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R) \mid \Sigma_{\partial\Delta_R}\big)(0)\\
&\quad = \frac{\exp\big((\pm\beta\delta^*\gamma^{-1}R)\big)}{Z_{\beta,\theta,\gamma,\Delta_R}(0)} \sum_{m^{\delta^*}(\Delta_R)\in\mathscr{M}_{\delta^*}(\Delta_R)} \mathbb{1}_{\{\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R)\}}\\
&\qquad \times\exp\left(-\frac{1}{\gamma}\Big\{\beta\hat{\mathscr{F}}\big(m_{\Delta_R}^{\delta^*},0\big) + \gamma\mathscr{G}\big(m_{\Delta_R}^{\delta^*}\big)\Big\}\right).
\end{aligned} \tag{5.3}$$

To estimate the stochastic part, we make a rough upper bound [see (3.15) and (3.16)] $|\mathscr{G}_{x,m^{\delta^*}}(x)(\lambda(x))| \le 2\beta\theta|D(x)|$ which corresponds to the situation where all the spins in $D^\lambda(x)$ are equal to $-\lambda(x)$. This gives us a factor,

$$\Xi(2\beta\theta, \Delta_R) \equiv \exp\Bigg\{\sum_{x\in\mathscr{C}_{\delta^*}(\Delta_R)} 2\beta\theta|D(x)|\Bigg\} \tag{5.4}$$

that we extract from the numerator in the left-hand side of (5.3).

To estimate the combinatorial factor that appears in $\hat{\mathscr{F}}$ [see (3.10)], we use the Stirling formula in the form given by Robbins [27] which is $\forall N \geq 1$, $N! = \sqrt{2\pi} N^{N+1/2} e^{-N} e^{\varepsilon_N}$ with $1/12N \leq \varepsilon_N \leq 1/(12N+1)$. Let us denote

$$\tilde{\mathscr{F}}\big(m^{\delta^*}_{\Delta_R}\big) = \frac{\delta^*}{2} \sum_{(x,y)\in \mathscr{C}^2_{1/\delta^*}(\Delta_R)} J_{\delta^*}(x-y)\big[\tilde{m}^{\delta^*}(x) - \tilde{m}^{\delta^*}(y)\big]^2 + \delta^* \sum_{x\in \mathscr{C}_{\delta^*}(\Delta_R)} f_{\beta,\theta}\big(m^{\delta^*}(x)\big), \tag{5.5}$$

where $f_{\beta,\theta}$ is the canonical free energy of the RFCW model [see (2.18)]. It is easy to see that restricting the configurations to those that are constantly equal to $m^{\delta^*}_\beta$, where $m^{\delta^*}_\beta$ is the nearest point to m_β belonging to the set $[-1, -1+4\gamma/\delta^*, -1+8\gamma/\delta^*, \dots, 1-4\gamma/\delta^*, 1]^2$, we get a lower bound for the normalization factor $Z_{\beta,\theta,\gamma,\Delta_R}(0)$. On the other hand, using the fact that

$$\sum_{(m^{\delta^*}(\pm 1, x))_{x\in \mathscr{C}_{\delta^*}(\Delta_R)}} 1 \leq \left(\frac{\delta^*}{2\gamma}\right)^{2R/\delta^*} = \exp\left(\frac{2R}{\delta^*}\log\frac{\delta^*}{2\gamma}\right) \tag{5.6}$$

to control the number of terms that occurs in the sum in (5.3), after the cancellation of some constants, we get

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\big(\mathscr{O}^{\delta_1,\zeta_1}_0(\Delta_R) \mid \Sigma_{\partial\Delta_R}\big)(0) \\ &\quad\leq \exp\left(\frac{\beta}{\gamma}\left(R\delta^* + 4 + 2R\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}\right)\right)\Xi(4\beta\theta, \Delta_R) \\ &\qquad \times \exp\left(-\frac{\beta}{\gamma}\inf_{m^{\delta^*}_{\Delta_R}\in \mathscr{O}^{\delta_1,\zeta_1}_0}\big\{\mathscr{F}\big(m^{\delta^*}_{\Delta_R}\big)\big\}\right), \end{aligned} \tag{5.7}$$

where $\mathscr{F}(m^{\delta^*}_{\Delta_R}) \equiv \tilde{\mathscr{F}}(m^{\delta^*}_{\Delta_R}) - \tilde{\mathscr{F}}(m^{\delta^*}_{\beta,\Delta_R})$.

To give a lower bound on the previous infimum, we use the fact that if x_i are positive numbers, bounded from above by a constant c, then if the arithmetic mean of N terms x_i is bounded from below by some $\zeta_1 \leq c$ there are at least $N\zeta_1/(2c-\zeta_1)$ terms x_i among the N, such that $x_i > \zeta_1/2$. Using (2.26) we get after some easy computations,

$$\inf_{m^{\delta^*}_{\Delta_R}\in \mathscr{O}^{\delta_1,\zeta_1}_0}\big\{\mathscr{F}\big(m^{\delta^*}_{\Delta_R}\big)\big\} \geq Rc(\beta,\theta)\zeta_1^3\delta\frac{1}{4(4-\zeta_1)^2} \geq Rc_1(\beta,\theta)\zeta_1^3\delta. \tag{5.8}$$

It remains to estimate $\Xi(4\beta\theta, \Delta_R)$. Let us denote $X(\Delta_R) \equiv 4\gamma\sum_{x\in\mathscr{C}_{\delta^*}(\Delta_R)} |D(x)|$. It is easy to see that $\mathbb{E}(X(\Delta_R)) \leq cR\sqrt{\gamma/\delta^*}$. Using Lemma 4.2, setting $t = 2\sqrt{R\gamma z_1}$, where z_1 is a positive real number, and regrouping, we get (5.1). □

With Proposition 5.1 we can control the Gibbs probability to have a run of $\eta^{\delta_1,\zeta_1} = 0$ anywhere on intervals that are rather long. However, their lengths depend on the parameters $\delta_1, \zeta_1, \delta^*$.

