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# PROBABILISTIC INTERPRETATION OF STICKY PARTICLE MODEL

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This work presents a construction of a solution for the nonlinear stochastic differential equation $X_t = X_0 + \int_0^t \mathbb{E}[u_0(X_0)|X_s]\,ds$, $t \geq 0$. The random variable $X_0$ with values in $\mathbb{R}$ and the function $u_0$ are given. We denote by $P_t$ the probability distribution of $X_t$ and $u(x,t) = \mathbb{E}[u_0(X_0)|X_t = x]$. We prove that $(P_t, u(\cdot,t), t \geq 0)$ is a weak solution for a system of conservation laws arising in adhesion particle dynamics.

**1. Introduction and main results.** Let us consider the system of conservation law

$$\begin{aligned} &\frac{\partial P(x,t)}{\partial t} + \frac{\partial (u(x,t)P(x,t))}{\partial x} = 0, \\ &\frac{\partial (u(x,t)P(x,t))}{\partial t} + \frac{\partial (u^2(x,t)P(x,t))}{\partial x} = 0 \end{aligned} \tag{1}$$

with initial value $P_0, u_0$. This system was studied by E, Rykov and Sinai (1996), and they have defined weak solutions of system (1) as follows.

DEFINITION. Let $(P_t, I_t)$ be a family of Borel measures, weakly continuous with respect to $t$, such that $I_t$ is absolutely continuously with respect to $P_t$ for each fixed $t$. Define $u(x,t) = (dI_t/dP_t)(x)$. Then $(P_t, I_t, u)_t$ is a weak solution of (1) with initial data $(P_0, u_0)$ if, for any $f, g \in C_0^1(\mathbb{R})$, the space of $C^1$-functions on $\mathbb{R}$ with compact support, and any $0 < t_1 < t_2$,

$$\int f(x)\,dP_{t_2}(x) - \int f(x)\,dP_{t_1}(x) = \int_{t_1}^{t_2}\int f'(x)\,dI_t(x)\,dt, \tag{D1}$$

$$\int g(x)\,dI_{t_2}(x) - \int g(x)\,dI_{t_1}(x) = \int_{t_1}^{t_2}\int g'(x)u(x,t)\,dI_t(x)\,dt \quad \text{and} \tag{D2}$$

$$P_t \to P_0,\ I_t \to I_0 \text{ weakly and as } t \to 0^+. \tag{D3}$$

E, Rykov and Sinai (1996) have constructed a weak solution under the following hypothesis.

(A1) The measure $P_0$ is positive Radon measure, either discrete or absolutely continuous with respect to the Lebesgue measure. In the latter case, they assume that $dP_0(x)/dx > 0$, for $x \in \mathrm{Supp}(P_0)$. If $\mathrm{Supp}(P_0)$ is unbounded, they assume additionally $\int_0^x s\,dP_0(s) \to \infty$ as $|x| \to \infty$.

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(A2) The function $u_0$ is continuous and for any $z > 0$,

$$\sup_{|x| \le z} |u_0(x)| \le b_0(z) \quad \text{and} \quad \lim_{|z| \to \infty} \frac{b_0(z)}{z} = 0.$$

Their construction is based on a connection between (1) and the following "sticky particle model" of Zeldovich (1970). Let us consider a system of particles $\{x_i^0\}$ on $\mathbb{R}$ with initial velocities $\{v_i^0\}$ and masses $\{m_i^0\}$. The particles move with constant velocities unless they collide. At collisions, the colliding particles stick and form a new massive particle. The mass and velocity of this new particle are given by the laws of conservation of mass and momentum. This model was proposed by Zeldovich (1970) to explain the formation of large scale structures in the universe. It was further developed by Kofman, Pogosyan and Shandarin (1990), Gurbatov, Malakhov and Saichev (1991), Shandarin and Zeldovich (1989), and Vergassola, Dubrulle, Frisch and Noullez (1994).

