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INTEGRATED BROWNIAN MOTION, CONDITIONED TO BE POSITIVE

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We study the two-dimensional process of integrated Brownian motion and Brownian motion, where integrated Brownian motion is conditioned to be positive. The transition density of this process is derived from the asymptotic behavior of hitting times of the unconditioned process. Explicit expressions for the transition density in terms of confluent hypergeometric functions are derived, and it is shown how our results on the hitting time distributions imply previous results of Isozaki–Watanabe and Goldman. The conditioned process is characterized by a system of stochastic differential equations (SDEs) for which we prove an existence and unicity result. Some sample path properties are derived from the SDEs and it is shown that $t \mapsto t^{9/10}$ is a "critical curve" for the conditioned process in the sense that the expected time that the integral part of the conditioned process spends below any curve $t \mapsto t^{\alpha}$ is finite for $\alpha < 9/10$ and infinite for $\alpha \geq 9/10$.

1. Introduction. Let (U, V) be the two-dimensional process of integrated Brownian motion (IBM) and Brownian motion (BM), where U represents IBM and V represents BM. This process is often called the *Kolmogorov diffusion* since its study was apparently initiated by [7].

It is well known (and easily verified by computing expectations and covariances of the Gaussian process involved) that the transition density of (U, V) is given by

$$
\begin{aligned}
&p_t(x, y; u, v)\\
&\quad = \frac{\sqrt{3}}{\pi t^2} \exp\left\{ -\frac{6(u-x-ty)^2}{t^3} + \frac{6(v-y)(u-x-ty)}{t^2} \right.\\
&\qquad\qquad \left. - \frac{2(v-y)^2}{t} \right\};
\end{aligned}
\tag{1.1}
$$

see [12]. Another way of writing this transition density (that often is useful)

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is

$$
\begin{aligned}
&p_t(x, y; u, v) \\
&\quad = \frac{\sqrt{3}}{\pi t^2} \exp\left\{ -\frac{6(u-x)^2}{t^3} + \frac{6(u-x)(v+y)}{t^2} \right. \\
&\qquad\qquad \left. - \frac{2(v^2 + vy + y^2)}{t} \right\}.
\end{aligned} \tag{1.2}
$$

We want to characterize the process (U, V), where U is conditioned to be positive and where $(U, V) = (0, 0)$ at time zero (U has slope zero at time zero). This process arises naturally in several contexts. Our motivation for studying this process originated in a study of the limiting behavior of the nonparametric maximum likelihood estimator of a convex density and nonparametric estimators of convex regression functions; see, for example, [6] and [11]. Another motivation can be found in the work of [16] on the convex hull of integrated Brownian motion with a parabolic drift. In both cases, one encounters excursions of integrated Brownian motion above certain curves at which the integrated Brownian motion touches at the endpoints of the excursion. Using the Cameron–Martin formula, these excursions can be described by excursions of integrated Brownian motion above a line. These excursions, in turn, can be related to integrated Brownian motion, conditioned to be positive, in a way that is somewhat analogous to the relation between Bessel(3) bridges and the Bessel(3) process for ordinary one-dimensional Brownian motion; see [5].

We determine the structure of this process in Sections 2 and 4. It is shown that the transition density of the process (U, V), where U is conditioned to be positive, is of the form

$$
h(x, y)^{-1} \bar{p}_t(x, y; u, v) h(u, v), \qquad x > 0,\ y \in \mathbb{R},
$$

where $\bar{p}_t$ is the transition density of the process (U, V), killed when U hits zero, and where $h(x, y)$ is proportional to

$$
\lim_{t \to \infty} t^{1/4} P_{(x, y)}\{\tau_0 > t\}, \tag{1.3}
$$

denoting by τ_0 the first time that U hits zero.