COROLLARY 5.2. *Given* $\beta > 1$ *and* $\beta\theta$ *that satisfies* (2.22), *then there exists a constant* $\tilde{c} = \tilde{c}(\beta, \theta)$ *such that, if* $\delta^* \log(1/\gamma) \downarrow 0$ *when* $\gamma \downarrow 0$, *for all* $\delta_1 > \delta^* > 0$, $\zeta_1 > 0$, *that satisfy*

$$\delta_1 \zeta_1^3 \geq \tilde{c}(\beta, \theta)\left(\sqrt{\frac{\gamma}{\delta^*}} \vee \delta^*\right) \tag{5.9}$$

for all $x > 0$, *for all intervals* Δ_R *of macroscopic length* R *that are included in a macroscopic interval* I *containing the origin, with* $|I| \leq \gamma^{-2}$ *and satisfy*

$$R \geq R_1 \equiv \frac{4\beta(1 + x)}{c(\beta, \theta)\delta_1 \zeta_1^3} \tag{5.10}$$

if $\gamma = 2^{-n}$, *with* $\mathbb{P}$-*probability* 1, *for all but a finite number of indices* n,

$$\mu_{\beta, \theta, \gamma}\big(\exists R\colon R_1 \leq |R| \leq |I| \,\exists \Delta_R \subset I\colon \mathcal{O}_0^{\delta_1, \zeta_1}(\Delta_R)\big) \leq \exp\left(-\frac{4\beta x}{\gamma}\right). \tag{5.11}$$

PROOF. Let us first remark that for a given R, the number of intervals Δ_R that are included in I is bounded from above by $|I|^2$; therefore if we take $z_1 = \sqrt{64(5 + \varepsilon)\log(1/\gamma)}$ for some positive ε, we get, using Lemma 4.2,

$$\begin{aligned} \mathbb{P}\Bigg[& \sup_{R\colon R_1 \leq |R| \leq |I|} \sup_{\Delta_R \subset I} \frac{1}{\sqrt{R}}\big(X(\Delta_R) - \mathbb{E}[X(\Delta_R)]\big) \\ & \geq \sqrt{64(5 + \varepsilon)\gamma \log\left(\frac{1}{\gamma}\right)}\Bigg] \leq \gamma^{1+\varepsilon} \end{aligned} \tag{5.12}$$

The $\mathbb{P}$-probabilistic statement follows from the first Borel–Cantelli lemma. Let us consider the term in the bracket in the exponent in the right-hand side of (5.1). Notice first that, since $(\delta^*/\gamma)\uparrow\infty$ when $\gamma \downarrow 0$, $\sqrt{\gamma/\delta^*} \geq (\gamma/\delta^*)\log(\delta^*/\gamma)$, if γ small enough, we can ignore the corresponding term in (5.1) and keep just the square root. To get a negative term in this exponent, we impose, since $\beta\theta$ is small,

$$c(\beta, \theta)\zeta_1^3\delta_1 - 4\left(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}\right) - 256(5 + \varepsilon)\beta\theta\sqrt{\gamma \log\frac{1}{\gamma}} \geq 0. \tag{5.13}$$

Using $\delta^* \log(1/\gamma) \downarrow 0$ when $\gamma \downarrow 0$, this becomes $c(\beta, \theta)\zeta_1^3\delta_1 - 4c(\delta^* \vee \sqrt{\gamma/\delta^*}) \geq 0$ by enlarging the constant c if necessary. To cancel the constant term 4β, in (5.1) and get the factor x in (5.11) we just impose (5.10). □

The second family of events we consider are, roughly speaking, those having two blocks, far apart but not too much, at the same equilibrium and somewhere between them there is a block of macroscopic length at least 1, close to the other equilibrium.

Let $\Delta_L = [l_1, l_2]$ with $l_i \in \mathbb{Z}$ for $i = 1, 2$ be a macroscopic interval of length L, and $\delta_2 > 0$, $\zeta_2 > 0$ be two real positive numbers; let us define for $\eta = +1$ or $\eta = -1$,

$$\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta) \equiv \left\{\eta^{\delta_2, \zeta_2}(l_1) = \eta^{\delta_2, \zeta_2}(l_2) = \eta, \exists\, \tilde{l}, l_1 < \tilde{l} < l_2, \eta^{\delta_2, \zeta_2}(\tilde{l}) = -\eta\right\} \tag{5.14}$$

and $\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L) \equiv \mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, +) \cup \mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, -)$. Our second result is the following.

PROPOSITION 5.3. *Given $\beta > 1$ and $\beta\theta$ that satisfies* (2.22), $\delta_2 > \delta^* > 0$, $\zeta_2 > 0$ *and* $z_2 > 0$, *then there exists* $\Omega_2 = \Omega_2(\gamma, \delta^*, \delta_2, \zeta_2, z_2, \Delta_L) \subset \Omega$ *such that on* Ω_2,

$$\mu_{\beta, \theta, \gamma}\big(\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L)\big) \le \exp\left(-\gamma^{-1}\left[\Delta\mathscr{F} - 4\beta\zeta_2 - 2L\left(\delta^* + \frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}\right) - 2L\beta\theta\sqrt{\frac{\gamma}{\delta^*}} - \sqrt{L\gamma}\,4\beta\theta z_2\right]\right) \tag{5.15}$$

for a strictly positive constant $\Delta\mathscr{F} = \Delta\mathscr{F}(\beta, \theta)$ *and*

$$\mathbb{P}[\Omega_1] \ge 1 - \exp - \big(z_2^2/64\big).$$

PROOF. The proof is similar to that of Proposition 5.1. We point out only the main differences. Let us call $\Delta_L^- = [l_1 + 1, l_2 - 1]$, and for $\eta = \pm 1$,

$$m_{\beta, \eta, \partial\Delta_l} = \left\{m^{\delta^*}(x); \forall\, x \in \mathscr{C}_{\delta^*}(\partial\Delta_L), m^{\delta^*}(x) = T^{1-\eta/2} m_\beta^{\delta^*}\right\}, \tag{5.16}$$

where if $m = (m_1, m_2)$, $T^0 m = m$ and $T^1 m = Tm = (-m_2, -m_1)$. An easy computation, using the fact that $\eta^{\delta_2, \zeta_2} = \eta$ leads to

$$\mu_{\beta, \theta, \gamma}\big(\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta)\big) \le \exp(4\beta c\zeta_2)\, \mu_{\beta, \theta, \gamma}\big(\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta) \mid \Sigma_{\partial\Delta_L}\big)(m_{\beta, \eta, \partial\Delta_L}). \tag{5.17}$$

Then making a block–spin transformation on the scale δ^* inside the volume Δ_L^-, denoting

$$\mathscr{F}\Big(m_{\Delta_L^-}^{\delta^*}, m_{\partial\Delta_L}^{\delta^*}\Big) = \mathscr{F}\Big(m_{\Delta_L^-}^{\delta^*}\Big) + \frac{\delta^*}{2} \sum_{\substack{x \in \mathscr{C}_{\delta^*}(\Delta_L^-) \\ y \in \mathscr{C}_{\delta^*}(\partial\Delta_L)}} J_{\delta^*}(x - y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y) \tag{5.18}$$

and, using the same arguments that lead to (5.7), give

$$\mu_{\beta, \theta, \gamma}\big(\mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta) \mid \Sigma_{\partial\Delta_L}\big)(m_{\beta, \eta, \partial\Delta_L}) \le \exp\left(\frac{\beta}{\gamma}\left(L\delta^* + 4\zeta_2 + 2L\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}\right)\right)\Xi(4\beta\theta, \Delta_L) \times \exp\left(-\frac{\beta}{\gamma}\inf_{m_{\Delta_L}^{\delta^*} \in \mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta)}\Big\{\mathscr{F}\Big(m_{\Delta_L}^{\delta^*}, m_{\beta, \eta, \partial\Delta_L}\Big)\Big\}\right). \tag{5.19}$$