The aim of the present work is to give a probabilistic interpretation of the "sticky particle model," when $P_0$ is the probability distribution of a random variable $X_0$ defined on some probability space $(\Omega, \mathbb{F}, \mu)$. The following theorem is the main result of our work.

THEOREM 1.1. *Let $u_0$ be a map from $\mathbb{R}$ to $\mathbb{R}$, with left and right limits, such that $P_0(\{x, u_0(x+) \neq u_0(x-)\}) = 0$, which satisfies $\lim_{|x|\to\infty}(u_0(x)/x) = 0$. Then there exists a process $(X_t)_{t\ge 0}$ on the probability space $(\Omega, \sigma(X_0), \mu)$, such that $\mu$ almost surely $t \in \mathbb{R}_+ \to X_t(\omega)$ is continuous, and for each fixed $t \ge 0$,*

$$X_t = X_0 + \int_0^t \mathbb{E}[u_0(X_0)|X_s]\, ds. \tag{2}$$

As a consequence we obtain the following corollary.

COROLLARY 1.1. *For each fixed $t \ge 0$, let $P_t$ be the probability distribution of $X_t$. We denote by $u(x,t) = \mathbb{E}[u_0(X_0)|X_t = x]$. Define the measure $I_t$ by $(dI_t/dP_t)(x) = u(x,t)$. Then $(P_t, I_t, u(\cdot,t))_{t\ge 0}$ is a weak solution for system (1) with initial data $(P_0, u_0)$.*

We finish this section by the proof of the corollary. We have, for $f, g \in C_0^1(\mathbb{R})$, $0 < t_1 < t_2$,

$$\int f(x)\, dP_{t_2}(x) - \int f(x)\, dP_{t_1}(x) = \mathbb{E}\big[f(X_{t_2}) - f(X_{t_1})\big]$$

and

$$\int g(x)u(x,t_2)\, dP_{t_2}(x) - \int g(x)u(x,t_1)\, dP_{t_1}(x)$$
$$= \mathbb{E}\big[g(X_{t_2})u_0(X_0) - g(X_{t_1})u_0(X_0)\big].$$

From (2) and the formula of change of variables, we have

$$f(X_{t_2}) - f(X_{t_1}) = \int_{t_1}^{t_2} f'(X_t)\mathbb{E}[u_0(X_0)|X_t]\,dt$$

and

$$g(X_{t_2})u_0(X_0) - g(X_{t_1})u_0(X_0) = \int_{t_1}^{t_2} g'(X_t)u_0(X_0)\mathbb{E}[u_0(X_0)|X_t]\,dt.$$

From that it is easy to show that $(P_t, I_t, u(\cdot, t))_{t>0}$ satisfies (D1) and (D2). The proof of (D3) is easy.

The next section presents some preliminary results in order to prove Theorem 1.1.

**2. Preliminary result.** Let us consider a finite number of particles with initial data $\{x_i^0, u_0(x_i^0), m_i^0\colon 1 \le i \le N\}$, where $\sum_i^N m_i^0 = 1$. So, the location $x_i^0$ can be seen as a realization of a random variable $X_0$, defined on some probability space $(\Omega, \mathbb{F}, \mu)$, with the distribution $P_0$ given by $\mu(X_0 = x_i^0) = P_0(\{x_i^0\}) = m_i^0$. The latter particles move following the "sticky particle model" defined in Section 1. The center of mass at time $t$ of a group of particles belonging to a subset $G$ of $\mathbb{R}$, is given by

$$C(G, t) = \mathbb{E}[X_0 + tu_0(X_0)|X_0 \in G]. \tag{3}$$

It is a linear function of $t$. If $G$ is a group of particles glued to a single one before or at time $t$, then from the conservation of mass and momentum, the location at time $t$ of this group is given by (3). In the sequel we denote by $\xi_t$ the partition of $\{x_i^0\colon 1 \le i \le N\}$, defined by the ordered groups $G_1(t), G_2(t), \dots, G_k(t)$, so that each group of particles is glued to a single one before or at time $t$, and different groups are at different locations at time $t$.

