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HOW OFTEN DOES A HARRIS RECURRENT MARKOV CHAIN RECUR?

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Let $\{X_n\}_{n\ge 0}$ be a Harris recurrent Markov chain with state space $(E, \mathscr{E})$, transition probability $P(x, A)$ and invariant measure π. Given a nonnegative π-integrable function f on E, the exact asymptotic order is given for the additive functionals

$$\sum_{k=1}^{n} f(X_k), \qquad n = 1, 2, \dots$$

in the forms of both weak and strong convergences. In particular, the frequency of $\{X_n\}_{n\ge 0}$ visiting a given set $A \in \mathscr{E}$ with $0 < \pi(A) < +\infty$ is determined by taking $f = I_A$. Under the regularity assumption, the limits in our theorems are identified. The one- and two-dimensional random walks are taken as the examples of applications.

1. Introduction. Let $\{X_n\}_{n\ge 0}$ be a Harris recurrent Markov chain with state space $(E, \mathscr{E})$, transition probability $P(x, A)$ and invariant measure π; by Harris recurrence, the invariant measure π uniquely (up to a constant multipler) exists. Throughout, we always assume that the σ-algebra $\mathscr{E}$ is countably generated. Without explanation, we adopt the notations which have already become standard in Markov chain context, such as P_μ for the Markovian probability with the initial distribution μ, E_μ for correspondent expectation and, $P^k(x, A)$ for the k-step transition of $\{X_n\}_{n\ge 0}$. The basic notions and facts of Markov chain used in this work can also be found in almost every standard book on Markov chains.

Recall that $\{X_n\}_{n\ge 0}$ is called *Harris recurrent* if it is irreducible and for any $A \in \mathscr{E}^+$, initial distribution μ,

$$P_\mu\{X_n \in A \text{ infinitely often}\} = 1,$$

where $\mathscr{E}^+ = \{A \in \mathscr{E};\ \pi(A) > 0\}$.

Given a set $A \in \mathscr{E}$ with $0 < \pi(A) < +\infty$, the Harris recurrence tells us that

$$\#\{k;\ 1 \le k \le n \text{ and } X_k \in A\} = \sum_{k=1}^{n} I_A(X_k) \to +\infty \quad \text{a.s. } (n \to +\infty). \tag{1.1}$$

The random sequence given in (1.1) is called *occupation time* in the literature.

How fast does the occupation time grow? Or, how often does the Harris recurrent Markov chain $\{X_n\}_{n\ge 0}$ visit A? When $\{X_n\}_{n\ge 0}$ is *positive recurrent*,

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that is, π is finite (otherwise $\{X_n\}_{n\geq 0}$ is said to be *null recurrent*), the law of large numbers shows that the occupation time has a linear growth rate.

We now consider the general situation. A well-known ratio limit theorem [see, e.g., Theorem 17.3.2 in Meyn and Tweedie (1993)] states that if the Markov chain $\{X_n\}_{n\geq 0}$ is Harris recurrent then for any $f, g \in \mathscr{L}^1(E, \mathscr{E}, \pi)$ with $\int g(x)\pi(dx) \neq 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(X_k) \Big/ \sum_{k=1}^{n} g(X_k) = \int f(x)\pi(dx) \Big/ \int g(x)\pi(dx) \quad \text{a.s.} \tag{1.2}$$

Without losing or gaining generality, therefore, the problem we raise is to determine the growth rate of the additive functional defined by

$$\sum_{k=1}^{n} f(X_k), \qquad n = 1, 2, \dots$$

for all $f \in \mathscr{L}^1(E, \mathscr{E}, \pi)$ with $\int f(x)\pi(dx) \neq 0$. The results in the case $\int f(x)\pi(dx) = 0$ will be reported elsewhere. From (1.2), we may focus our attension to those with $f \geq 0$.

To normalize our additive functional, we need the concept of D-set. Recall [Orey (1971)] that a subset $D \in \mathscr{E}$ with $0 < \pi(D) < +\infty$ is called a *D-set* of a Harris recurrent Markov chain $\{X_n\}_{n\geq 0}$, if for any $A \in \mathscr{E}^+$,

$$\sup_{x\in E} E_x\left(\sum_{k=1}^{\tau_A} I_D(X_k)\right) < +\infty,$$

where we define the *hitting time* of A as

$$\tau_A = \inf\{n \geq 1;\ X_n \in A\}, \qquad A \in \mathscr{E}^+.$$

In Revuz (1975) and Nummelin (1984), D-set is also called *special set*. According to Theorem 6.2 in Orey (1971), D-sets not only exist, but exist in abundance.

The only reason we introduce D-set in this paper is the following ergodic theorem [see, e.g., Theorem 2, Chapter 2, Orey (1971)]:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mu P^k(C) \Big/ \sum_{k=1}^{n} \nu P^k(D) = \frac{\pi(C)}{\pi(D)}, \tag{1.3}$$

where C, D are D-sets and μ, ν are arbitrary probability measures on $(E, \mathscr{E})$. We mention the fact [Example 1.1 in Krengel (1966)] that C and D in (1.3) cannot be replaced by arbitrary sets with positive but finite invariant measures.

We now let the D-set D and the probability measure ν be fixed. Define

$$a(t) = \pi(D)^{-1} \sum_{k=1}^{[t]} \nu P^k(D), \qquad t \geq 0. \tag{1.4}$$

Clearly, $a(t)$ is a nonnegative, nondecreasing function ($a(t)$ is called *truncated Green function* in the literature). By the recurrence we have that $a(t) \to +\infty$

as $t \to +\infty$. In view of (1.3), the asymptotic order of $a(t)$ (as $t \to +\infty$) depends only on the Markov chain $\{X_n\}_{n\geq 0}$ [or its transition $P(x, A)$]. Hence the following concept of regular chain comes in a natural way. Namely, a Harris recurrent Markov chain $\{X_n\}_{n\geq 0}$ is called *regular*, if the function $a(t)$ is regularly varying at infinity: the limit

$$\lim_{\lambda\to+\infty} a(\lambda t)/a(\lambda)$$

exists for all $t > 0$. It is easy to see that there exists a real number $p \geq 0$ such that

$$\lim_{\lambda\to+\infty} a(\lambda t)/a(\lambda) = t^p \qquad \forall\, t > 0. \tag{1.5}$$

It is a standard fact [see, e.g., Chapter 18 in Meyn and Tweedie (1993)] that the truncated Green function $a(t)$ has a linear increasing rate if $\{X_n\}_{n\geq 0}$ is positive recurrent, or a sublinear increasing rate if $\{X_n\}_{n\geq 0}$ is null recurrent. Hence, we always have $0 \leq p \leq 1$. We call p the *regular index* of $\{X_n\}_{n\geq 0}$. To emphasize its regular index, sometimes a regular Harris recurrent Markov chain with index p is simply called p-regular. We notice that the regularity adopted here is similar in spirit to hypothesis (C) introduced in Touati (1990). It should also be pointed out (see the proof of Theorems 2.3 and 2.4 below) that in the "atomic" context, a Harris recurrent chain is p-regular if and only if the hitting time to an atom is in the domain of attraction of a stable law with p being the stable index. A strengthened version of such a stability is introduced in Csáki and Csörgő (1995) as a condition for a strong invariant principle for null recurrent Markov chains given in their paper.

By definition, a positive recurrent Markov chain is 1-regular. We will see in section 6 that an one-dimensional random walk is 1/2-regular and a two-dimensional random walk is 0-regular, provided they are centered and square integrable.

2. Main results.

We first deal with the general situation without assuming regularity.