This motivates the study of the asymptotic behavior of $P_{(x, y)}\{\tau_0 > t\}$, as $t \to \infty$. This has been studied by [4], but we give a simple direct approach to this problem in Section 2, avoiding the use of Laplace transforms, Tauberian theorems and separation of cases. Our approach leads to an integral representation of h, valid for all values of the arguments. In fact, we obtain the asymptotic behavior of the density

$$
P_{(x, y)}\left\{\tau_0 \in dt,\ V(\tau_0)\big/\sqrt{t} \in -dz\right\}\Big/(dt\, dz), \tag{1.4}
$$

as $t \to \infty$, showing that this joint density asymptotically behaves as the product of the density of τ_0 and the density of $V(\tau_0)/\sqrt{t}$ on $(-\infty, 0)$, as

$t \to \infty$. Moreover, we show that the function (1.3) has an explicit representation in terms of confluent hypergeometric functions; see part (iii) of Theorem 2.1 and part (ii) of Lemma 2.1.

We also study the behavior of the density

$$P_{(x,y)}\{\tau_0 \in dt, V(\tau_0) \in -dz\}\big/(dt\,dz), \tag{1.5}$$

if t is fixed and $z \downarrow 0$, showing that this density is of order $z^{3/2}$, as $z \downarrow 0$; see part (ii) of Theorem 2.1. This result provides us with the transition density of an "excursion" of the process (U, V), where $(U(0), V(0)) = (U(1), V(1)) = (0, 0)$ and $U(t) > 0$, $t \in (0, 1)$ [see (2.26)].

In Section 3 we discuss how our results on the asymptotic behavior of (1.4) can be specialized to yield previous results of [2] and [4]. The latter comparison reveals at the same time a curious relation between the hypergeometric function ${}_2F_1$ and gamma functions that was unknown to us and seems to be nonstandard. This comparison also reveals that Goldman's result seems to be off by a factor 6.

Using the results of Section 2, we determine the marginal density of the conditioned process in Section 4. Next we show in Section 5 that the conditioned process can be characterized by a system of stochastic differential equations (SDEs), and derive from the structure of these equations that U will not hit zero after time zero and will drift off to ∞, as $t \to \infty$. The SDEs, together with the analytic properties of the function h, yield a very simple tool for proving these facts.

Finally, we deduce in Section 6 from Theorem 4.1 in Section 4 another sample path property of the process $\tilde{U}$, where we denote the conditioned process by $(\tilde{U}, \tilde{V})$. This is the property that the curve $t \mapsto t^{9/10}$ is a "critical curve" for the process $\tilde{U}$ in the sense that the expected amount of time the process $\tilde{U}$ spends below any curve $t \mapsto t^{\alpha}$ is finite for $\alpha < 9/10$ and is infinite for $\alpha \geq 9/10$.

2. The asymptotic behavior of $P_{(x,y)}\{\tau_0 > t\}$ for large t. By [8], Théorème 1, page 388, we have, for $x, z > 0$,

$$\begin{aligned}
&P_{(x,y)}\Big\{\tau_0 \in dt, V(\tau_0)\big/\sqrt{t} \in -dz\Big\}\Big/(dt\,dz)\\
&\quad = z\sqrt{t}\,\bigg\{p_t(x, y; 0, -z\sqrt{t})\\
&\qquad - \int_{s=0}^{t}\int_{w=0}^{\infty} p_{t-s}(x, y; 0, w)\, P_{(0,-z\sqrt{t})}\{\tau_0^+ \in ds, V(\tau_0^+) \in dw\}\bigg\}\sqrt{t} \qquad (2.1)\\
&\quad = tz\bigg\{q_t(x, y; 0, -z\sqrt{t})\\
&\qquad - \int_{s=0}^{t}\int_{w=0}^{\infty} q_{t-s}(x, y; 0, w)\, P_{(0,-z\sqrt{t})}\{\tau_0^+ \in ds, V(\tau_0^+) \in dw\}\bigg\},
\end{aligned}$$