Throughout this section and Section 3 we shall assume that the probability $P_0$ is concentrated on a finite set. The following lemmas are due to E, Rykov and Sinai (1996).

LEMMA 2.1. *Let $G_1$ and $G_2$ be two neighboring groups of particles such that $C(G_1, t) < C(G_2, t)$ for $t < \tau$, and $C(G_1, \tau) = C(G_2, \tau)$. Then for $t > \tau$,*

$$C(G_2, t) < C(G_1 \cup G_2, t) < C(G_1, t).$$

PROOF. Since both $C(G_1, t)$ and $C(G_2, t)$ are linear functions of $t$, we have for $t > \tau$,

$$C(G_1, t) > C(G_2, t).$$

We have for $\alpha = \mathbb{P}(X_0 \in G_1)/\mathbb{P}(X_0 \in G_1 \cup G_2)$,

$$\begin{aligned} C(G_1 \cup G_2, t) &= \mathbb{E}[X_0 + tu_0(X_0)|X_0 \in G_1 \cup G_2] \\ &= \alpha C(G_1, t) + (1 - \alpha)C(G_2, t). \end{aligned}$$

The latter equality achieves the proof. $\square$

LEMMA 2.2. *Let* $G = \{x_i^0: j' \leq i \leq j''\} \in \xi_t$. *If* $I_1 = [x_{j'}^0, x]$ *and* $I_2 = (x, x_{j''}^0]$, *for* $x_{j'}^0 < x < x_{j''}^0$, *then*

$$C(I_1, t) \geq C(I_2, t). \tag{4}$$

PROOF. Assume on the contrary that $C(I_1, t) < C(I_2, t)$. Since $C(G, t) = \alpha C(I_1, t) + (1 - \alpha) C(I_2, t)$ for some $\alpha \in (0, 1)$, we have

$$C(I_1, t) < C(G, t). \tag{5}$$

Let us consider the evolution of the set of particles $I_1$. Each time, the set is hit from the right by a particle or a cluster of particles, we add them to our set. In this way we obtain a growing family of sets $I_1(s) = \{x_j^0: j' \leq j \leq i(s)\}$. From Lemma 2.1, we have, for all $s \leq t$,

$$C(I_1(s), s) < C(I_1, s).$$

From the assumption of Lemma 2.2 we have $i(t) = j''$. Hence we have

$$C(G, t) < C(I_1, t),$$

contradicting (5).

LEMMA 2.3. *A particle* $x$ *is the left endpoint*, *respectively*, *the right endpoint*, *of an element of the partition* $\xi_t$ *iff*

$$\max_{y<x} C([y, x), t) < \min_{z \geq x} C([x, z], t), \tag{6}$$

*respectively*, $\max_{y < x} C([y, x], t) < \min_{z \geq x} C((x, z], t)$.

PROOF. The proofs of both cases are similar. Let $x$ be a particle satisfying (6), and belonging to the group $G = \{x_i^0, \dots, x_j^0\}$. Assume that $x_i^0 < x$. From (4) we have

$$C\big([x_i^0, x), t\big) \geq C\big([x, x_j^0], t\big),$$

which contradicts (6).

Assume now that $x$ is the left endpoint of an element of $\xi_t$. For any $y < x < z$, we want to show that $C([y, x), t) < C([x, z], t)$. Let $I_1, \dots, I_l$ be consecutive elements of $\xi_t$ to the left of $x$, and $y \in I_1 = \{x_i^0: i_1 \leq i \leq i_2\}$. Let $J_1, \dots, J_r$ be the consecutive elements of $\xi_t$ to the right of $x$, and $x \in J_1 = \{x, \dots, x'\}$, and $z \in J_r = \{x_i^0: j_1 \leq i \leq j_2\}$.