THEOREM 2.1. *Let $\{X_n\}_{n\geq 0}$ be a Harris recurrent Markov chain with state space E, transition probability $P(x, A)$ and invariant measure π. Then for every nonnegative function $f \in \mathscr{L}^1(E, \mathscr{E}, \pi)$ with $\int f(x)\pi(dx) > 0$ and every initial distribution μ, both the sequences*

$$\left\{\sum_{k=1}^n f(X_k)\Big/a(n)\right\}_{n\geq 1} \quad \textit{and} \quad \left\{\left(\sum_{k=1}^n f(X_k)\Big/a(n)\right)^{-1}\right\}_{n\geq 1}$$

are bounded in probability, where the random variables in the second sequence are allowed to take the value ∞.

The normalizer for the strong limit theorems takes a different form. Define the function $L_2\lambda$ ($\lambda \geq 0$) by

$$L_2\lambda = \log\log\max\{\lambda, e^e\}, \qquad \lambda \geq 0.$$

THEOREM 2.2. *Let $\{X_n\}_{n\geq 0}$ be a Harris recurrent Markov chain with state space E, transition probability $P(x, A)$ and invariant measure π. Then there exists a constant $0 < L < +\infty$ such that*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} f(X_k) \Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) = L \int f(x)\pi(dx) \quad a.s. \tag{2.1}$$

for every nonnegative function $f \in \mathscr{L}^1(E, \mathscr{E}, \pi)$.

Theorem 2.1 and Theorem 2.2 give the exact increaing rates for our additive functionals in the weak sense and strong sense, respectively. According to Theorem 17.3.2 in Meyn and Tweedie (1993), the algebra $\mathscr{A}$ of invariant sets is a.s. trivial when $\{X_n\}_{n\geq 0}$ is Harris recurrent: for every $A \in \mathscr{A}$, $P_{\bullet}(A)$ is identically zero or one. Consequently, the statement of a strong limit law like (2.1) is independent of the choice of the initial distribution.

With regularity assumption, one can also identify the limiting parameters in our results.

THEOREM 2.3. *Let $\{X_n\}_{n\geq 0}$ be a regular Harris recurrent Markov chain with state space E, transition probability $P(x, A)$ and invariant measure π. Then for every nonnegative function $f \in \mathscr{L}^1(E, \mathscr{E}, \pi)$, the sequence of distributions*

$$\mathscr{L}_{P_\mu}\left(\sum_{k=1}^{n} f(X_k) \Big/ a(n)\right), \qquad n = 1, 2, \ldots$$

weakly converges for every initial distribution μ. Moreover, we have:

(i) *When the regular index $p = 0$, the limit distribution is the exponential distribution with parameter $\int f(x)\pi(dx)$.*

(ii) *When the regular index p satisfies $0 < p < 1$, the limit distribution is*

$$\mathscr{L}\left(G_p^{-p} \int f(x)\pi(dx)\right),$$

where G_p is a stable random variable with Laplace transform

$$E\exp\{-tG_p\} = \exp\left\{-\frac{t^p}{\Gamma(p+1)}\right\}, \qquad t \geq 0.$$

(iii) *When the regular index $p = 1$,*

$$\sum_{k=1}^{n} f(X_k) \Big/ a(n) \to \int f(x)\pi(dx) \quad \text{in probability.}$$

THEOREM 2.4. *Let $\{X_n\}_{n\geq 0}$ be a p-regular ($0 \leq p \leq 1$) Harris recurrent Markov chain with state space E, transition probability $P(x, A)$ and invariant* *measure* π. *Then*,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} f(X_k) \Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) \tag{2.2}$$
$$= \frac{\Gamma(p+1)}{p^p (1-p)^{1-p}} \int f(x)\pi(dx) \quad a.s.$$

for every nonnegative function $f \in \mathscr{L}^1(E, \mathscr{E}, \pi)$, *where we interpret* $p^p = (1-p)^{1-p} = 1$ *if* $p = 0$ *or* 1.

The weak laws for the Markovian additive functionals of our type (where $f \geq 0$) have been studied in Darling and Kac (1957) by a method different from what we propose in this paper. The special cases of random walks, Brownian motions and Lèvy processes have been studied in a number of papers. A good resource to find these results is the book by Révész (1990). Here we would like to mention the work by Marcus and Rosen (1994b) on the strong laws for local times of recurrent random walks and Lévy process, where they connect trancated Green functions with the growth rate of the local times. A nice way of computing truncated Green functions in the case of random walks is given in Proposition 2.4 [and consequently, (2.j)] of Le Gall and Rosen (1991). In Section 6, we will apply Theorems 2.3 and 2.4 to one-dimensional and two-dimensional random walks by making use of their results.

Our method consists of three steps: regeneration, splitting and sampling. The concept of split chain is independently introduced by Athreya and Ney (1978) and Nummelin (1978). The idea is to construct an atom which allows for regeneration. This powerful tool has been developed and utilized by quite a number of works in the study of Markov chains with general state spaces. We refer to the books by Nummelin (1984), Meyn and Tweedie (1993), Duflo (1997) and some references quoted in these books as part of the literature. The three-step scheme of regeneration, splitting and sampling is considerably exploited in Touati (1990). Our way of utilizing this method is slightly different, which can also be found in de Acosta and Chen (1998) and Chen (1998). As an important feature of this paper, our strong laws rely on a large deviation estimation for Markov chains, which takes a form different from those known in the literature. This progress is partially inspired by the observation made in Csáki and Csörgő (1995) on the stability of the hitting times.

3. Some results related to an atom. The main idea of the regeneration method is to analyze the Markov chain by dividing it into independent and identically distributed (i.i.d.) random blocks, with which the concept of atom plays a crucial role in this paper. A set $\alpha \in \mathscr{E}^+$ is called an *atom* of $\{X_n\}_{n\geq 0}$ [or its transition $P(x, A)$] if

$$P(x, \cdot) = P(y, \cdot) \quad \text{for all } x, y \in \alpha.$$

Noticing that $P_x = P_y$ for $x, y \in \alpha$ on the σ-algebra generated by $\{X_n\}_{n\geq 1}$, we denote the common value by P_α. Notations like $P(\alpha, \cdot)$ and E_α are also used in the obvious way. In the rest of the section we suppose that the Harris recurrent Markov chain $\{X_n\}_{n\geq 0}$ has an atom α. Let

$$i_\alpha(n) = \sum_{k=1}^{n} I_\alpha(X_k), \qquad n = 1, 2, \dots, \tag{3.1}$$

$$\varphi(t) = \sum_{k=1}^{[t]} P^k(\alpha, \alpha), \qquad t \geq 0. \tag{3.2}$$

According to Proposition 5.13(iii) in Nummelin (1984), α is a D-set. In view of (1.3) we have

$$\lim_{t\to+\infty} \varphi(t)/a(t) = \pi(\alpha). \tag{3.3}$$

Define

$$\tau_\alpha(0) = 0 \quad \text{and} \quad \tau_\alpha(1) = \tau_\alpha = \inf\{n \geq 1;\ X_n \in \alpha\},$$
$$\tau_\alpha(k+1) = \inf\{n > \tau_\alpha(k);\ X_n \in \alpha\}, \qquad k \geq 1.$$

The assumption of Harris recurrence ensures that for each k, $\tau_\alpha(k) < +\infty$ a.s. P_μ for each initial distribution μ.