where τ_0^+ denotes the first time U passes zero *after* time zero and

$$q_t(x,y;u,v) = p_t(x,y;u,v) - p_t(x,y;u,-v). \tag{2.2}$$

The function q_t was already an important tool in [2] (who called it p^*). In [12] the joint density of τ_0^+ and $V(\tau_0^+)$ under $P_{(0,-z)}$ is derived: for $z > 0$,

$$\begin{aligned} &P_{(0,-z)}\{\tau_0^+ \in ds, V(\tau_0^+) \in dw\} \\ &\quad = \frac{3w}{\pi\sqrt{2\pi}s^2} \exp\left\{-\frac{2}{s}(z^2 - zw + w^2)\right\} \\ &\qquad \times \int_0^{4zw/s} \xi^{-1/2} \exp\left\{-\frac{3}{2}\xi\right\} d\xi\, dw\, ds. \end{aligned} \tag{2.3}$$

It will be shown that the dominating asymptotic behavior, as $t \to \infty$, but also if $z \downarrow 0$ and t is fixed, is coming from the double integral on the right-hand side of (2.1). We first state a preliminary result, giving the integral representation of the crucial function h and also its relation to a function of one argument g that can be expressed in terms of standard confluent hypergeometric functions.

LEMMA 2.1. *Let the functions* $g\colon \mathbb{R} \to (0,\infty)$ *and* $h\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ *be defined by*

$$\begin{aligned} h(x,y) &= \int_{s=0}^{\infty}\int_{w=0}^{\infty} w^{3/2} q_s(x,y;0,-w)\, ds\, dw \\ &= \frac{2\sqrt{3}}{\pi}\int_{s=0}^{\infty}\int_{w=0}^{\infty} w^{3/2}\exp\{-6x^2s^3 - 6xys^2 - 2(y^2+w^2)s\} \\ &\qquad \times \sinh(6xws^2 + 2yws)\, ds\, dw \end{aligned} \tag{2.4}$$

and

$$g(y) = \int_{s=0}^{\infty}\int_{w=0}^{\infty} w^{3/2} q_s(1,y;0,-w)\, ds\, dw = h(1,y) \tag{2.5}$$

and write $\mathscr{D}_{(x,y)}$ *for the differential operator*

$$\mathscr{D}_{(x,y)} = y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial y^2}. \tag{2.6}$$

Note that

$$h(x,y) = x^{1/6} g(yx^{-1/3}). \tag{2.7}$$

Then:

(i) *The function* h *is harmonic for* $\mathscr{D}_{(x,y)}$ *in the sense that* $\mathscr{D}_{(x,y)}h(x,y) = 0$, *and the function* g *is analytic on* $\mathbb{R}$ *and satisfies the second-order differential equation*

$$g''(y) = \tfrac{2}{3}y^2 g'(y) - \tfrac{1}{3}yg(y), \qquad y \in \mathbb{R}. \tag{2.8}$$

(ii) *The function g has the representation*

$$g(y) = \left(\tfrac{2}{9}\right)^{1/6} yU\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{2}{9}y^3\right), \qquad y > 0, \tag{2.9}$$

$$g(y) = -\left(\tfrac{2}{9}\right)^{1/6} \tfrac{1}{6}yV\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{2}{9}y^3\right), \qquad y < 0, \tag{2.10}$$

$$g(0) = \lim_{y\to 0} g(y) = \left(\tfrac{2}{9}\right)^{-1/6}\Gamma\left(\tfrac{1}{3}\right)/\Gamma\left(\tfrac{1}{6}\right), \tag{2.11}$$

where U and V are the confluent hypergeometric functions, as defined on page 256 *of* [13].