We have first

$$C(I_1, t) < C(I_2, t) < \cdots < C(J_1, t) < \cdots < C(J_r, t).$$

From Lemma 2.1 and Lemma 2.2, we have

$$C\big((y, x_{i_2}], t\big) < C(I_1, t) < C\big([x_{i_1}^0, y], t\big)$$

and

$$C\big((z, x_{j_2}^0], t\big) < C(J_r, t) < C\big([x_{j_1}^0, z), t\big).$$

Since

$$C([y,x),t) = \alpha_1 C\Big(\big[y, x^0_{i_2}\big], t\Big) + \alpha_2 C(I_2, t) + \cdots + \alpha_l C(I_l, t)$$

and

$$C([x,z],t) = \beta_1 C([x,x'],t) + \beta_2 C(J_2,t) + \cdots + \beta_r C\Big(\big[x^0_{j_1}, z\big], t\Big),$$

where $\sum \alpha_i = \sum \beta_i = 1$, and $\alpha_i \geq 0$, $\beta_i \geq 0$, we must have

$$C([y,x),t) < C([x,z],t).$$

*Some consequences*. Let $I_1, \dots, I_j, \dots$ be the successive groups of particles glued to a single one before or at time $t$. For $x \in I_j$, we set

$$\varphi(t,x) = \mathbb{E}\big[X_0 + t u_0(X_0) | X_0 \in I_j\big]$$

and we extend the definition of $\varphi(t,\cdot)$ to the whole line by putting $\varphi(t,x) = \varphi(t, x_i^0)$ if $x_i^0 \leq x < x_{i+1}^0$, $\varphi(t,x) = \varphi(t, x_1^0)$ if $x < x_1^0 = \min_i x_i^0$, $\varphi(t,x) = \varphi(t, x_N^0)$ if $x \geq x_N^0 = \max_i x_i^0$.

For all $t \geq 0$, the map $x \in \mathbb{R} \to \varphi(t,x)$ is increasing. The map $t \in \mathbb{R}_+ \to \varphi(t,x)$, for $x \in \mathbb{R}$, is Lipschitz continuous and satisfies the following property:

$$\varphi(t,x) = \mathbb{E}\big[X_0 + t u_0(X_0) | \varphi(t, X_0) = \varphi(t,x)\big]. \tag{7}$$

THEOREM 2.1. *If $u_0$ is a function bounded on any compact set of $\mathbb{R}$ and such that $\lim_{|x|\to\infty}(u_0(x)/x) = 0$, then for all $t > 0$ and for all finite intervals $(a,b)$, which intercept the image of $\varphi(t,\cdot)$, the set $\{x\colon \varphi(t,x) \in (a,b)\}$ is uniformly bounded with respect to the class of probabilities $P_0$ supported by finite sets, and $t \in [0,T]$, for all $T > 0$.*

PROOF. Let $x_{\min} = \min\{x\colon \varphi(t,x) \in (a,b)\}$, and $x_{\max} = \max\{x\colon \varphi(t,x) \in (a,b)\}$. Obviously $x_{\min}$ (respectively, $x_{\max}$) has to be the left endpoint (respectively, the right endpoint) of an element $I_j$ in the partition $\xi_t$. From Lemma 2.2, we have

$$x_{\min} + t u_0(x_{\min}) \geq \varphi(t, x_{\min}) \geq a.$$

Now, using the hypothesis under $u_0$, we get that $x_{\min}$ is uniformly bounded from below with respect to $t \in [0,T]$ and $P_0$ belongs to the class of probabilities supported by finite sets. Similarly, we have

$$x_{\max} + t u_0(x_{\max}) \leq \varphi(t, x_{\max}) \leq b.$$

Again the hypothesis under $u_0$ yields an upper bound. $\square$

THEOREM 2.2. (i) *Let $I_j$ be an element of the partition $\xi_t$, $x_l = \min I_j$ and $x_r = \max I_j$; then*

$$x_l + s u_0(x_l) = x_r + s u_0(x_r) \quad \textit{for some } s \leq t.$$

*If $a < b$, and $T > 0$, then:*

(ii) *The set $\varphi(t, [a, b])$ is uniformly bounded with respect to the class of probabilities $P_0$ supported by finite sets, and $t \in [0, T]$.*