It is not too difficult to check that there exists a constant $\Delta\mathscr{F} = \Delta\mathscr{F}(\beta, \theta)$, depending neither on $\eta = \pm 1$ nor on L, which is strictly positive if $\beta > 1$ and $\beta\theta$ satisfies (2.22), such that

$$\inf_{m^{\delta^*}_{\Delta_L} \in \mathscr{W}^{\delta_2, \zeta_2}(\Delta_L, \eta)} \left\{ \mathscr{F}\left(m^{\delta^*}_{\Delta_L}, m_{\beta, \eta, \partial\Delta_L} \right) \right\} \geq \Delta\mathscr{F}. \tag{5.20}$$

Now $\Xi(4\beta\theta, \Delta_L)$ can be estimated as before and this ends the proof of Proposition 5.3. □

By similar computations to the proof of Corollary 5.2, making the choice $z_2 = z_1$ it is easy to check the following corollary.

COROLLARY 5.4. *There exists a constant $\tilde{c} = \tilde{c}(\beta, \theta)$ such that, if $\delta^* \log(1/\gamma) \downarrow 0$ where $\gamma \downarrow 0$, for all $\delta_2 > \delta^* > 0$, $\zeta_2 > 0$, for all $x > 0$, that satisfies*

$$\Delta\mathscr{F}(1 - x) - \tilde{c}(\beta, \theta)\zeta_2 > 0 \tag{5.21}$$

for all intervals Δ_L of macroscopic length L that are included in an interval I that contains the origin, with $|I| \leq \gamma^{-2}$ and satisfy

$$L \leq L_2 \equiv \frac{\Delta\mathscr{F}(1 - x) - \tilde{c}(\beta, \theta)\zeta_2}{c(\beta, \theta)\left(\delta^* \vee \sqrt{\gamma/\delta^*}\right)} \tag{5.22}$$

if $\gamma = 2^{-n}$, with $\mathbb{P}$-probability 1, for all but a finite number of indices n,

$$\mu_{\beta, \theta, \gamma}\left(\exists\, L\colon 2 \leq |L| \leq L_2\, \exists\, \Delta_L \subset I\colon \mathscr{W}^{\delta_2, \zeta_2}(\Delta_L)\right) \leq \exp\left(-\frac{\beta x \Delta\mathscr{F}}{\gamma}\right). \tag{5.23}$$

The third family of events describes fluctuations around one equilibrium.

Let $\Delta_L = [l_1, l_2]$ with $l_i \in \mathbb{Z}$ for $i = 1, 2$ be a macroscopic interval of length L and $\delta_4 > \delta_1 > 0$, $\zeta_4 > \zeta_1 > 0$ be four real positive numbers. Let us define for $\eta = +1$ or $\eta = -1$,

$$\begin{aligned} &\mathscr{R}^{\delta_1, \zeta_1, \delta_4, \zeta_4}_{0, \eta}\left(\Delta_L, \tilde{L}\right) \\ &\quad \equiv \Big\{ \eta^{\delta_1, \zeta_1}(l_1) = \eta^{\delta_1, \zeta_1}(l_2) = \eta, \forall\, l \in (l_1, l_2), \eta^{\delta_1, \zeta_1}(l) = 0, \\ &\qquad \exists\, \tilde{l}_1, \tilde{l}_2, \tilde{l}_2 - \tilde{l}_1 = \tilde{L}, \\ &\qquad l_1 < \tilde{l}_1 < \tilde{l}_2 \leq l_2, \eta^{\delta_4, \zeta_4}(\tilde{l}) = 0\, \forall\, \tilde{l}\colon \tilde{l}_1 \leq \tilde{l} \leq \tilde{l}_2 \Big\} \end{aligned} \tag{5.24}$$

and $\mathscr{R}^{\delta_1, \zeta_1, \delta_4, \zeta_4}_{0}(\Delta_L, \tilde{L}) \equiv \mathscr{R}^{\delta_1, \zeta_1, \delta_4, \zeta_4}_{0,+}(\Delta_L, \tilde{L}) \cup \mathscr{R}^{\delta_1, \zeta_1, \delta_4, \zeta_4}_{0,-}(\Delta_L, \tilde{L})$.

PROPOSITION 5.5. *Given $\beta > 1$ and $\beta\theta$ that satisfies* (2.22), $\delta_4 > \delta_1 > \delta^*$, $\zeta_4 > \zeta_1 > 0$ *and $z_3 > 0$, then there exists*

$$\Omega_3 = \Omega_3\left(\gamma, \delta^*, \delta_1, \delta_4, \zeta_1, \zeta_4, z_3, \Delta_L, \tilde{L}\right)$$

such that on Ω_3,

$$\mu_{\beta,\theta,\gamma}\Big(\mathscr{R}_{0,\eta}^{\delta_1,\zeta_1,\delta_4,\zeta_4}\big(\Delta_L,\tilde{L}\big)\Big)$$

$$\leq \exp\Bigg(-\gamma^{-1}\Bigg[c(\beta,\theta)\big(\zeta_4^3\delta_4\tilde{L}+\zeta_1^3\delta_1(L-\tilde{L})\big)-4\beta\zeta_1 -2L\bigg(\delta^*+\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}\bigg)-2L\beta\theta\sqrt{\frac{\gamma}{\delta^*}}-\sqrt{L\gamma}\,4\beta\theta z_3\Bigg]\Bigg) \tag{5.25}$$

for some positive constants $c(\beta,\theta)$ *and* c *and* $P[\Omega_3]\geq 1-\exp(-z_3^2/64)$.

The proof is similar to the proofs of Propositions 5.1 and 5.3.

An immediate consequence of this result is the following corollary.