It is well known and easy to verify (via strong Markov property) that under the law P_α, the random blocks $\{B_k\}_{k\geq 0}$ given by

$$B_k = \left\{X_{\tau_\alpha(k)+1}, \dots, X_{\tau_\alpha(k+1)}\right\}, \qquad k = 0, 1, 2, \dots .$$

is an i.i.d. sequence with the common distribution

$$\mathscr{L}_{P_\alpha}\left(\{X_1, \dots, X_{\tau_\alpha}\}\right).$$

In particular, under P_α the real random sequence $\{\tau_\alpha(k) - \tau_\alpha(k-1)\}_{k\geq 1}$ is an i.i.d. with the common distribution

$$P_\alpha\{\tau_\alpha = n\}, \qquad n = 1, 2, \dots .$$

LEMMA 3.1. *For each* $n \geq 1$,

$$E_\alpha \min\{\tau_\alpha, n\} \geq \frac{n}{1 + \varphi(n)}. \tag{3.4}$$

PROOF. By definition we have

$$\tau_\alpha(i_\alpha(n)) = \max\{k;\ 0 \leq k \leq n \quad \text{and} \quad X_k \in \alpha\}. \tag{3.5}$$

Therefore,

$$\begin{aligned}
1 = P_\alpha\left\{0 \leq \tau_\alpha(i_\alpha(n)) \leq n\right\} &= \sum_{k=0}^{n} P_\alpha\left\{\tau_\alpha(i_\alpha(n)) = k\right\} \\
&= \sum_{k=0}^{n} P_\alpha\left\{X_k \in \alpha,\ X_{k+1} \notin \alpha, \dots, X_n \notin \alpha\right\} \\
&= \sum_{k=0}^{n} P_\alpha\left\{X_k \in \alpha\right\} P_\alpha\left\{\tau_\alpha \geq n - k + 1\right\}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
n &= \sum_{k=0}^{n-1} \sum_{j=1}^{k} P_\alpha\{X_j \in \alpha\} P_\alpha\{\tau_\alpha \ge k - j + 1\} \\
&= \sum_{j=0}^{n-1} P_\alpha\{X_j \in \alpha\} \sum_{k=1}^{n-j} P_\alpha\{\tau_\alpha \ge k\} \\
&\le (1 + \varphi(n-1)) \sum_{k=1}^{n} P_\alpha\{\tau_\alpha \ge k\} \\
&\le (1 + \varphi(n)) E_\alpha \min\{\tau_\alpha, n\},
\end{aligned}
$$

which gives (3.4). □

LEMMA 3.2. (i) *For each* $n \ge 1$ *and* $t > 0$,

$$E_\alpha \exp\big\{ - \big(\log E_\alpha \exp\{-t\tau_\alpha\}\big) i_\alpha(n) \big\} \ge E_\alpha \exp\{-t\tau_\alpha\} e^{nt}.$$

(ii) *For each* $n \ge 1$, $t > 0$ *and* $\lambda > 0$,

$$E_\alpha \exp\big\{ - \big(\log E_\alpha \exp\big\{ - t \min\{\tau_\alpha, \lambda\}\big\}\big) i_\alpha(n) \big\} \le \exp(t(\lambda + n)).$$

PROOF. Consider the random sequence $\{M_n\}_{n\ge 0}$ given by

$$M_n = \exp\big\{ - t\tau_\alpha(n) - \big(\log E_\alpha \exp\{-t\tau_\alpha\}\big) n \big\}, \qquad n = 0, 1, 2, \dots .$$

Under the law P_α, $\{M_n\}_{n\ge 0}$ is a Martingale w.r.t. the filtration

$$\sigma\{X_k;\ k \le \tau_\alpha(n)\}, \qquad n = 0, 1, 2, \dots .$$

On the other hand, by definition,

$$i_\alpha(n) + 1 = \min\{k \ge 1;\ \tau_\alpha(k) > n\}. \tag{3.6}$$

Hence $i_\alpha(n) + 1$ is a stopping time w.r.t. $\{M_n\}_{n\ge 0}$. By a well-known Doob's stopping rule,

$$E_\alpha \exp\big\{ - t\tau_\alpha(i_\alpha(n) + 1) - (\log E_\alpha \exp\{-t\tau_\alpha\})(i_\alpha(n) + 1) \big\} = 1.$$

Hence the claim (1) follows from the fact that $\tau_\alpha(i_\alpha(n) + 1) > n$ [see (3.6)].

With $\{\tau_\alpha(k) - \tau_\alpha(k-1)\}_{k\ge 1}$ being replaced by the i.i.d. sequence $\{\xi_k\}_{k\ge 1}$,

$$\xi_k = \min\big\{\tau_\alpha(k) - \tau_\alpha(k-1), \lambda\big\}, \qquad k = 1, 2, \dots$$

and with the similar argument, one can prove

$$E_\alpha \exp\left\{ -t \sum_{k=1}^{i_\alpha(n)+1} \xi_k - \big(\log E_\alpha \exp\big\{ - t \min\{\tau_\alpha, \lambda\}\big\}\big)\big(i_\alpha(n) + 1\big) \right\} = 1.$$

Hence the claim (2) follows from the following observation:

$$\sum_{k=1}^{i_\alpha(n)+1} \xi_k \le \lambda + \sum_{k=1}^{i_\alpha(n)} \xi_k \le \lambda + \tau_\alpha\big(i_\alpha(n)\big) \le \lambda + n,$$

where the last step follows from (3.5). □

LEMMA 3.3. *Assume that* $\{b_n\}_{n\geq 1}$ *is a sequence of nondecreasing positive numbers such that*

$$\sum_k P_\alpha\big\{i_\alpha(n_k) \geq b_{n_k}\big\} = +\infty \tag{3.7}$$

for some subsequence $\{n_k\}$ *satisfying*

$$b_{n_{k+1}} \geq r b_{n_k} \quad \text{for sufficiently large } k \tag{3.8}$$

with some $r > 1$. *Then*

$$\limsup_{n\to\infty} \frac{i_\alpha(n)}{b_n} \geq 1 \quad a.s. \tag{3.9}$$

PROOF. Although the situation is different, some of the ideas in the proof come from Kuelbs (1981). We may assume that $\{b_n\}_{n\geq 1}$ are integers, for otherwise we consider $\{[b_n]\}_{n\geq 1}$ instead. For each $k \geq 1$, define

$$\sigma_k = \inf\{n \geq n_k;\ \tau_\alpha(b_n) \leq n\}.$$

Given $\varepsilon > 0$, from (3.8) there exists an integer $s \geq 1$ such that

$$b_{n_{k+1}}/b_{n_{k+s}} < \varepsilon \qquad \forall\ k \geq 1.$$

We have

$$\begin{aligned} D_k &\equiv \big\{\tau_\alpha(b_n) > n \text{ for all } n \geq n_{k+s},\ \sigma_k \in [n_k, n_{k+1})\big\} \\ &\supset \big\{\tau_\alpha(b_n) - \tau_\alpha(b_{\sigma_k}) > n \text{ for all } n \geq n_{k+s},\ \sigma_k \in [n_k, n_{k+1})\big\}. \end{aligned}$$

By independence,

$$\begin{aligned} P_\alpha(D_k) &\geq \sum_{j\in[n_k,\, n_{k+1})} P_\alpha\{\sigma_k = j\} P_\alpha\big\{\tau_\alpha(b_n - b_j) > n \text{ for all } n \geq n_{k+s}\big\} \\ &\geq P_\alpha\big\{\sigma_k \in [n_k, n_{k+1})\big\} P_\alpha\big\{\tau_\alpha([(1-\varepsilon)b_n]) > n \text{ for all } n \geq n_s\big\}. \end{aligned}$$

From the definition of D_k, we easily see that at most s of the events D_k can occur at a single time, so

$$\begin{aligned} s &\geq \sum_k P_\alpha(D_k) \\ &\geq P_\alpha\big\{\tau_\alpha([(1-\varepsilon)b_n]) > n \text{ for all } n \geq n_s\big\} \sum_k P_\alpha\big\{\sigma_k \in [n_k, n_{k+1})\big\}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_k P_\alpha\big\{\sigma_k \in [n_k, n_{k+1})\big\} &\geq \sum_k P_\alpha\big\{\sigma_k = n_k\big\} \\ &= \sum_k P_\alpha\big\{\tau_\alpha(b_{n_k}) \leq n_k\big\} \\ &= \sum_k P_\alpha\big\{i_\alpha(n_k) \geq b_{n_k}\big\} = +\infty, \end{aligned}$$

where the third step follows from the fact

$$\{\tau_\alpha(k) \le n\} = \{i_\alpha(n) \ge k\} \qquad \forall\, n,\ k \ge 1. \tag{3.10}$$