(iii) *The set $\partial\varphi(t, [a, b])/\partial t$, defined $dt$ a.e., is uniformly bounded with respect to the class of probabilities $P_0$ supported by finite sets, and $t \in [0, T]$.*

PROOF. (i) From Lemma 2.2 we have, for all $y \in [x_l, x_r]$,

$$C([x_l, y], t) \geq C((y, x_r], t).$$

We deduce that

$$x_l + tu_0(x_l) \geq x_r + tu_0(x_r).$$

Since $x_l \leq x_r$, we have

$$x_l + su_0(x_l) = x_r + su_0(x_r) \quad \text{for some } s \in (0, t].$$

(ii) Let $a < b$ and $T > 0$. Since $\lim_{|y| \to \infty}(u_0(y)/y) = 0$, there exist $y_1 < a$ and $y_2 > b$, such that for all $t \in [0, T]$,

$$y + tu_0(y) < -1 \quad \text{for } y < y_1, \tag{8}$$

and

$$z + tu_0(z) > 1 \quad \text{for } z > y_2. \tag{9}$$

Let $x \in [a, b]$ and $I_j$ be the element of $\xi_t$ which contains $x$. Let $x_l$ and $x_r$ be, respectively, the left and right endpoints of $I_j$. From the assertion (i) of the theorem, and (8), (9), we have $[x_l, x_r] \subset [y_1, y_2]$. From that we have the proof of assertion (ii).

(iii) For $x \in [a, b]$, we have from (7),

$$\frac{\partial\varphi(t, x)}{\partial t} = \mathbb{E}[u_0(X_0) | \varphi(t, X_0) = \varphi(t, x)]\, dt \quad \text{a.e.}$$

It follows from assertion (ii) and Theorem 2.1, that

$$\left| \frac{\partial\varphi(t, x)}{\partial t} \right| \leq \max_{y \in K} |u_0(y)|,$$

where $K$ is some compact set which depends on $a$, $b$, $T$ and $u_0$. □

**3. Proof of Theorem 1.1 in the finite case.** We will show that the process $(X_t := \varphi(t, X_0), t \geq 0)$ satisfies Theorem 1.1. Let $T_1 = \min\{t > 0: q_i + tu_0(q_i) = q_j + tu_0(q_j)$, for some $i \neq j\}$ be the first time when collisions arrive. From the definition of $\varphi$ we have, for $0 \leq t < T_1$, $X_t = X_0 + tu_0(X_0)$. From the conservation of mass and momentum, we can show that

$$\lim_{\varepsilon \to 0+} \frac{X_{T_1 + \varepsilon} - X_{T_1}}{\varepsilon} = \mathbb{E}\left[u_0(X_0) | X_{T_1}\right] := u_1(X_{T_1}).$$

Let $T_2$ be the second time when collisions arrive. At $t$, such that $T_1 \leq t < T_2$, $X_t = X_{T_1} + (t - T_1)\mathbb{E}[u_0(X_0) | X_{T_1}]$. By induction we construct the successive times of collisions $T_1 < T_2 < \cdots < T_M < T_{M+1} = \infty$. The time $T_M$ is the last time when collisions arrive.

PROPOSITION 3.1. *At $t$, such that $T_n \le t < T_{n+1}$ and $1 \le n \le M$,*

$$\sigma(X_t) = \sigma(X_{T_n}) \quad \text{and} \quad X_t = X_{T_n} + (t - T_n)\mathbb{E}\big[u_0(X_0)|X_{T_n}\big].$$

PROOF. First for $t < T_1$ the events $[\varphi(t, X_0) = \varphi(t, q_i)]$, $1 \le i \le N$ do not intersect and span the $\sigma$-field $\sigma(X_t)$. Since $\sigma(X_0)$ is spanned by $[X_0 = q_i]$, $1 \le i \le N$, $\sigma(X_t) \subset \sigma(X_0)$ and $\text{card}(\sigma(X_t)) = \text{card}(\sigma(X_0))$ then both $\sigma$-fields coincide. The proof of the case $T_n \le t < T_{n+1}$ is the same and can be obtained by induction.