COROLLARY 5.6. *Given* $\beta>1$ *and* $\beta\theta$ *that satisfies* (2.22), *there exist two constants* $\tilde{c}_i=\tilde{c}_i(\beta,\theta)$ *for* $i=1,2$ *such that if* $\delta^*\log(1/\gamma)\downarrow 0$ *when* $\gamma\downarrow 0$, *for all* $\delta_4>\delta_1>\delta^*$, $\zeta_4>\zeta_1>0$ *that satisfy*

$$\delta_4\zeta_4^3\geq\delta_1\zeta_1^3\geq\tilde{c}_1\bigg(\sqrt{\frac{\gamma}{\delta^*}}\vee\gamma\log\frac{1}{\gamma}\bigg) \tag{5.26}$$

and $\delta_4\zeta_4^3\geq\tilde{c}_1\zeta_1$, *for all* $1>x>0$, *for all intervals* Δ_L *of macroscopic length* L *that are included in an interval* I *that contains the origin, with* $|I|\leq\gamma^{-2}$ *if* $\gamma=2^{-n}$, *with* $\mathbb{P}$-*probability* 1, *for all but a finite number of indices* n, *for all* $\tilde{L}\geq 1$,

$$\mu_{\beta,\theta,\gamma}\Big(\exists\, L\colon 2\leq L\leq|I|\,\exists\,\Delta_L\subset I\colon \mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}\big(\Delta_L,\tilde{L}\big)\Big) \leq\exp\bigg(-\frac{\tilde{c}_2(\beta,\theta)\tilde{L}x\zeta_4^3\delta_4}{\gamma}\bigg). \tag{5.27}$$

Therefore if we denote

$$\mathscr{O}_0^{\delta_1,\zeta_1}(I)\equiv\bigcup_{R\colon R_1\leq R\leq|I|}\ \bigcup_{\Delta_R\subset I}\mathscr{O}_0^{\delta_1,\zeta_1}(\Delta_R), \tag{5.28}$$

$$\mathscr{W}^{\delta_2,\zeta_2}(I)\equiv\bigcup_{L\colon 2\leq L\leq L_2}\ \bigcup_{\Delta_L\subset I}\mathscr{W}^{\delta_2,\zeta_2}(\Delta_L) \tag{5.29}$$

and

$$\mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}(I)\equiv\bigcup_{L\colon 2\leq L\leq|I|}\ \bigcup_{\Delta_L\subset I}\ \bigcup_{\tilde{L}\colon 1\leq\tilde{L}\leq L}\mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}\big(\Delta_L,\tilde{L}\big), \tag{5.30}$$

then, for an appropriate choice of various parameters, δ_i,ζ_i for $i\colon 1\leq i\leq 4$, as a consequence of Corollaries 5.2, 5.4 and 5.6, all the previous sets have a Gibbs probability that goes to zero, $\mathbb{P}$-almost surely. It is convenient to make the choices $\delta_1=\delta_2$, $\zeta_2=\zeta_1$, $\zeta_4>\zeta_1$ and $\delta_4\geq\delta_1$. We note that $\eta^{\delta_1,\zeta_1}(l)=\eta$ implies $\eta^{\delta_4,\zeta_4}(l)=\eta$. Therefore, on the complementary of the unions of the previous sets we can only have runs of length at most R_1 of $\eta^{\delta_1,\zeta_1} = 0$ followed by runs of length at least L_2 of equilibrium $\eta^{\delta_4,\zeta_4}(l) = \eta$.

Namely, blocks $\eta^{\delta_1,\zeta_1}(l) = 0$ between adjacent blocks of the same equilibrium can be only $\eta^{\delta_4,\zeta_4} = \eta$, since (5.27).

The next step is to prove that the length of the previous run of $\eta^{\delta_4,\zeta_4} = \eta$, which is at least L_2, is in fact bounded from below by a much larger quantity.

We define [see (5.14)] for $\eta \in \{+1, -1\}$, $l_1 < \tilde{l}_1 < \tilde{l}_2 < l_2$ with $2 \le \tilde{l}_1 - l_1 \le R_1$, $l_2 - \tilde{l}_2 \le R_1$,

$$
\begin{aligned}
&\tilde{\mathscr{W}}_\eta^{\delta_4,\zeta_4}\big(l_1, \tilde{l}_1, \tilde{l}_2, l_2\big)\\
&\quad\equiv \Big\{\eta^{\delta_4,\zeta_4}(l_1) = \eta^{\delta_4,\zeta_4}(l_2) = \eta, \eta^{\delta_4,\zeta_4}\big(\tilde{l}_1 - 1\big)\\
&\quad\quad = \eta^{\delta_4,\zeta_4}\big(\tilde{l}_1\big) = -\eta,\\
&\quad\quad \eta^{\delta_4,\zeta_4}(l) = -\eta \ \forall\, l\colon \tilde{l}_1 + 1 \le l \le \tilde{l}_2 - 1,\\
&\quad\quad \eta^{\delta_4,\zeta_4}\big(\tilde{l}_2\big) = \eta^{\delta_4,\zeta_4}\big(\tilde{l}_2 + 1\big) = -\eta\Big\}.
\end{aligned}
\tag{5.31}
$$

In the following proposition we will show that, uniformly in the choices of $\tilde{l}_1$, $\tilde{l}_2$, l_1 and l_2 in a fixed interval $\mathscr{I}$ of suitable length, this set of events has small probability.

PROPOSITION 5.7. *Given $\beta > 1$, $0 < x < 1$, $p > 1$, $\hat{c} > 0$, $\rho > 0$, if $\theta \le x^2\Delta\mathscr{F}/48\sqrt{\hat{c}(p+1+2\rho)}$ then there exist $\gamma_0 > 0$ and $c_0 > 0$ such that for $\gamma \le \gamma_0$, if $\zeta_4 g_2(1/\zeta_4) \le (x^2\Delta\mathscr{F}/96)(1 \wedge \beta/\sqrt{\hat{c}(p+1+2\rho)})$, for all $\delta_4 > \delta^* = c_0\gamma \log\log(1/\gamma)$, for all intervals $I = [l_1, l_2]$ such that $|I| \le \hat{c}(\gamma \log\log(1/\gamma))^{-1}$, and for any $I \subset \mathscr{I}$, $|\mathscr{I}| = \tilde{c}\gamma^{-1}(\log 1/\gamma)^p$ for some positive constant $\tilde{c}$, on a set $\Omega_4 = \Omega_4(\mathscr{I}, \beta, \theta, \gamma)$ that satisfies*

$$
\mathbb{P}[\Omega_4] \ge 1 - \frac{2\tilde{c}}{\hat{c}}\left(\log\frac{1}{\gamma}\right)^{p+\rho} \exp\left(-\left(\log\log\frac{1}{\gamma}\right)(p + 2\rho + 1)\right),
\tag{5.32}
$$

we have, uniformly on all intervals $[\tilde{l}_1, \tilde{l}_2] \subset I$ and uniformly on $I \subset \mathscr{I}$,

$$
\mu_{\beta,\theta,\gamma}\Big(\tilde{\mathscr{W}}_\eta^{\delta_4,\zeta_4}\big(l_1, \tilde{l}_1, \tilde{l}_2, l_2\big)\Big) \le \exp\left[-\frac{\beta}{\gamma}x(1-x)\Delta\mathscr{F}\right]
\tag{5.33}
$$

for $\eta = \pm 1$.