Hence,

$$P_\alpha\{\tau_\alpha([(1-\varepsilon)b_n]) > n \text{ for all } n \ge n_s\} = 0.$$

Since $s \ge 1$ can be arbitrarily large, we have proved

$$P_\alpha\{\tau_\alpha([(1-\varepsilon)b_n]) \le n \text{ i.o.}\} = 1.$$

In view of (3.10) we have

$$\limsup_{n\to\infty} \frac{i_\alpha(n)}{b_n} \ge 1 - \varepsilon \quad \text{a.s.}$$

Letting $\varepsilon \to 0^+$ gives (3.9). □

LEMMA 3.4. *Let* $\{X_n\}_{n\ge 0}$ *be p-regular* $(0 \le p \le 1)$. *Then*

$$\log E_\alpha \exp(-t\tau_\alpha) \sim -\big(\varphi(t^{-1})\Gamma(p+1)\big)^{-1} \quad \text{as } t \to 0^+. \tag{3.11}$$

PROOF. By (8.10), page 177 in Meyn and Tweedie (1993), for each $t > 0$,

$$\sum_{n=1}^{\infty} P^n(\alpha,\alpha)e^{-tn} = E_\alpha \exp(-t\tau_\alpha) + E_\alpha \exp(-t\tau_\alpha) \sum_{n=1}^{\infty} P^n(\alpha,\alpha)e^{-tn}.$$

Hence,

$$\log E_\alpha \exp(-t\tau_\alpha) = \log \frac{\sum_{n=1}^{\infty} P^n(\alpha,\alpha)e^{-tn}}{1 + \sum_{n=1}^{\infty} P^n(\alpha,\alpha)e^{-tn}} \sim -\left(\sum_{n=1}^{\infty} P^n(\alpha,\alpha)e^{-tn}\right)^{-1}$$

as $t \to 0^+$, where the second step follows from the fact that

$$\sum_{n=1}^{\infty} P^n(\alpha,\alpha) = +\infty.$$

Hence, (3.11) follows from Tauberian's theorm. □

4. Proof of Theorems 2.1 and 2.2.

We only prove Theorem 2.1 and Theorem 2.2 in the case when $\{X_n\}_{n\ge 0}$ posseses an atom α. We also assume that $\{X_n\}_{n\ge 0}$ starts from α. In the proof of Theorems 2.3 and 2.4 later we shall show how the general situation can be reduced to this special case. Let $i_\alpha(n)$ and $\varphi(t)$ be given in (3.1) and (3.2), respectively. By (1.2) and (1.3) we may take $f = I_\alpha$ and $D = \alpha$ [Recall that D is given in the definition of $a(t)$.] Therefore the boundedness in probability of $\{i_\alpha(n)/\varphi(n)\}_{n\ge 1}$ follows from the Chebyshev inequality.

To prove Theorem 2.1 in the reduced case, it remains to show the boundedness in probability of the sequence $\{(i_\alpha(n)/\varphi(n))^{-1}\}_{n\ge 1}$. That is, for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$P_\alpha\{i_\alpha(n) \ge \delta\varphi(n)\} \ge 1 - \varepsilon \tag{4.1}$$

for sufficiently large n. By (2.4) in Lemma 1 of Chen (1998), for any $s > 0$ and $n \geq 1$,

$$E_\alpha\big(i_\alpha(n) I_{\{i_\alpha(n)\geq s\}}\big) \leq P_\alpha\big\{i_\alpha(n) \geq s\big\}(s + 1 + \varphi(n)).$$

Taking $s = \delta\varphi(n)$ gives

$$\begin{aligned}(1-\delta)\varphi(n) &\leq E_\alpha\big(i_\alpha(n) I_{\{i_\alpha(n)\geq \delta\varphi(n)\}}\big)\\ &\leq P_\alpha\big\{i_\alpha(n) \geq \delta\varphi(n)\big\}((1+\delta)\varphi(n) + 1).\end{aligned}$$

Therefore,

$$P_\alpha\big\{i_\alpha(n) \geq \delta\varphi(n)\big\} \geq \frac{(1-\delta)\varphi(n)}{(1+\delta)\varphi(n)+1},$$

from which one can see how (4.1) is satisfied for small $\delta > 0$.

To have Theorem 2.2 in the atomic case, we need to prove

$$0 < \limsup_{n\to\infty} i_\alpha(n) \Big/ \varphi\Big(\frac{n}{L_2\varphi(n)}\Big) L_2\varphi(n) < +\infty \quad \text{a.s.} \tag{4.2}$$

We first establish the upper bound. Taking $t = L_2\varphi(n)/n$ and $\lambda = n/L_2\varphi(n)$ in Lemma 3.2(ii) gives

$$\begin{aligned}&E_\alpha \exp\bigg\{-\bigg(\log E_\alpha \exp\bigg\{-\frac{L_2\varphi(n)}{n}\min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\bigg\}\bigg) i_\alpha(n)\bigg\}\\ &\quad\leq \exp(1 + L_2\varphi(n))\end{aligned}$$

for every $n \geq 1$.

On the other hand,

$$\begin{aligned}&E_\alpha \exp\bigg\{-\frac{L_2\varphi(n)}{n}\min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\bigg\}\\ &\quad\leq 1 - \frac{L_2\varphi(n)}{n} E_\alpha \min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\\ &\qquad + \frac{1}{2}\bigg(\frac{L_2\varphi(n)}{n}\bigg)^2 E_\alpha\bigg(\min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\bigg)^2\\ &\quad\leq 1 - \frac{L_2\varphi(n)}{2n} E_\alpha \min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\\ &\quad\leq 1 - \frac{L_2\varphi(n)}{2n}\frac{[n/L_2\varphi(n)]}{1+\varphi([n/L_2\varphi(n)])},\end{aligned}$$

where the last step follows from Lemma 3.1. Hence,

$$-\log E_\alpha \exp\bigg\{-\frac{L_2\varphi(n)}{n}\min\bigg\{\tau_\alpha, \frac{n}{L_2\varphi(n)}\bigg\}\bigg\} \geq \frac{1}{4\varphi(n/L_2\varphi(n))}$$

for sufficiently large n. We have thus proved

$$E_\alpha \exp\bigg\{i_\alpha(n) \Big/ 4\varphi\bigg(\frac{n}{L_2\varphi(n)}\bigg)\bigg\} \leq e \exp\{L_2\varphi(n)\}$$

for sufficiently large n. Therefore,

$$
\begin{aligned}
&P_\alpha\left\{i_\alpha(n) \geq 12\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right\} \\
(4.3)\qquad &\leq \exp\{-3L_2\varphi(n)\}E_\alpha \exp\left\{i_\alpha(n)\Big/4\varphi\left(\frac{n}{L_2\varphi(n)}\right)\right\} \\
&\leq e\exp\big\{-2L_2\varphi(n)\big\} \quad \text{for large } n.
\end{aligned}
$$

Define the subsequence $\{n_k\}$ by

$$n_k = \inf\big\{n;\ \varphi\big(n/L_2\varphi(n)\big) \geq 2^k\big\}, \qquad k = 1, 2, \dots .$$

We hence have

$$L_2\varphi(n_k) \sim \log k \quad \text{as } k \to \infty. \tag{4.4}$$

In view of (4.3) we have

$$\sum_k P_\alpha\left\{i_\alpha(n_k) \geq 12\varphi\left(\frac{n_k}{L_2\varphi(n_k)}\right)L_2\varphi(n_k)\right\} < +\infty.$$

By the Borel–Cantelli lemma,

$$\limsup_{k\to\infty} i_\alpha(n_k)\Big/\varphi\left(\frac{n_k}{L_2\varphi(n_k)}\right)L_2\varphi(n_k) \leq 12 \quad \text{a.s.}$$

So the upper bound in (4.2) follows from a standard argument.