Let us prove the second part. We have, for $T_n \le t < T_{n+1}$,

$$X_t = X_0 + \sum_{i=0}^{n-1} (T_{i+1} - T_i) u_i(X_{T_i}) + (t - T_n) u_n(X_{T_n}),$$

where $u_i(X_{T_i})$ is the velocity of the system when time $s \in [T_i, T_{i+1})$. The conservation of mass and momentum gives that

$$u_i(X_{T_i}) = \mathbb{E}\big[u_{i-1}(X_{T_{i-1}})|X_{T_i}\big].$$

From the fact that $\sigma(X_{T_i}) \subset \sigma(X_{T_{i-1}})$, and by induction, we have

$$u_i(X_{T_i}) = \mathbb{E}\big[u_0(X_0)|X_{T_i}\big].$$

Adding all terms, we obtain

$$X_t = X_{T_n} + (t - T_n)\mathbb{E}\big[u_0(X_0)|X_{T_n}\big].$$

Now from Proposition 3.1 we can show that the process $(X_t, t \ge 0)$ satisfies Theorem 1.1. □

**4. The general case.** In the general case, we will prove Theorem 1.1 via discrete approximations. Let us consider a system of particles on $\mathbb{R}$ with initial distribution $P_0$ and velocity function $u_0$. Let $X_0$ be a random variable with probability distribution $P_0$. Take a sequence of random variables $X_0^{(n)}$; each $X_0^{(n)}$ takes its values in a finite set, such that a.s. $X_0^{(n)} \to X_0$ as $n \to \infty$. The initial velocity of the particle $X_0^{(n)}$ is equal to $u_0(X_0^{(n)})$. Using Section 2, we construct the corresponding trajectories $X_t^{(n)} = \varphi^{(n)}(t, X_0^{(n)})$, $t \ge 0$. The key of the rest of the proof is based on the following improvement of Lemmas 4 and 4′ in E, Rykov and Sinai (1996).

THEOREM 4.1. *As $n \to \infty$, the sequence $(\varphi^{(n)}; n \ge 1)$, converges uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}$ to some map $(\varphi(t, x), t \in \mathbb{R}_+, x \in \mathbb{R})$.*

PROOF. Assume to the contrary that $\varphi^{(n)}$ do not converge uniformly on some bounded set, say $[0, T] \times (c, d)$. Then there exists $\varepsilon > 0$ and sequences $(t_n, y_n) \in (0, T) \times (c, d)$ such that

$$\big|\varphi^{(n)}(t_n, y_n) - \varphi^{(m)}(t_m, y_m)\big| > \varepsilon. \tag{10}$$

From Theorem 2.2, we can choose $y_n, t_n$ such that $y_n \to y$, $t_n \to t$, $\varphi^{(n)}(t_n, y_n) \to x$, which contradicts (10) and achieves the proof. □

Let $\varphi$ be the limit of $\varphi^{(n)}$. We derive, from Theorem 4.1 and Theorem 2.2, that the function $\varphi$ satisfies the following important properties.

P1. For all $x \in \mathbb{R}$ the map $t \in [0, \infty) \to \varphi(t, x)$ is Lipschitz continuous.
P2. For all compact set $K$ there exists $c(K) > 0$ such that for all $n \in \mathbb{N}$; $x \in K$,

$$\left|\frac{\partial \varphi(t,x)}{\partial t}\right| + \left|\frac{\partial \varphi^{(n)}(t,x)}{\partial t}\right| < c(K)\,dt \quad \text{almost everywhere.} \tag{11}$$

P3. For all $T > 0$, and for all compact set $K$, the inverse images $\{x: \varphi^{(n)}(t, x) \in K\}$ are uniformly bounded with respect to $n \geq 1$ and $t \in [0, T]$.