PROOF. The first step is to restrict ourselves to a finite volume Gibbs measure. Since $\eta^{\delta_4,\zeta_4}(l_1) = \eta^{\delta_4,\zeta_4}(l_2) = \eta$, we get

$$
\begin{aligned}
&\mu_{\beta,\theta,\gamma}\Big(\tilde{\mathscr{W}}_\eta^{\delta_4,\zeta_4}\big(l_1, \tilde{l}_1, \tilde{l}_2, l_2\big)\Big)\\
&\quad \le \exp\left(4\beta\frac{\zeta_4}{\gamma}\right)\mu_{\beta,\theta,\gamma}\Big(\tilde{\mathscr{W}}_\eta^{\delta_4,\zeta_4}\big(l_1, \tilde{l}_1, \tilde{l}_2, l_2\big) \mid \Sigma_{\partial\Delta_L}\Big)(0).
\end{aligned}
\tag{5.34}
$$

Using the fact that $\eta^{\delta_4,\zeta_4}(\tilde{l}_1) = \eta^{\delta_4,\zeta_4}(\tilde{l}_1 - 1)$ and $\eta^{\delta_4,\zeta_4}(\tilde{l}_2 + 1) = \eta^{\delta_4,\zeta_4}(\tilde{l}_2)$ we can also decouple the interval $[\tilde{l}_1 - 1, \tilde{l}_2 + 1]$ from the interval $[l_1, l_2]$. This will produce three adjacent intervals. We associate, the interaction between the first and the second interval to the first term and the interaction between the second and the third interval to the third term. This will give, up to a factor $\exp(8\beta(\zeta_4/\gamma))$, a product of three terms, each one being localized on one of the three intervals. We make a rough estimate for the random magnetic field for the terms corresponding to the first and the third interval. Applying an argument similar to the one given in Corollary 5.4, we get that, with a $\mathbb{P}$-probability 1, uniformly with respect to all intervals $[\tilde{l}_1, \tilde{l}_2]$ included in an interval $\mathscr{I}$ containing the origin, with $|\mathscr{I}| \le 1/\gamma^2$,

$$\mu_{\beta,\theta,\gamma}\Big(\tilde{\mathscr{W}}_{\eta}^{\delta_4,\zeta_4}\big(l_1, \tilde{l}_1, \tilde{l}_2, l_2\big) \mid \Sigma_{\partial\Delta_L}\Big)(0)$$

$$\le \exp\left(12\beta\frac{\zeta_4}{\gamma}\right)\exp\left(-\frac{\beta x \Delta\mathscr{F}}{\gamma}\right) \tag{5.35}$$

$$\times \exp\Big(\beta\gamma^{-1}\Big[\delta^*\big(\tilde{l}_2 - \tilde{l}_1\big)\Big]\Big)\frac{Z_{-\eta,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}{Z_{\eta,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)},$$

where the last term is similar to the one defined in (4.5), with $\mathscr{R}(\tau) = \mathscr{R}^{\delta_4,\zeta_4}(\tilde{l}_1, \tilde{l}_2, \tau)$. Writing in way similar to (4.10), with self-explanatory notations, we have

$$\frac{Z_{-\eta,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}{Z_{\eta,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)} = \exp\Big(\Delta\mathscr{G}\big(m_{\beta,\tilde{I}_{12}}^{\delta^*}, \varepsilon\big)\Big)\frac{Z_{-\eta,0,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}{Z_{\eta,0,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}. \tag{5.36}$$

Using the estimate (4.28) we get

$$\mathbb{P}\left[\left|\log\frac{Z_{-\eta,0,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}{Z_{\eta,0,\delta_4,\zeta_4}\big(\tilde{I}_{12}\big)}\right| \ge \frac{\varepsilon}{\gamma}\right] \le \exp\left(-\frac{\varepsilon^2}{212\gamma|\tilde{I}_{12}|\beta\theta\zeta_4 g_2(1/\zeta_4)}\right). \tag{5.37}$$

To get a result which is true *uniformly* with respect to all subintervals $\tilde{I}_{12}$ of I, and for any I in a given interval $\mathscr{I}$ of length $\tilde{c}(\gamma)^{-1}(\log 1/\gamma)^p$ containing the origin, we need a modification of the Ottaviani inequality [31] that takes into account that we do not have a sum of random variables, that is, not an additive process but merely an approximate additive process.

To simplify notations, given an interval $\tilde{I} \subset I$, let us call $Y(\tilde{I}) \equiv \log(Z_{-\eta,0,\delta_4,\zeta_4}(\tilde{I})/Z_{\eta,0,\delta_4,\zeta_4}(\tilde{I}))$.

LEMMA 5.8. *For any given interval I,*

$$\mathbb{P}\left[\max_{\tilde{I}_{1,2}\subset I}|Y\big(\tilde{I}_{12}\big)| \ge \beta\frac{4\varepsilon + 12\zeta_4}{\gamma}\right] \le \frac{\mathbb{P}\big[|Y(I)| \ge \beta(\varepsilon/\gamma)\big]}{\inf_{\tilde{I}_{12}\subset I}\mathbb{P}\Big[|Y\big(\tilde{I}_{12}\big)| \le \beta(\varepsilon/\gamma)\Big]}. \tag{5.38}$$

PROOF. Recall that $[l_1, l_2] \equiv I$ and intervals $\tilde{I}_{12} = [\tilde{l}_1, \tilde{l}_2]$. Using the fact that for all $\tilde{I}_{12} \subset I$, $|Y(\tilde{I}_{12})| \le |Y([l_1, \tilde{l}_1])| + |Y([l_1, \tilde{l}_2])| + \beta(4\zeta_4/\gamma)$, we get

$|Y(\tilde{I}_{12})| \le 2\max_{l_1 \le \tilde{l} \le l_2}|Y([l_1, \tilde{l}])| + \beta(4\zeta_4/\gamma)$. Therefore,

$$\begin{aligned}\mathbb{P}\left[\max_{\tilde{I}_{1,2}\subset I}|Y(\tilde{I}_{12})| \ge \beta\frac{4\varepsilon + 12\zeta_4}{\gamma}\right] \\ \le \mathbb{P}\left[\max_{l_1\le\tilde{l}\le l_2}|Y([l_1, \tilde{l}])| \ge \beta\frac{2\varepsilon + 4\zeta_4}{\gamma}\right].\end{aligned} \tag{5.39}$$