It remains to establish the lower bound. Let

$$b_n = \left[\lambda\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right], \qquad n = 1, 2, \dots,$$

where the constant $\lambda > 0$ will be specified later. By (3.10),

$$P_\alpha\left\{i_\alpha(n) \geq \lambda\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right\} = P_\alpha\big\{\tau_\alpha(b_n) \leq n\big\}.$$

Let

$$p_n = \big[b_n/L_2\varphi(n)\big] \quad \text{and} \quad q_n = [b_n/p_n], \qquad n = 1, 2, \dots .$$

Then for each $n \geq 1$,

$$
\begin{aligned}
P_\alpha\big\{\tau_\alpha(b_n) \leq n\big\} &\geq P_\alpha\big\{\tau_\alpha(p_n(q_n+1)) \leq n\big\} \\
&\geq P_\alpha\left(\bigcap_{j=1}^{q_n+1}\left\{\tau_\alpha(jp_n) - \tau_\alpha((j-1)p_n) \leq \frac{n}{q_n+1}\right\}\right) \\
&= \left(P_\alpha\left\{\tau_\alpha(p_n) \leq \frac{n}{q_n+1}\right\}\right)^{q_n+1} \\
&= \left(P_\alpha\left\{i_\alpha\left(\left[\frac{n}{q_n+1}\right]\right) \geq p_n\right\}\right)^{q_n+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&P_\alpha\left\{i_\alpha(n) \geq \lambda\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right\} \\
&\qquad \geq \left(P_\alpha\left\{i_\alpha\left(\left[\frac{n}{q_n+1}\right]\right) \geq p_n\right\}\right)^{q_n+1}.
\end{aligned}
\tag{4.5}
$$

Notice that

$$
p_n \sim \lambda\varphi\left(\frac{n}{L_2\varphi(n)}\right), \qquad q_n \sim L_2\varphi(n) \quad \text{and} \quad \frac{n}{q_n+1} \sim \frac{n}{L_2\varphi(n)}
\tag{4.6}
$$

as $n \to \infty$. By (4.1), for given $0 < u < 1$ we can take $\lambda > 0$ so small that

$$
P_\alpha\left\{i_\alpha\left(\left[\frac{n}{q_n+1}\right]\right) \geq p_n\right\} \geq e^{-u}
$$

holds eventually. Combining (4.4) and (4.5) gives

$$
\sum_k P_\alpha\left\{i_\alpha(n_k) \geq \lambda\varphi\left(\frac{n_k}{L_2\varphi(n_k)}\right)L_2\varphi(n_k)\right\} = +\infty.
$$

By Lemma 3.3 we thus have

$$
\limsup_{n\to\infty} i_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n) \geq \lambda > 0 \quad \text{a.s.}
$$

which gives a desired lower bound. □

5. Proof of Theorems 2.3 and 2.4.

STEP 1. We first prove Theorems 2.3 and 2.4 under the additional assumption that $\{X_n\}_{n\geq 0}$ possesses an atom α such that

$$
P(\alpha, \alpha) > 0.
\tag{5.1}
$$

Similarly, we take $f = I_\alpha$ and $D = \alpha$. Hence, Theorem 2.3 is restated as:

(1′) When the regular index $p = 0$,

$$
i_\alpha(n)/\varphi(n) \to \text{Exp in distribution,}
$$

where Exp is a random variable having the exponential distribution with parameter 1.

(2′) When the regular index $0 < p < 1$, $\{i_\alpha(n)/\varphi(n)\}_{n\geq 1}$ weakly converges to $\mathscr{L}(G_p^{-p})$.

(3′) When the regular index $p = 1$,

$$
i_\alpha(n)/\varphi(n) \to 1 \text{ in probability.}
$$

An equivalent form of Theorem 2.4 in the atomic case is

$$
\limsup_{n\to\infty} i_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n) = \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}} \quad \text{a.s.}
\tag{5.2}
$$

We now prove (1′), (2′) and (3′). From (5.1), $\{X_n\}_{n\geq 0}$ is aperiodic [see, e.g., Meyn and Tweedie (1993) for details]. By Orey's convergence theorem [see, e.g., Theorem 18.1.2 in Meyn and Tweedie (1993)], we may assume that $\{X_n\}_{n\geq 0}$ starts from α. Suppose $0 < p \leq 1$ and let $\psi(t)$ be the inverse function of φ (defined in the usual way). From Lemma 3.4 we have that for all $t > 0$,

$$E_\alpha \exp\left\{-t\frac{\tau_\alpha(n)}{\psi(n)}\right\} \to \exp\left\{-\frac{t^p}{\Gamma(p+1)}\right\} \quad \text{as } n \to \infty.$$

Hence,

$$\tau_\alpha(n)/\psi(n) \to G_p \quad \text{in distribution if } 0 < p < 1 \tag{5.3}$$

and

$$\tau_\alpha(n)/\psi(n) \to 1 \quad \text{in probability if } p = 1. \tag{5.4}$$

From (3.10) one can see how (5.3) and (5.4) imply (2′) and (3′), respectively.

Suppose $p = 0$. Then, $\varphi(t)$ is slowly varying at infinity,

$$\lim_{\lambda\to+\infty} \varphi(\lambda t)/\varphi(\lambda) = 1 \qquad \forall t > 0.$$

By Lemma 3.4,

$$1 - E_\alpha \exp(-t\tau_\alpha) \sim \frac{1}{\varphi(t^{-1})} \quad \text{as } t \to 0^+.$$

According to Tauberian's theorem [see, e.g., Feller (1971) Section XIII. 5], this is equivalent to

$$P_\alpha\{\tau_\alpha > \lambda\} \sim \frac{1}{\varphi(\lambda)} \quad \text{as } \lambda \to +\infty.$$

Therefore [Darling (1952)],

$$\varphi(\tau_\alpha(n))/n \to 1/\mathrm{Exp} \text{ in distribution,}$$

which, by (3.10), implies (1′).

We now prove (5.2). We first prove the following.

Claim 1. When $p = 0$,

$$\lim_{n\to\infty} \frac{1}{L_2\varphi(n)} \log E_\alpha \exp\left\{ti_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)\right\} = 0$$

for every $0 < t < 1$.

Claim 2. When $0 < p \leq 1$,

$$\lim_{n\to\infty} \frac{1}{L_2\varphi(n)} \log E_\alpha \exp\left\{ti_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)\right\} = (\Gamma(p+1)t)^{1/p}$$

for every $t > 0$.

Given $s > 0$ and $\varepsilon > 0$, taking $t = sL_2\varphi(n)/n$ and $\lambda = \varepsilon n$ in Lemma 3.2(ii) gives

$$E_\alpha \exp\biggl\{-\biggl(\log E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\min\{\tau_\alpha, \varepsilon n\}\biggr\}\biggr)i_\alpha(n)\biggr\} \le \exp\{(1+\varepsilon)sL_2\varphi(n)\} \qquad \forall\, n \ge 1. \tag{5.5}$$

On the other hand,

$$\begin{aligned} &E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\min\{\tau_\alpha, \varepsilon n\}\biggr\} \\ &\quad = E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\tau_\alpha\biggr\} + o(1) \\ &\quad = (1+o(1))E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\tau_\alpha\biggr\}, \qquad n \to \infty. \end{aligned}$$

By regularity and Lemma 3.4, therefore,

$$-\log E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\min\{\tau_\alpha, \varepsilon n\}\biggr\} \sim \frac{s^p}{\varphi(n/L_2\varphi(n))\Gamma(p+1)}, \qquad n \to \infty. \tag{5.6}$$

If $p = 0$, from (5.6) for given $0 < t < 1$,

$$-\log E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\min\{\tau_\alpha, \varepsilon n\}\biggr\} > t\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)$$

for sufficiently large n. By (5.5),

$$E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \le \exp\{(1+\varepsilon)sL_2\varphi(n)\}$$

eventually holds, which gives

$$\limsup_{n\to\infty} \frac{1}{L_2\varphi(n)}\log E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \le (1+\varepsilon)s. \tag{5.7}$$

Letting $s \to 0^+$ proves Claim 1.