PROOF. (i) The infinitesimal generator of the process (U, V) is given by the partial differential operator $\mathscr{D}_{(x,y)}$, defined by (2.6), and therefore, as noted in, for example, [10], page 1302, the transition density $p_t(x, y; u, v)$ of the process (U, V) satisfies the (backward) Kolmogorov equation

$$\mathscr{D}_{(x,y)} p_t(x, y; u, v) = \frac{\partial}{\partial t} p_t(x, y; u, v), \tag{2.12}$$

implying that also

$$\mathscr{D}_{(x,y)} q_t(x, y; u, v) = \frac{\partial}{\partial t} q_t(x, y; u, v). \tag{2.13}$$

Hence we get, if $x > 0$,

$$\begin{aligned}\mathscr{D}_{(x,y)} h(x, y) &= \int_{w=0}^{\infty} w^{3/2} \int_{s=0}^{\infty} \frac{\partial}{\partial s} q_s(x, y; 0, -w)\, ds\, dw \\ &= \int_{w=0}^{\infty} w^{3/2} \lim_{s\to\infty} q_s(x, y; 0, -w)\, dw \\ &\quad - \int_{w=0}^{\infty} w^{3/2} \lim_{s\downarrow 0} q_s(x, y; 0, -w)\, dw \\ &= 0.\end{aligned}$$

This implies, by (2.7),

$$\begin{aligned}\frac{\partial^2}{\partial y^2} h(x, y) &= x^{-1/2} g''(yx^{-1/3}) = -2y \frac{\partial}{\partial x} h(x, y) \\ &= \frac{2}{3} y^2 x^{-7/6} g'(yx^{-1/3}) - \frac{1}{3} yx^{-5/6} g(yx^{-1/3}).\end{aligned} \tag{2.14}$$

Evaluating this for $x = 1$, we get (2.8). The analyticity of the function g (on $\mathbb{R}$) follows from (2.5) together with the integral representation (2.4) of h.

(ii) Let $\mathbf{M}$ be the (standardized) confluent hypergeometric function (a version of the so-called Kummer function), defined by (9.04) on page 255 of [13]. A straightforward computation, using the fact that $\mathbf{M}$ and U satisfy the confluent hypergeometric equation (see, e.g., [13], page 254), shows that any solution of the differential equation (2.8) is of the form

$$y\left\{A\cdot\mathbf{M}\left(\tfrac{1}{6},\tfrac{4}{3},\tfrac{2}{9}y^3\right)+B\cdot U\left(\tfrac{1}{6},\tfrac{4}{3},\tfrac{2}{9}y^3\right)\right\},\tag{2.15}$$

for constants A and B. We are going to specify this function to our function g (i.e., determine the constants A and B) at $+\infty$, since determining A and B at a finite point (like 0) seems much harder in this case! In fact, by determining the behavior at ∞ we will find a relation between special (hypergeometric) functions at zero, allowing us to compare the results in [2] with those in [4] (showing that there is in fact a discrepancy; see the end of Section 3).

Denoting $\frac{2}{9}y^3$ by z, we have, by (10.07), [13], page 257,

$$\mathbf{M}\left(\tfrac{1}{6},\tfrac{4}{3},\tfrac{2}{9}y^3\right)\sim e^z z^{-7/6}/\Gamma\left(\tfrac{1}{6}\right),\qquad y\to\infty\tag{2.16}$$

and, by (10.01), [13], page 256,

$$U\left(\tfrac{1}{6},\tfrac{4}{3},\tfrac{2}{9}y^3\right)\sim z^{-1/6},\qquad y\to\infty.\tag{2.17}$$

On the other hand, using the change of variables $s\to ys/3$ and $w\to wy$, we have

$$\begin{aligned}
g(y)&=h(1,y)\\
&=\frac{2\sqrt{3}}{\pi}\int_{s=0}^{\infty}\int_{w=0}^{\infty}w^{3/2}\exp\{-6s^3-6ys^2-2y^2s-2w^2s\}\\
&\qquad\qquad\times\sinh(6ws^2+2yws)\,ds\,dw\\
&=\frac{2}{\pi\sqrt{3}}y^{7/2}\int_{s=0}^{\infty}\int_{w=0}^{\infty}w^{3/2}\exp\left\{-\frac{2}{9}y^3s(s^2+3s+3+3w^2)\right\}\\
&\qquad\qquad\times\sinh\left(\frac{2}{3}y^3ws(s+1)\right)ds\,dw\\
&\sim\frac{2y^{7/2}}{\pi\sqrt{3}}\int_{s=0}^{\infty}\int_{w=0}^{\infty}w^{3/2}\exp\left\{-\frac{2}{3}y^3s(1+w^2)\right\}\sinh\left(\frac{2}{3}y^3sw\right)ds\,dw,
\end{aligned}$$