Let $\tau = \inf\{t \ge l_1; |Y([l_1, t])| \ge \beta(2\varepsilon + 4\zeta_4/\gamma)\}$, $\inf(\emptyset) = \infty$. Since, for all $k \in [l_1, l_2]$, $|Y(I)| \ge |Y([l_1, k])| - |Y([k+1, l_2])| - \beta(4\zeta_4/\gamma)$, we have

$$\{\tau = k\} \cap \left\{|Y([k+1, l_2])| \le \beta\frac{\varepsilon}{\gamma}\right\} \subset \left\{|Y(I)| \ge \beta\frac{\varepsilon}{\gamma}\right\}. \tag{5.40}$$

Therefore, making a partition over the possible values of τ and using independence, we get

$$\mathbb{P}\left[|Y(I)| \ge \beta\frac{\varepsilon}{\gamma}\right] \ge \inf_{l_1\le k\le l_2}\mathbb{P}\left[|Y([k+1, l_2])| \le \beta\frac{\varepsilon}{\gamma}\right]\sum_{k=l_1}^{l_2}\mathbb{P}[\tau = k]. \tag{5.41}$$

Using the definition of τ, we get (5.38). □

We assume without loss of generality that $\mathcal{I}$ is centered at the origin and that $|I| = \hat{c}(1/\gamma\log\log(1/\gamma))$ for a given $\hat{c}$. We make a block decomposition of the interval $\mathcal{I}$ into blocks of length $\hat{c}(2\gamma\log\log(1/\gamma))^{-1}$, that is, $\mathcal{I} = \bigcup_{-j_1\le j\le j_1}\hat{I}_j$ with $2j_1 + 1 = [(2\tilde{c}/\hat{c})(\log 1/\gamma)^p\log\log(1/\gamma)]$. Note that any interval I we consider is included in the union of three consecutive intervals $\hat{I}_{[j,j+2]} \equiv \hat{I}_j \cup \hat{I}_{j+1} \cup \hat{I}_{j+2}$ for some $-j_1 \le j \le j_1 - 2$. Therefore, we get, denoting $\max^*_{I\subset\mathcal{I}}$ the maximum over the intervals I such that $|I| = \hat{c}(\gamma\log\log(1/\gamma))^{-1}$ that are in $\mathcal{I}$, for all $\varepsilon > 0$, setting $\tilde{\varepsilon} = 4\varepsilon + 12\zeta_4$, we have

$$\begin{aligned}\mathbb{P}\left[\max^*_{I\subset\mathcal{I}}\max_{\tilde{I}_{12}\subset I}|Y(\tilde{I}_{12})| \ge \beta\frac{\tilde{\varepsilon}}{\gamma}\right] \\ \le \frac{2\tilde{c}(\log 1/\gamma)^p\log\log(1/\gamma)}{\hat{c}}\mathbb{P}\left[\max_{\tilde{I}_{12}\subset\hat{I}_{[0,2]}}|Y(\tilde{I}_{12})| \ge \beta\frac{\tilde{\varepsilon}}{\gamma}\right].\end{aligned} \tag{5.42}$$

Using (5.37) and (5.38), we have

$$\begin{aligned}\mathbb{P}\left[\max^*_{I\subset\mathcal{I}}\max_{\tilde{I}_{12}\subset I}\left|\log\frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})}\right| \ge \beta\frac{\tilde{\varepsilon}}{\gamma}\right] \\ \le \frac{2\tilde{c}(\log(1/\gamma))^{p+\rho}}{\hat{c}}\frac{\exp(-u\log\log(1/\gamma))}{1 - \exp(-u\log\log(1/\gamma))},\end{aligned} \tag{5.43}$$

where $u \equiv \tilde{\varepsilon}^2\beta^2/212\hat{c}\beta\theta\zeta_4 g_2(1/\zeta_4)$ and $\rho > 0$ is small as we want. We assume for the moment that the various parameters are chosen such that $u \ge p + 1 + 2\rho$. Using the first Borel–Cantelli lemma, recalling that $\gamma = 2^{-n}$, we get that with a $\mathbb{P}$-probability 1, for all but a finite number of indices n,

$$\max_{I\subset\mathcal{I}}{}^{*} \max_{\tilde I_{12}\subset I} \frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde I_{12})} \le \exp\left(\beta\frac{4\varepsilon+12\zeta_4}{\gamma}\right). \tag{5.44}$$

It remains to estimate the first term in the right-hand side of (5.36).

We have $\Delta\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}}) = -\eta\sum_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}X(x)$ where

$$\begin{aligned} X(x) &\equiv -2\beta\theta\lambda(x)|D(x)|\left[m^{\delta^*}_{\beta,1}+m^{\delta^*}_{\beta,2}+\Xi(x,\beta\theta,\alpha)\right]\\ &\quad -\lambda(x)\log\frac{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,2}}\Psi_{0,0,m^{\delta^*}_{\beta,1}}}{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,1}}\Psi_{0,0,m^{\delta^*}_{\beta,2}}} \end{aligned} \tag{5.45}$$

with $\Xi(x,\beta\theta,\alpha)\equiv[\hat\varphi(m^{\delta^*}_{\beta,1},\lambda(x)\beta\theta,\alpha)-\hat\varphi(m^{\delta^*}_{\beta,2},\lambda(x)\beta\theta,\alpha)]$.

Therefore we need to estimate from above the probability of

$$\mathcal{A}\equiv\left\{\max_{I\subset\mathcal{I}}{}^{*}\max_{\tilde I_{12}\subset I}\left|\gamma\sum_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}X(x)\right|\ge s\right\} \tag{5.46}$$

for $s>0$. For our purpose it is enough to prove (5.46) for $s\le s_0$, for a given s_0. This will be done in two steps that are similar to the proof of (5.43). First we give an estimate for a fixed $\tilde I_{12}$ and then we make a block decomposition of $\mathcal{I}$ into blocks of length $\hat c(2\gamma\log\log(1/\gamma))^{-1}$. Arguing as before, we apply the usual Ottaviani inequality. All of this is standard and it is just an adaptation of the proof of the upper bound in the law of the iterated logarithm given by de Acosta [14]. It follows from the exponential Markov inequality and independence that, for all $\lambda\ge 0$,

$$\mathbb{P}\left[\gamma\sum_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}X(x)\ge s\right]\le e^{-st}\prod_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}\mathbb{E}\left[\exp(t\gamma X(x))\right]. \tag{5.47}$$