Suppose $0 < p \le 1$ and let $t > 0$ be fixed. The similar argument gives (5.7) for all $s > (\Gamma(p+1)t)^{1/p}$. Letting $s \to (\Gamma(p+1)t)^{1/p}$ and $\varepsilon \to 0^+$ gives

$$\limsup_{n\to\infty} \frac{1}{L_2\varphi(n)}\log E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \le (\Gamma(p+1)t)^{1/p}. \tag{5.8}$$

On the other hand, taking $t = sL_2\varphi(n)/n$ in Lemma 3.2(i) gives

$$\begin{aligned} &E_\alpha \exp\biggl\{-\biggl(\log E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\tau_\alpha\biggr\}\biggr) i_\alpha(n)\biggr\} \\ &\quad\geq E_\alpha \exp\biggl\{-\frac{sL_2\varphi(n)}{n}\tau_\alpha\biggr\} \exp(sL_2\varphi(n)) \\ &\quad= (1+o(1))\exp(sL_2\varphi(n)) \quad \text{as } n\to\infty. \end{aligned}$$

Applying Lemma 3.4 and regularity we see that for arbitrary $0 < s < (\Gamma(p+1)t)^{1/p}$,

$$E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \geq (1+o(1))\exp\{sL_2\varphi(n)\}$$

eventually holds, which gives

$$\liminf_{n\to\infty} \frac{1}{L_2\varphi(n)} \log E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \geq (\Gamma(p+1)t)^{1/p}. \tag{5.9}$$

Hence, Claim 2 follows from (5.8) and (5.9).

We now prove (5.2) in the case $0 < p < 1$. Note that $E_\alpha i_\alpha(n) = \varphi(n)$ $(n \geq 1)$. Given $t \leq 0$, by Jensen's inequality,

$$E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} \geq \exp\biggl\{t\varphi(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\}$$

for each $n \geq 1$. Since $p < 1$, one can see that

$$\varphi(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr) = o(L_2\varphi(n)) \quad \text{as } n\to\infty.$$

Hence we have

$$\lim_{n\to\infty} \frac{1}{L_2\varphi(n)} \log E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} = 0 \quad \forall\, t \leq 0.$$

Combining this with the previous observation gives

$$\lim_{n\to\infty} \frac{1}{L_2\varphi(n)} \log E_\alpha \exp\biggl\{ti_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)\biggr\} = \Lambda(t) \qquad \forall\, t \in \mathbf{R},$$

where

$$\Lambda(t) = \begin{cases} (\Gamma(p+1)t)^{1/p}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

By the Gärtner–Ellis theorem [Theorem 2.3.6 in Dembo and Zeitouni (1993)] on the large deviations, the distribution sequence

$$\mathscr{L}_{P_\alpha}\biggl(i_\alpha(n)\Big/\varphi\biggl(\frac{n}{L_2\varphi(n)}\biggr)L_2\varphi(n)\biggr), \qquad n = 1, 2, \ldots$$

satisfies the large deviation principle governed by the rate function $\Lambda^*(\lambda)$ given by

$$\Lambda^*(\lambda) = \sup_{t\in\mathbf{R}} \{t\lambda - \Lambda(t)\} = \begin{cases} p^{p(1-p)^{-1}}(1-p)\left(\dfrac{\lambda}{\Gamma(p+1)}\right)^{(1-p)^{-1}}, & \lambda \geq 0, \\ +\infty, & \lambda < 0. \end{cases}$$

In particular, for every $\lambda > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{L_2\varphi(n)} \log P_\alpha\left\{i_\alpha(n) \geq \lambda\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right\} \\ &\qquad = -p^{p(1-p)^{-1}}(1-p)\left(\frac{\lambda}{\Gamma(p+1)}\right)^{(1-p)^{-1}}. \end{aligned} \tag{5.10}$$

Let $r > 1$ be fixed but arbitrary and define subsequence $\{n_k\}_k$ by

$$n_k = \inf\{n;\ \varphi(n/L_2\varphi(n)) \geq r^k\}, \qquad k = 1, 2, \dots .$$

One can see that $L_2\varphi(n_k) \sim \log k$ as $k \to \infty$. Since the right-hand side of (5.10) is greater or less than -1 depending on

$$\lambda < \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}} \quad \text{or} \quad \lambda > \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}},$$

the series

$$\sum_k P_\alpha\left\{i_\alpha(n_k) \geq \lambda\varphi\left(\frac{n_k}{L_2\varphi(n_k)}\right)L_2\varphi(n_k)\right\}$$

diverges or converges accordingly. By Lemma 3.3,

$$\limsup_{n\to\infty} i_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n) \geq \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}} \quad \text{a.s.} \tag{5.11}$$

On the other hand, by the Borel–Cantelli lemma,

$$\limsup_{k\to\infty} i_\alpha(n_k)\Big/\varphi\left(\frac{n_k}{L_2\varphi(n_k)}\right)L_2\varphi(n_k) \leq \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}} \quad \text{a.s.}$$

Notice that $r > 1$ can be arbitrarily close to 1. A standard argument gives

$$\limsup_{n\to\infty} i_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n) \leq \frac{\Gamma(p+1)}{p^p(1-p)^{1-p}} \quad \text{a.s.} \tag{5.12}$$

Hence, (5.2) follows from (5.11) and (5.12) in the case $0 < p < 1$.

It remains to prove (5.2) for $p = 0$ and $p = 1$, which takes the form

$$\limsup_{n\to\infty} i_\alpha(n)\Big/\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n) = 1 \quad \text{a.s.}$$

Observing the previous proof we need only to show that for any $0 < \varepsilon < 1$,

$$\limsup_{n\to\infty} \frac{1}{L_2\varphi(n)} \log P_\alpha\left\{i_\alpha(n) \geq (1+\varepsilon)\varphi\left(\frac{n}{L_2\varphi(n)}\right)L_2\varphi(n)\right\} < -1, \tag{5.13}$$

$$\liminf_{n\to\infty} \frac{1}{L_2\varphi(n)} \log P_\alpha\left\{ i_\alpha(n) \ge (1-\varepsilon)\varphi\left(\frac{n}{L_2\varphi(n)}\right) L_2\varphi(n)\right\} > -1. \tag{5.14}$$

The upper bound (5.13) is a direct consequence of Claim 1 and Claim 2. We now prove the lower bound (5.14). Because of similarity we take the case $p = 0$ for an example. In view of (4.5) and (4.6), in which we take $\lambda = 1 - \varepsilon$, we have

$$P_\alpha\left\{ i_\alpha(n) \ge (1-\varepsilon)\varphi\left(\frac{n}{L_2\varphi(n)}\right) L_2\varphi(n)\right\} \ge \left(P_\alpha\left\{ i_\alpha\left(\left[\frac{n}{q_n+1}\right]\right) \ge p_n\right\}\right)^{q_n+1}.$$

Notice that $\{i_\alpha(n)/\varphi(n)\}_{n\ge1}$ weakly converges to the exponential distribution with parameter 1. Therefore,

$$\begin{aligned}\liminf_{n\to\infty} &\frac{1}{L_2\varphi(n)} \log P_\alpha\left\{ i_\alpha(n) \ge (1-\varepsilon)\varphi\left(\frac{n}{L_2\varphi(n)}\right) L_2\varphi(n)\right\} \\ &\ge \log \int_{1-\varepsilon}^{+\infty} e^{-t}\,dt = -1+\varepsilon > -1.\end{aligned}$$

STEP 2. We now prove Theorem 2.3 and Theorem 2.4 under the assumption that there exists a $C \in \mathscr{E}^+$ such that

$$P(x, A) \ge bI_C(x)\nu(A), \qquad x \in E,\ A \in \mathscr{E} \tag{5.15}$$

for some $b > 0$ and probability measure ν on $(E, \mathscr{E})$ with $\nu(C) > 0$.