as $y\to\infty$. However, the last displayed expression equals

$$\begin{aligned}
&\frac{\sqrt{3y}}{2\pi}\int_0^{\infty}w^{3/2}\left\{\frac{1}{1-w+w^2}-\frac{1}{1+w+w^2}\right\}dw\\
&\qquad=\frac{\sqrt{3y}}{\pi}\int_0^{\infty}\frac{w^{5/2}}{1+w^2+w^4}\,dw=\sqrt{y}\,.
\end{aligned}$$

Because of (2.16) and (2.17), it now follows that, in the representation (2.15) of g on $(0,\infty)$, the coefficient A has to be zero and hence that

$$g(y)=\left(\tfrac{2}{9}\right)^{1/6}yU\left(\tfrac{1}{6},\tfrac{4}{3},\tfrac{2}{9}y^3\right),$$

for $y \in (0, \infty)$. On the other hand, we have (using 13.5.8, page 508 of [1])

$$\lim_{y \downarrow 0} yU\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{2}{9}y^3\right) = \left(\tfrac{2}{9}\right)^{-1/3} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}.$$

The obvious candidate for the analytic continuation at zero is provided by the confluent hypergeometric function V, defined by

$$V(a, c, z) = e^z U(c - a, c, -z), \tag{2.18}$$

since V also satisfies the confluent hypergeometric equation and has the desired vanishing behavior at ∞. In fact, it is immediate from 13.5.8, page 508, [1], that

$$\Gamma\left(\tfrac{1}{6}\right) \lim_{y \downarrow 0} yU\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{2}{9}y^3\right) = \Gamma\left(\tfrac{7}{6}\right) \lim_{y \downarrow 0} (-y) V\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{2}{9}y^3\right),$$

and, using the integral representations of U and V, it is easily verified that equality also holds at the level of the derivative. Hence we have a complete representation of the function g in terms of the confluent hypergeometric functions U and V. $\square$

With this preliminary result in hand, we are prepared for the following theorem.

THEOREM 2.1. *Let τ_0 be the first time that U hits zero, if the process (U, V) starts at (x, y) at time zero, where $x > 0$. Then*:

(i) *As $t \to \infty$ we have, for any $z > 0$,*

$$\begin{aligned} &P_{(x,y)}\{\tau_0 \in dt, V(\tau_0)/\sqrt{t} \in -dz\} \big/ (dt\, dz) \\ &\qquad \sim \frac{3 \cdot 2^{3/2}}{\pi^{3/2}} \frac{z^{3/2} \exp\{-2z^2\}}{t^{5/4}} h(x, y). \end{aligned} \tag{2.19}$$

(ii) *As $z \downarrow 0$ through strictly positive values of z, we have, for any $t > 0$,*

$$\begin{aligned} &P_{(x,y)}\{\tau_0 \in dt, V(\tau_0) \in -dz\} \big/ (dt\, dz) \\ &\qquad \sim \frac{4\sqrt{3}\, z^{3/2}}{\sqrt{2\pi}} \int_{s=0}^{t} \int_{w=0}^{\infty} w^{3/2} s^{-1/2} p_s(0, w; 0, 0) \\ &\qquad\qquad \times q_{t-s}(x, y; 0, -w)\, ds\, dw. \end{aligned} \tag{2.20}$$

(iii)

$$P_{(x,y)}\{\tau_0 > t\} \sim \frac{3\Gamma(1/4)}{2^{3/4}\pi^{3/2}} \frac{h(x, y)}{t^{1/4}} \quad \text{as } t \to \infty. \tag{2.21}$$

PROOF. (i) and (ii) The elementary but somewhat technical proofs of these properties are given in the Appendix.