To estimate the previous Laplace transform, we use $e^x\le 1+x+(x^2/2)e^{|x|}$, $\forall\, x\in\mathbb{R}$. Using the fact that $\mathbb{E}(X)=0$, we get

$$\mathbb{E}\left[\exp(t\gamma X(x))\right]\le 1+(t\gamma)^2\frac{\mathbb{E}\left[X^2(x)\right]}{2}\exp(t\gamma\|X(x)\|_\infty). \tag{5.48}$$

Using Proposition 3.1, if γ is small enough, and how small depends on $\beta\theta$ to absorb the last term in (3.25), we have for some positive constant c, $\|X(x)\|_\infty\le 4\beta\theta(\delta^*/\gamma)(1+c\beta\theta)$. On the other hand, it is easy to check that, calling $\mathbb{E}[|D(x)|^2]=V^2(x)=\delta^*/\gamma$, we have also for some positive constant c, if γ is small enough, $\mathbb{E}[X^2(x)]\le 16(\beta\theta)^2(1+c\beta\theta)^2(\delta^*/\gamma)$. Using $1+x\le e^x$ $\forall x\in\mathbb{R}$ and $|\mathcal{C}_{\delta^*}(\tilde I_{12})|=|\tilde I_{12}|/\delta^*$, we easily get

$$\begin{aligned}&\prod_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}\mathbb{E}\left[\exp(tX(x))\right]\\ &\qquad\le\exp\left[\gamma 8(t\beta\theta)^2(1+c\beta\theta)|\tilde I_{12}|\exp(t\delta^*4\beta\theta(1+c\beta\theta))\right].\end{aligned} \tag{5.49}$$

The choice of t depends on $|\tilde{I}_{12}|$. If $\gamma|\tilde{I}_{12}| \geq \delta^*/g_3(\gamma)$ with $\lim_{\gamma\downarrow 0} g_3(\gamma) = 0$ as slowly as we want, we choose $t = s/16\gamma|\tilde{I}_{12}|(\beta\theta)^2(1 + c\beta\theta)$. If $\gamma|\tilde{I}_{12}| \leq \delta^*/g_3(\gamma)$, we choose $t = (s \log\log(1/\gamma)/32\hat{c}(\beta\theta)^2(1 + c\beta\theta))s$. Assuming that $g_3(\gamma)$ is such that $\gamma(\log\log(1/\gamma))^2 \leq (g_3(\gamma))^2$, in both cases, we get

$$\begin{aligned} &\mathbb{P}\left[\left|\gamma \sum_{x\in\mathscr{C}_{\delta^*}(\tilde{I}_{12})} X(x)\right| \geq \beta s\right] \\ &\qquad \leq 2\exp\left[-\frac{s^2\log\log(1/\gamma)(1-2s_0cg_3(\gamma))}{32\hat{c}(\theta)^2(1+c\beta\theta)}\right] \end{aligned} \tag{5.50}$$

for $s \leq s_0$ and for some constant c. [Note that, given $s_0 > 0$, it is always possible to find $\gamma_0 > 0$ such that for $\gamma \leq \gamma_0$, the quantity $(1 - 2s_0cg_3(\gamma))$ is strictly positive.] To get uniformity with respect to all subintervals that are in I, we write simply

$$\max_{\tilde{I}_{12}\subset I}\left|\gamma \sum_{x\in\mathscr{C}_{\delta^*}(\tilde{I}_{12})} X(x)\right| \leq 2\max_{l_1\leq l\leq l_2}\left|\gamma\sum_{x=l_1}^{l} X(x)\right|. \tag{5.51}$$

Therefore, using the Ottaviani inequality

$$\mathbb{P}\left[\max_{l_1\leq l\leq l_2}\left|\gamma\sum_{x=l_1}^{l} X(x)\right| \geq 2\beta s\right] \leq \frac{\mathbb{P}\left[|\gamma\sum_{x=l_1}^{l_2} X(x)| \geq \beta s\right]}{\inf_{l_1\leq l\leq l_2}\mathbb{P}\left[|\gamma\sum_{x=l_1}^{l} X(x)| \leq \beta s\right]}, \tag{5.52}$$

we get, setting $\tilde{u} = (1 - 2s_0cg_3(\gamma))/32\hat{c}(\theta)^2(1 + c\beta\theta)$ by an argument similar to the one that gives (5.42),

$$\begin{aligned} &\mathbb{P}\left[\max_{I\subset\mathscr{I}}{}^{*}\max_{\tilde{I}_{12}\subset I}\left|\gamma\sum_{x\in\mathscr{C}_{\delta^*}(I)} X(x)\right| \geq 2\beta s\right] \\ &\qquad \leq \frac{4\tilde{c}(\log(1/\gamma))^{p+\rho}}{\hat{c}}\exp\left(-s^2\tilde{u}\log\log\frac{1}{\gamma}\right). \end{aligned} \tag{5.53}$$

We then collect (5.34), (5.35), (5.36), (5.44), obtaining

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\left(\mathscr{W}_{\eta}^{\delta_1,\zeta_1,\delta_4,\zeta_4}\left(l_1,\tilde{l}_1,\tilde{l}_2,l_2\right)\right) \\ &\qquad \leq \exp\left[-\frac{\beta}{\gamma}\left(x\Delta\mathscr{F} - 24\zeta_4 - 4\varepsilon - 4s - \delta^*|I|\right)\right]. \end{aligned} \tag{5.54}$$

We make the following choices: $s \leq s_0 = x^2\Delta\mathscr{F}/16$, $\varepsilon = \frac{1}{16}x^2\Delta\mathscr{F}$, $c_0 = x^2\Delta\mathscr{F}/4\hat{c}$, $\zeta_4 \leq x^2\Delta\mathscr{F}/96$; this will give us (5.33). We take γ_0 such that $(1 - 2s_0cg_3(\gamma_0)) = \frac{1}{2}$. To be able to satisfy $s^2\tilde{u} \geq p + 2\rho + 1$ *and* $s \leq s_0$, we impose $\theta \leq x^2\Delta\mathscr{F}/48\sqrt{\hat{c}(p+2\rho+1)}$ and we can take $s = 16\theta\sqrt{\hat{c}(p+2\rho+1)}$. Recalling that we need also $u \geq p+2\rho+1$, we impose that ζ_4 is such that $\zeta_4 g_2(1/\zeta_4) \leq x^2\beta\Delta\mathscr{F}/72\sqrt{\hat{c}(p+2\rho+1)}$, that is, with the condition above we assume $\zeta_4 g_2(1/\zeta_4) \le x^2 \Delta\mathscr{F}/96[1 \wedge \beta/\sqrt{\hat{c}(p+2\rho+1)}]$ and we get (5.32). This ends the proof of Proposition 5.7. □