Our approuch is to create an atom by making use of so-called split chain technique which initially belongs to Nummelin (1978, 1984) and Athreya and Ney (1978). According to Section 4.4 in Nummelin (1984) [with $s(x) = I_C(x)$], under (5.15) one can embed $\{X_n\}_{n\ge0}$ into a larger probability space on which a sequence $\{Y_n\}_{n\ge0}$ of $\{0,1\}$-valued random variables is defined in such a way that $\{(X_n, Y_n)\}_{n\ge0}$ becomes a Markov chain with state space $E \times \{0,1\}$ and an atom α^* given by

$$\alpha^* = E \times \{1\}.$$

Such a Markov chain is called *split chain* in the literature. By Proposition 4.8 in Nummelin (1984), $\{(X_n, Y_n)\}_{n\ge0}$ is Harris recurrent. We also have

$$\pi^*(\alpha^*) = b\pi(C) \quad \text{and} \quad \int f(x)\pi^*(d(x,y)) = \int f(x)\,dx, \tag{5.16}$$

where π^* is the invariant measure of $\{(X_n, Y_n)\}_{n\ge0}$ such that the marginal of π^* on E is π [see Section 4.4 in Nummelin (1984) for details]. Moreover [(4.19), page 63, Nummelin (1984)],

$$\tilde{P}^k(\alpha^*, \alpha^*) = b\nu P^{k-1}(C), \qquad k = 1, 2, \dots$$

where $\tilde{P}^k$ is the k-step transition of $\{(X_n, Y_n)\}_{n\ge0}$. Taking $k = 1$, one can see that the condition (5.1) applies to the split chain and its atom α^*. Further, from the first part of (5.16) we have

$$
\begin{aligned}
\tilde{a}(n) &\equiv \pi^*(\alpha^*)^{-1} \sum_{k=1}^{n} \tilde{P}^k(\alpha^*, \alpha^*) \\
&\sim \pi(C)^{-1} \sum_{k=1}^{n} \nu P^k(C) \sim a(n), \qquad n \to \infty,
\end{aligned}
\tag{5.17}
$$

where the second "$\sim$" follows from (1.3) and the fact [Proposition 5.13(iii) in Nummelin (1984)] that C is a D-set of $\{X_n\}_{n\geq 0}$. In particular, $\{(X_n, Y_n)\}_{n\geq 0}$ is p-regular. Applying the result achieved at the previous step to the split chain $\{(X_n, Y_n)\}_{n\geq 0}$, function f (viewed as a function on $E\times\{0,1\}$) and the sequence $\{\tilde{a}(n)\}_{n\geq 1}$, and combining (5.16), (5.17) we have the desired conclusion.

STEP 3. We now extend our results to the general situation. Because of similarity we only prove Theorem 2.4. Let $0 < t < 1$ be fixed but arbitrary. Define the transition probability $P_t(x, A)$ on $(E, \mathscr{E})$ as follows:

$$
P_t(x, A) = (1-t) \sum_{k=1}^{\infty} t^{k-1} P^k(x, A), \qquad x \in E,\ A \in \mathscr{E}.
$$

According to Proposition 8.2.13(1) in Duflo (1997), $P_t(x, A)$ is Harris recurrent. One can directly verify that π is an invariant measure of $P_t(x, A)$. By Proposition 5.4.5(ii) of Meyn and Tweedie (1993), $P_t(x, A)$ satisfies the additional condition assumed at the second step.

Let $\{\beta_n\}_{n\geq 1}$ be an i.i.d. Bernoulli random variables with the common law,

$$
P\{\beta_1 = 0\} = t \quad \text{and} \quad P\{\beta_1 = 1\} = 1 - t.
$$

We always assume independence between $\{\beta_n\}_{n\geq 1}$ and $\{X_n\}_{n\geq 0}$. Define a renewal sequence $\{\sigma(k)\}_{k\geq 0}$ as follows:

$$
\sigma(0) = 0 \quad \text{and} \quad \sigma(k) = \inf\{n > \sigma(k-1);\ \beta_n = 1\}, \qquad k \geq 1.
$$

One can easily see that $\{\sigma(k) - \sigma(k-1)\}_{k\geq 1}$ is an i.i.d. sequence with the common law given by

$$
P\{\sigma(1) = k\} = (1-t)t^{k-1}, \qquad k = 1, 2, \ldots .
$$

By (5.9) in de Acosta (1988), the random sequence $\{X_{\sigma(n)}\}_{n\geq 0}$ is a Markov chain with the transition $P_t(x, A)$. On the other hand, we have

$$
\sum_{k=1}^{n} f(X_{\sigma(k)}) = \sum_{k=1}^{\sigma(n)} \beta_k f(X_k), \qquad n = 1, 2, \ldots .
\tag{5.18}
$$

In particular, taking expectation in (5.18) one can obtain

$$
a_t(n) \equiv \pi(D)^{-1} \sum_{k=1}^{n} \nu P_t^k(D) = (1-t) E a(\sigma(n)).
$$

By the law of large numbers, $\{\sigma(n)/n\}_{n\ge 1}$ tends to $(1-t)^{-1}$ in each L-norm as well as in the sense of almost sure convergence. By regularity and the dominated convergence theorem,

$$a_t(n) \sim (1-t)a(n)E\left(\frac{\sigma(n)}{n}\right)^p \sim (1-t)^{1-p}a(n) \quad \text{as } n\to\infty. \tag{5.19}$$

Notice that a D-set of $P(x, A)$ is also a D-set of its resolvent chain $P_t(x, A)$. Hence, the Markov chain $P_t(x, A)$ is p-regular.

Applying the conclusion achieved in the second step to $P_t(x, A)$ gives

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=1}^{\sigma(n)} \beta_k f(X_k)\Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) \\ = (1-t)^{1-p}\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \quad \text{a.s.}\end{aligned}$$

Given $0<\lambda_1<1-t<\lambda_2$, by regularity we have

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=1}^{\sigma([\lambda_i n])} \beta_k f(X_k)\Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) \\ = \lambda_i^p(1-t)^{1-p}\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \quad \text{a.s., } i=1,2.\end{aligned}$$

On the other hand, by the law of large numbers

$$\sigma([\lambda_1 n])/n \to \frac{\lambda_1}{1-t} < 1 \quad \text{and} \quad \sigma([\lambda_2 n])/n \to \frac{\lambda_2}{1-t} > 1 \quad \text{a.s.} \tag{5.20}$$

Therefore,

$$\begin{aligned}&\lambda_1^p(1-t)^{1-p}\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \\ &\quad\le \limsup_{n\to\infty} \sum_{k=1}^{n} \beta_k f(X_k)\Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) \\ &\quad\le \lambda_2^p(1-t)^{1-p}\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \quad \text{a.s.}\end{aligned}$$