(iii) Integrating w.r.t. dz in (2.19), we get

$$P_{(x,y)}\{\tau_0 \in dt\}/dt \sim \frac{3\Gamma(1/4)}{2^{3/4}\pi^{3/2}} \frac{h(x,y)}{4t^{5/4}}. \tag{2.22}$$

From this we get (2.21) by integrating w.r.t. t. Note that the positivity of $P_{(x,y)}\{\tau_0 > t\}$ for all $x > 0$ and $y \in \mathbb{R}$ implies

$$g(y) > 0 \quad \text{for all } y \in \mathbb{R}. \tag{2.23}$$

□

We introduce the following notation for the result in part (ii) of Theorem 2.1. Let $\bar{h}$ be defined by

$$\begin{aligned}\bar{h}(t,x,y) &= \frac{4\sqrt{3}}{\sqrt{2\pi}} \int_{s=0}^{t} \int_{w=0}^{\infty} w^{3/2} s^{-1/2} p_s(0,w;0,0) \\ &\qquad \times q_{t-s}(x,y;0,-w)\, ds\, dw.\end{aligned} \tag{2.24}$$

Since $q_{t-s}(x,y;u,v)$ satisfies the backward Kolmogorov equation for the process (U,V), for all $x > 0$, and $\lim_{s\downarrow 0} q_s(x,y;0,w) = 0$, if $x > 0$, it follows that the function

$$t \mapsto \bar{h}(1-t,x,y), \qquad t \in [0,1),$$

is "space–time harmonic" on $[0,1)$ in the sense that

$$\left(\frac{\partial}{\partial t} + \mathscr{D}_{(x,y)}\right)\bar{h}(1-t,x,y) = 0,$$

where $\mathscr{D}_{(x,y)}$ is defined by (2.6). Since $\bar{h}(1-t,x,y)$ has the interpretation

$$\lim_{z\downarrow 0} z^{-3/2} P_{(t,x,y)}\{\tau_0 \in du, V(\tau_0) \in -dz\}/(du\,dz)\big|_{u=1}, \tag{2.25}$$

where $P_{(t,x,y)}\{\tau_0 \in du,\ V(\tau_0) \in -dz\}$ denotes the probability that $\tau_0 \in du$ and $V(\tau_0) \in -dz$, if the value of the process is (x,y) at time t, the transition density of the "bridge" of (U,V) on $[0,1]$, starting at $(0,0)$, where U is conditioned to be positive and where $(U(1),V(1)) = (0,0)$, is given by

$$\bar{h}(1-s,x,y)^{-1}\bar{p}_{t-s}(x,y;u,v)\bar{h}(1-t,u,v), \tag{2.26}$$

if $0 < s < t < 1$. Here $\bar{p}$ is the transition density of the process (U,V), killed when U hits zero, and can be written for $x, u > 0$ as

$$\begin{aligned}\bar{p}_t(x,y;u,v) &= p_t(x,y;u,v) \\ &\quad - \int_{s=0}^{t}\int_{w=0}^{\infty} p_{t-s}(0,-w;u,v) P_{(x,y)}\{\tau_0 \in ds, V(\tau_0) \in -dw\};\end{aligned}$$

see [9] relation (3), page 1054. In particular, since $p_t(x,y;u,v) = p_t(u,-v;x,-y)$ and $\bar{p}(x,y;u,v) = \bar{p}_t(u,-v;x,-y)$ (see [9], relation (4), page 1054), letting $u \downarrow 0$, we get, for $x, z > 0$,

$$
\begin{aligned}
\bar{p}(x, y; 0, -z) &= \bar{p}(0, z; x, -y) \\
&= p_t(x, y; 0, -z) \\
&\qquad - \int_{s=0}^{t} \int_{w=0}^{\infty} p_{t-s}(x, y; 0, w) \\
&\qquad\qquad \times P_{(0,z)}\{\tau_0 \in ds, V(\tau_0) \in -dw\} \\
&= z^{-1} P_{(x,y)}\{\tau_0 \in ds, V(\tau_0) \in -dz\} \big/ dt\, dz.
\end{aligned} \tag{2.27}
$$