Now we return to the proof of Theorem 1.1. We put, for each $t \geq 0$, $X_t = \varphi(t, X_0)$. To show that $(X_t, t \geq 0)$ satisfies Theorem 1.1, we need the following lemma.

LEMMA 4.1. (i) *Let $T_n$ be a sequence of continuous maps from $\mathbb{R}_+ \to \mathbb{R}$, such that*:

(a) *$T_n$ converges uniformly to 0 on any compact set of $\mathbb{R}_+$.*
(b) *The weak derivative $dT_n(t)/dt$ are measurable maps, uniformly bounded on every compact set of $\mathbb{R}_+$.*

*Then for all $G \in C_0(\mathbb{R}_+)$, we have, as $n \to \infty$,*

$$\int_0^\infty G(t)\frac{dT_n(t)}{dt}\,dt \to 0.$$

(ii) *For all $f \in C_0(\mathbb{R})$, $g \in C_0(\mathbb{R}_+)$ we have*

$$\lim_{n\to\infty} \int g(t) f\big(X_t^{(n)}\big)\mathbb{E}\big[u_0\big(X_0^{(n)}\big)|X_t^{(n)}\big]\,dt = \int g(t) f(X_t)\frac{dX_t}{dt}\,dt$$

*and*

$$\int g(t)\mathbb{E}[f(X_t)u_0(X_0)]\,dt = \int g(t)\mathbb{E}\left[f(X_t)\frac{dX_t}{dt}\right]dt.$$

PROOF. (i) Let $G \in C_0(\mathbb{R}_+)$ and $(G_m)$ be a sequence such that for all $m \geq 1$:

(a) $G_m \in C^1([0, L])$, for some $L > 0$.
(b) $\int_0^\infty |G_m(t) - G(t)|\,dt \to 0$.

From (b) there exists $c > 0$, such that

$$\left|\int G(t)\frac{dT_n(t)}{dt}\,dt\right| \leq c\int |G(t) - G_m(t)|\,dt + \left|\int G_m(t)\frac{dT_n(t)}{dt}\,dt\right|.$$

From (a) we have for all $m$, $|\int G_m(t)(dT_n(t)/dt)\,dt| \to 0$ as $n \to \infty$. We conclude that $\int G(t)(dT_n(t)/dt)\,dt \to 0$ as $n \to \infty$.

(ii) From the fact that $\varphi^{(n)}$ converges uniformly to $\varphi$, we can show, for all Lipschitz functions $h$, that a.s. the sequence of processes $t \to h(X_t^{(n)})$ converges uniformly on compact subsets of $\mathbb{R}_+$ to the process $t \to h(X_t)$. It follows that a.s., $t \to X_t^{(n)}$ converges in the distribution sense to $t \to X_t$. Hence, $dX_t^{(n)}/dt$ converges in the distribution sense to $dX_t/dt$.

Let us prove that for all $g \in C_0(\mathbb{R}_+)$ and $f \in C_0(\mathbb{R})$,

$$I(n) = \int f\big(X_t^{(n)}\big) \frac{dX_t^{(n)}}{dt} g(t)\, dt \to \int f(X_t) \frac{dX_t}{dt} g(t)\, dt = I,$$

as $n \to \infty$. From the triangular inequality and (11) we have for some constant $c > 0$,

$$\begin{aligned} |I(n) - I| \le c \int \big| f\big(X_t^{(n)} - f(X_t)\big| |g(t)|\, dt \\ + \left| \int g(t) f(X_t) \left\{ \frac{dX_t^{(n)}}{dt} - \frac{dX_t}{dt} \right\} dt \right|. \end{aligned}$$