PROOF OF THEOREM 2.4. We prove that the complementary of the set $\mathscr{R}^{\delta_4,\zeta_4}(l_1,l_2,R_1) \cup \mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1,l_2)$ has Gibbs probability that goes to zero as $\exp(-c_4(\beta,\theta)\delta_4\zeta_4^3/\gamma)$. We decompose

$$\begin{aligned}\mathscr{A} &\equiv \left(\mathscr{R}^{\delta_4,\zeta_4}(l_1,l_2,R_1,+)\right)^c \cap \left(\mathscr{R}^{\delta_4,\zeta_4}(l_1,l_2,R_1,-)\right)^c\\ &= \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \cup \mathscr{A}_4,\end{aligned} \tag{5.55}$$

where [see (5.30)] $\mathscr{A}_1 \equiv \mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+2R_1, l_2-2R_1])$ and [see (5.29)] $\mathscr{A}_2 \equiv \mathscr{W}^{\delta_4,\zeta_4}([l_1+2R_1, l_2-2R_1])$. While $\mathscr{A}_1$ and $\mathscr{A}_2$ refer to the behavior of the profiles in the bulk of the interval, $\mathscr{A}_3$ and $\mathscr{A}_4$ consider the behavior of the profiles in a region close to the boundaries. Namely, for a given $\eta \in \{-1,+1\}$, we can be in $(\mathscr{R}^{\delta_4,\zeta_4}(l_1,l_2,R_1,\eta))^c$ just because we have $\eta^{\delta_4,\zeta_4}(l_1+2R_1) \ne \eta$ or $\eta^{\delta_4,\zeta_4}(l_2-2R_1) \ne \eta$. Let us define

$$\mathscr{A}_3^{\eta}(l) \equiv \left\{m^{\delta^*} : \eta^{\delta_4,\zeta_4}(l) \ne \eta\right\} \tag{5.56}$$

and

$$\mathscr{A}_3 = \bigcup_{\eta,\eta' \in \{-1,+1\}^2} \mathscr{A}_3^{\eta}(l_1+2R_1) \cup \mathscr{A}_3^{\eta'}(l_2-2R_1). \tag{5.57}$$

Suppose that a profile is in $\mathscr{A}_3^{\eta}(l_1+2R_1)$; then we can have four alternatives.

The block l_1+2R_1 has $\eta^{\delta_4,\zeta_4}(l_1+2R_1)=0$ or $\eta^{\delta_4,\zeta_4}(l_1+2R_1)=-\eta$ and it is sandwiched at a distance smaller than $2R_1$ by two blocks with the same η's or with different η's. In this last case, the profiles are fronts.

It is easy to see that

$$\begin{aligned}&\mathscr{A}_3 \cap \left(\mathscr{O}_0^{\delta_1,\zeta_1}([l_1+R_1,l_2-R_1])\right)^c \cap \mathscr{A}\\ &\quad\subset \mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+R_1,l_2-R_1]) \cup \mathscr{W}^{\delta_4,\zeta_4}([l_1+R_1,l_2-R_1])\\ &\qquad \cup \mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+R_1,l_2-R_1]).\end{aligned} \tag{5.58}$$

It remains to consider what is left in $\mathscr{A}$. The presence of $\mathscr{A}_4$ comes from the fact that in the definition of $\mathscr{A}_1$, there are four parameters δ_4, ζ_4, δ_1, ζ_1 and since $\delta_4\zeta_4^3 \ge \tilde{c}_1(\beta,\theta)\zeta_1$, we can have blocks such that $\eta^{\delta_1,\zeta_1}=0$ but $\eta^{\delta_4,\zeta_4}=1$. Let us define

$$\mathscr{A}_4^{\eta}(l) \equiv \left\{m^{\delta^*} : \eta^{\delta_4,\zeta_4}(l)=\eta, \eta^{\delta_1,\zeta_1}(l)=0\right\} \tag{5.59}$$

and

$$\mathscr{A}_4 = \bigcup_{\eta,\eta' \in \{-1,+1\}^2} \mathscr{A}_4^{\eta}(l_1+2R_1) \cup \mathscr{A}_4^{\eta'}(l_2-2R_1). \tag{5.60}$$

Arguing as before, we get

$$\begin{aligned}&\mathscr{A}_4 \cap \left(\mathscr{O}^{\delta_1,\zeta_1}([l_1+R_1, l_2-R_1])\right)^c \cap \mathscr{A}\\ &\quad\subset \mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+R_1, l_2-R_1])\\ &\qquad\cup \mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+R_1, l_2-R_1]).\end{aligned}\tag{5.61}$$

It is now clear that we have

$$\begin{aligned}&\mathscr{A} \cap \left(\mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1, l_2)\right)^c \cap \left(\mathscr{O}_0^{\delta_1,\zeta_1}([l_1+R_1, l_2-R_1])\right)^c\\ &\quad\subset \mathscr{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([l_1+R_1, l_2-R_1])\\ &\qquad\cup \mathscr{W}^{\delta_4,\zeta_4}([l_1+R_1, l_2-R_1])\end{aligned}\tag{5.62}$$

and (2.36) follows immediately from Corollaries 5.2, 5.4, 5.6 and Proposition 5.7. □

PROOF OF THEOREM 2.5. Taking into account (2.36), we must check that for $l_1 \le l_2 \le l_3$ that belongs to $\mathscr{I}$, an event of the form

$$\mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_1, l_2, \eta) \cap \mathscr{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(l_2, l_3, \eta)\tag{5.63}$$

with $l_2 - l_1 \le l_{\hat{c}}(\gamma)$ and $l_3 - l_2 \le l_{\hat{c}}(\gamma)$ has small Gibbs probability and moreover that this is true with a very high $\mathbb{P}$-probability, uniformly for $l_1 \le l_2 \le l_3$ in $\mathscr{I}$. But it is immediate to see that those events are controlled by Proposition 5.7.

Using Theorem 2.3, denoting by $c_2 = [c(x, \rho, \gamma)/(\beta\theta)^2(m_{\beta,1} + m_{\beta,2})^2]$ [see (2.30)], we end the proof of Theorem 2.5. □

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M. CASSANDRO
DIPARTIMENTO DI FISICA
UNIVERSITÁ DI ROMA
LA SAPIENZA
P.LE A. MORO
00185 ROMA
ITALY
E-MAIL: cassandro@roma1.infn.it

E. ORLANDI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÁ DI ROMA TRE
L.GO S. MURIALDO 1
00156 ROMA
ITALY
E-MAIL: orlandi@matrm3.mat.uniroma3.it

P. PICCO
CPT-CNRS, LUMINY
CASE 907, F-13288
MARSEILLE CEDEX 9
FRANCE
E-MAIL: picco@cpt.univ-mrs.fr