The final step follows from (2.1). Now (2.26) can be checked as follows. Due to the Markov property, the transition density of the bridge equals

$$
\begin{aligned}
&P\{(U(t), V(t)) \in du\, dv \mid (U(s), V(s)) = (x, y), \\
&\qquad\qquad (U(1), V(1)) = (0, 0), \tau_0 > 0\}/du\, dv \\
&\quad = \bar{p}_t(x, y; u, v) \lim_{z \downarrow 0} \frac{\bar{p}_{1-t}(u, v; 0, z)}{\bar{p}_{1-s}(x, y; 0, z)} \\
&\quad = \bar{p}_t(x, y; u, v) \lim_{z \downarrow 0} \left. \frac{P_{(t,u,v)}\{\tau_0 \in dw, V(\tau_0) \in -dz\}}{P_{(s,x,y)}\{\tau_0 \in dw, V(\tau_0) \in -dz\}} \right|_{w=1},
\end{aligned}
$$

which proves (2.26) according to (2.25).

Similarly, by (i) of Lemma 2.1, the function h defined by (2.4) is harmonic for the differential operator $\mathscr{D}_{(x,y)}$. Since $h(x, y)$ is proportional to

$$\lim_{t \to \infty} t^{1/4} P_{(x,y)}\{\tau_0 > t\},$$

this function gives the transition density of the process (U, V), where U is conditioned to be positive,

$$h(x, y)^{-1} \bar{p}_{t-s}(x, y; u, v) h(u, v),$$

for $x, u > 0$. This process is characterized by a system of stochastic differential equations in Section 5, where it will be shown that U (as first component of the conditioned process) will drift off to ∞ and will never hit zero after time zero.

3. The results of Isozaki–Watanabe and Goldman.

Set

$$f(r, a, A) \equiv P_{(0,0)}(U(t) < r + at \text{ for all } 0 \le t \le A)$$

and

$$g(r, a, \sigma) \equiv P_{(0,0)}\big(U(t) < r + at + \sigma t^2 \text{ for all } 0 \le t \le \infty\big).$$

Sinai [16] showed that

$$f(r, a, A) \asymp A^{-1/4} \quad \text{as } A \to \infty$$

and

$$g(r, a, \sigma) \asymp \sigma^{1/2} \quad \text{as } \sigma \to 0,$$

where $f \asymp g$ means that f/g lies between two positive and finite constants. Isozaki and Watanabe [4] sharpen these results to

$$f(r, a, A) \sim C(r, a) A^{-1/4} \quad \text{as } A \to \infty \tag{3.1}$$

and to

$$g(r, a, \sigma) \sim D(r, a) \sigma^{1/2} \quad \text{as } \sigma \to 0, \tag{3.2}$$

where they give explicit formulas for $C(r, a)$ and $D(r, a)$. They do this by deriving an asymptotic expression for $1 - E_{(x, y)} \exp(-a\sigma^2 \tau_0 - b\sigma V(\tau_0))$ for all $a \geq 0$, $b \geq 0$, and $(x, y) \in \mathbb{R}^2$ with $x \leq 0$. The result (3.1), where $r > 0$ and $a \in \mathbb{R}$, is directly related to (2.21). Indeed, due to the symmetry of Brownian motion started at 0, it is easily seen that

$$f(r, a, A) = P_{(r, a)}\{\tau_0 > A\}.$$

Thus, (1.5) of [4] says that, changing the notation to agree with (2.21),

$$P_{(x, y)}(\tau_0 > t) \sim t^{-1/4} \frac{1}{\Gamma(3/4)} \frac{3\sqrt{2}}{\sqrt{\pi 2\sqrt{2}}\, \Gamma(1/6)} x^{1/6} \psi(y x^{-1/3}),$$