Now we use assertion (i), with $T_n(t) = X_t^{(n)} - X_t$ and $G(t) = g(t) f(X_t)$. We get

$$\int g(t) f(X_t) \left\{ \frac{dX_t^{(n)}}{dt} - \frac{dX_t}{dt} \right\} dt \to 0;$$

on the other hand, $\int | f(X_t^{(n)} - f(X_t)| \, |g(t)|\, dt$ goes also to 0 as $n \to \infty$, which yields $I(n) \to I$ as $n \to \infty$. Now, from the fact that $dX_t^{(n)}/dt = \mathbb{E}[u_0(X_0^{(n)}) | X_t^{(n)}]$, combined with the dominated convergence theorem, we have

$$\int g(t) \mathbb{E}\left[ f(X_t) \frac{dX_t}{dt} \right] dt = \int g(t) \mathbb{E}[ f(X_t) u_0(X_0)]\, dt,$$

which achieves the proof of the lemma. □

Now we return to the proof of Theorem 1.1. From Lemma 4.1 we have that $\mathbb{E}[u_0(X_0)|X_t] = \mathbb{E}[(dX_t/dt)|X_t]\, dt \otimes \mu$ almost everywhere. From the property $\varphi^{(n)}(s, X_0^{(n)}) = \varphi^{(n)}(s - t, X_t^{(n)})$; $t \le s$, we derive $\sigma(X_s : t \le s) = \sigma(X_t)$. Now it is easy to see that $dX_t/dt$ is $\sigma(X_t)$-measurable. We conclude that $\mathbb{E}[(dX_t/dt)|X_t] = dX_t/dt = \mathbb{E}[u_0(X_0)|X_t]$, which yields Theorem 1.1. □

*Concluding remark.* It is important at the end of this work to discuss the connection with E, Rykov and Sinai (1996). The first essential result of these authors is the following principle for the construction, for each $t > 0$, of a partition $\xi_t$ of $\mathbb{R}$ using the initial data $(P_0, u_0)$.

The generalized variational principal (GVP): $y \in \mathbb{R}$ is the left endpoint of an element of $\xi_t$ iff for any $y^-, y^+ \in \mathbb{R}$, such that $y^- < y < y^+$, the following holds:

$$\frac{\int_{[y^-,y)}(\eta + tu_0(\eta))\,dP_0(\eta)}{\int_{[y^-,y)}dP_0(\eta)} < \frac{\int_{[y,y^+]}(\eta + tu_0(\eta))\,dP_0(\eta)}{\int_{[y,y^+]}dP_0(\eta)}.$$

They have constructed a weak solution of system (1) using $(\xi_t,\ t > 0)$. In our work we have proved the existence of a stochastic process $(X_t, t \geq 0)$ which satisfies

$$X_t = X_0 + \int_0^t \mathbb{E}[u_0(X_0)|X_s]\,ds, \qquad t \geq 0$$

and where $X_0$ is such that $P_0 = \text{law}(X_0)$. We have showed, for all probability measure $P_0$ and without resorting to GVP at the continuous level, that $(P_t = \text{law}(X_t),\ u(\cdot, t) = \mathbb{E}[u_0(X_0)|X_t = \cdot],\ t \geq 0)$ is a weak solution of system (1). If $P_0$ satisfies the condition (A1) given in the introduction, then our weak solution coincides with the weak solution given by E, Rykov and Sinai (1996). Our probabilistic interpretation can also be extended to the multidimensional version of system (1), and to the system

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} + \frac{\partial(u(x,t)P(x,t))}{\partial x} &= 0, \\ \frac{\partial(u(x,t)P(x,t))}{\partial t} + \frac{\partial(u^2(x,t)P(x,t))}{\partial x} &= -\frac{\partial g}{\partial x}P(x,t), \\ \frac{\partial^2 g}{\partial^2 x} &= P, \end{aligned}$$

which already has been studied by E, Rykov and Sinai (1996).

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