Letting $\lambda_1, \lambda_2 \to 1-t$ gives

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=1}^{n} \beta_k f(X_k)\Big/ a\left(\frac{n}{L_2 a(n)}\right) L_2 a(n) \\ = (1-t)\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \quad \text{a.s.}\end{aligned} \tag{5.21}$$

With $\{1-\beta_k\}_{k\geq 1}$ instead of $\{\beta_k\}_{k\geq 1}$ in (5.21) we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=1}^{n}(1-\beta_k)f(X_k)\Big/a\left(\frac{n}{L_2a(n)}\right)L_2a(n) \\ = t\frac{\Gamma(p+1)}{p^p(1-p)^{1-p}}\int f(x)\pi(dx) \quad \text{a.s.}\end{aligned} \tag{5.22}$$

Finally, the desired conclusion follows from (5.21), (5.22) and from letting $t\to 0^+$ in the following decomposition:

$$\sum_{k=1}^{n} f(X_k) = \sum_{k=1}^{n}\beta_k f(X_k) + \sum_{k=1}^{n}(1-\beta_k)f(X_k), \qquad n = 1,2,\dots. \qquad \square$$

6. Applications. As an application of our results, we study the limit laws for occupation times of Harris recurrent random walks. Recall that a sequence $\{S_n\}_{n\geq 0}$ of $\mathbf{R}^d$-valued ($d\geq 1$) random variables is called a random walk if $\{S_n - S_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence. We assume that

$$E(S_1 - S_0) = 0 \quad \text{and} \quad E|S_1 - S_0|^2 < +\infty \tag{6.1}$$

where $|\cdot|$ is the Euclidean norm. The limit theorems for occupation times of recurrent random walks with lattice values have been extensively studied. In this section we consider the random walks with nonlattice values. The distribution F of the increment $S_1 - S_0$, or the random walk $\{S_n\}_{n\geq 0}$, is called *spread out*, if there is an integer $k\geq 1$ such that the kth convolution F^{*k} is not singular to Lebesgue measure λ on $\mathbf{R}^d$. "This class is much larger than the class of absolutely continuous probability measures," as pointed out in Revuz [(1975), page 91]. Being spread out, $\{S_n\}_{n\geq 0}$ is Harris recurrent (with Lebesgue measure as its invariant measure) if and only if $d = 1$ or 2 [see, e.g., Chapter 3, Sections 4 and 5, Revuz (1975)].

So we focus on the case $d\leq 2$ in the following discussion. Given a measurable set $A\subset \mathbf{R}^d$ with $0<\lambda(A)<+\infty$, let $\xi_n(A)$ be the occupation time of A up to time n, that is,

$$\xi_n(A) = \sum_{k=1}^{n} I_A(S_n) = \#\{k;\ 1\leq k\leq n \quad \text{and} \quad S_n\in A\}, \qquad n = 1,2,\dots.$$

By Harris recurrence $\xi_n(A)\to +\infty$ a.s. We intend to find the exact order for the sequence $\{\xi_n(A)\}_{n\geq 1}$

Let D be a D-set of $\{S_n\}_{n\geq 0}$. In particular, $0<\lambda(D)<+\infty$. The argument proving Proposition 2.4 [and therefore (2.j)] in Le Gall and Rosen (1991) gives

$$a(n) \equiv \frac{1}{\lambda(D)}\sum_{k=1}^{n} P^k(0,D) \sim \begin{cases} \sqrt{\dfrac{2n}{\pi E(S_1 - S_0)^2}}, & \text{if } d = 1,\\[2ex] \dfrac{\log n}{2\pi|\Gamma|^{1/2}}, & \text{if } d = 2, \qquad n\to\infty,\end{cases}$$

where Γ is the covariance matrix of $S_1 - S_0$. Notice that $|\Gamma| > 0$ and $E(S_1 - S_0)^2 > 0$ if $\{S_n\}_{n\geq 0}$ is spread out. In particular, $\{S_n\}_{n\geq 0}$ is regular in this case with the regular index $p = 1/2$ when $d = 1$ and $p = 0$ when $d = 2$.

Notice the fact [see, e.g., Feller (1971), Section VI.2] that

$$G_{1/2}^{-1/2} =^d \sqrt{\frac{\pi}{2}}|U|,$$

where U is a $N(0, 1)$ normal random variable.

Applying Theorem 2.3 and Theorem 2.4 to $\{S_n\}_{n\geq 0}$ gives the following results.

THEOREM 6.1. *Let $\{S_n\}_{n\geq 0}$ be a random walk on $\mathbf{R}^1$, which is spread out and which satisfies the condition* (6.1). *For each measurable set $A \subset \mathbf{R}^1$ with $0 < \lambda(A) < +\infty$,*

$$\lim_{n\to\infty} P\{\xi_n(A) \leq x\sqrt{n}\} = \frac{\sqrt{2\sigma^2}}{\sqrt{\pi}\lambda(A)} \int_o^x \exp\left\{-\frac{\sigma^2 t^2}{2\lambda(A)^2}\right\} dt \qquad \forall\, x > 0, \tag{6.2}$$

$$\limsup_{n\to\infty} \frac{\xi_n(A)}{\sqrt{2n \log\log n}} = \frac{\lambda(A)}{\sigma} \quad a.s., \tag{6.3}$$

where $\sigma^2 = E(S_1 - S_0)^2$.

THEOREM 6.2. *Let $\{S_n\}_{n\geq 0}$ be random walk on $\mathbf{R}^2$, which is spread out and which satisfies the condition* (6.1). *For each measurable set $A \subset \mathbf{R}^2$ with $0 < \lambda(A) < +\infty$,*

$$\lim_{n\to\infty} P\{\xi_n(A) \leq x \log n\} = 1 - \exp\left\{-\frac{2\pi|\Gamma|^{1/2}x}{\lambda(A)}\right\} \qquad \forall\, x > 0, \tag{6.4}$$

$$\limsup_{n\to\infty} \frac{\xi_n(A)}{\log n \log\log\log n} = \frac{\lambda(A)}{2\pi|\Gamma|^{1/2}} \quad a.s., \tag{6.5}$$

where Γ is the covariance matrix of $S_1 - S_0$.

REMARK 6.3. Theorems 6.1 and 6.2 can be further generalized by dropping the condition (6.1). Instead, we assume that $\{S_n\}_{n\geq 0}$ is in the domain of attraction of a stable law G: there is a positive sequence $\{b(n)\}_{n\geq 1}$ such that $S_n/b(n) \to_d G$. It is well known that the sequence $\{b(n)\}_{n\geq 1}$ must be regularly varying with index $1/2 \leq \beta < +\infty$. We also assume that it satisfies

$$\sum_n \frac{1}{b(n)^d} = +\infty.$$

Indeed, from (2.j) in Le Gall and Rosen (1991),

$$a(n) \sim p(0) \sum_{k=1}^n \frac{1}{b(k)^d} \to +\infty, \qquad n \to \infty, \tag{6.6}$$

where $p(\cdot)$ is the density of G. By Theorem 4.11 in Revuz (1975), $\{S_n\}_{n\geq 0}$ is Harris recurrent. Although we cannot claim regularity conclusively, (6.6) suggests that it is true most of the time.

REMARK 6.4. In the case when $\{S_n\}_{n\geq 0}$ take lattice values, all forms given in Theorems 6.1 and 6.2 have been obtained under varous conditions, where Lebesgue measure is replaced by counting measure. See, for example, Révész (1990) for a collection of the results and Marcus and Rosen (1994a, b) for some later developments. It should be pointed out that our results for Markov chains contain and (in some cases) slightly modify these results. [The symmetry assumed in Marcus and Rosen (1994a, b), for example, can be dropped by utilizing our results.] The weak laws in (6.2) and (6.4) in the continuous value case go back at least to Darling and Kac (1957). As far as we know, the strong laws given in (6.3) and (6.5) are new in the nonlattice context.

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