where the function ψ is defined by

$$\psi(y) = \begin{cases} \displaystyle\int_0^\infty v^{-5/6} \left(y^3 + \frac{9}{2} v \right)^{1/6} \exp(-v)\, dv, & y > 0, \\ \displaystyle\exp\left(-\frac{2|y|^3}{9} \right) \int_0^\infty \left(\frac{9}{2} v \right)^{1/6} \left(v + \frac{2}{9}|y|^3 \right)^{-5/6} \exp(-v)\, dv, & y \leq 0. \end{cases}$$

It is possible to express ψ in terms of the hypergeometric functions U and V (see e.g., [1], (13.2.5), page 505 and (2.18) in Section 2),

$$\psi(y) = \begin{cases} \Gamma\left(\frac{1}{6}\right)\left(\frac{2}{9}\right)^{1/6} y U\left(\frac{1}{6}, \frac{4}{3}, \frac{2}{9} y^3\right), & y > 0, \\ -\Gamma\left(\frac{7}{6}\right)\left(\frac{2}{9}\right)^{1/6} y V\left(\frac{1}{6}, \frac{4}{3}, \frac{2}{9} y^3\right), & y \leq 0. \end{cases}$$

Hence this corresponds to the results found in Section 2.

Before specializing our results to that in Goldman [2], we note that there is a factor $1/6$ missing in his Proposition 2. Following the indicated steps between Goldman's (3.1) and (3.2), it becomes clear that the factor 3 in front of (3.2) should not be there: it disappears when the substitution $t - s \to w^{-1/3}$ is performed. Moreover, reducing the series of multiple integrals just before Proposition 2 to the expression involving the hypergeometric function ${}_2F_1$, a factor $1/2$ is lost. Therefore, using our notation, Goldman's Proposition 2 should actually read

$$\begin{aligned} P_{(x, 0)}&(\tau_0 \in dt)/dt \\ &\sim \frac{x^{1/6}}{t^{5/4}} \frac{3 \cdot 6^{1/12}}{8\pi^2} \frac{\Gamma(5/4)\Gamma(7/4)\Gamma(5/12)}{\Gamma(3/2)} {}_2F_1(5/12, 7/4; 3/2; 3/4) \\ &= \frac{x^{1/6}}{t^{5/4}} \frac{3^{25/12}\Gamma(5/12)}{2^{65/12}\pi\sqrt{\pi}} {}_2F_1(5/12, 7/4; 3/2; 3/4) \end{aligned}$$

for $x > 0$ as $t \to \infty$. Substituting $y = 0$ in (2.22) and using (2.11), our corresponding result reads,

$$P_{(x,0)}(\tau_0 \in dt)\big/dt \sim \frac{x^{1/6}}{t^{5/4}} \frac{3^{4/3}\Gamma(5/4)\Gamma(1/3)}{2^{11/12}\Gamma(1/6)\pi\sqrt{\pi}}.$$

Equality of these two asymptotic expressions leads to the following result, which we were unable to locate in the literature on special functions:

$${}_2F_1\left(\frac{5}{12}, \frac{7}{4}; \frac{3}{2}; \frac{3}{4}\right) = \frac{\Gamma(5/4)\Gamma(1/3)2^{9/2}}{\Gamma(1/6)\Gamma(5/12)3^{3/4}}.$$

Numerical verification shows that both sides are equal to $2.0353\ldots$.

We finally show how the quantity

$${}_2F_1(5/12, 7/4; 3/2, 3/4)$$

of Goldman's Proposition 2 emerges from our integral representation in (2.19), since this might not be immediately obvious. This follows by writing the integral as a power series, using the power series for the sinh-function,

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} w^{3/2} \exp\{-6x^2s^3 - 2sw^2\} \sinh(6xws^2)\, dw\, ds \\
&\quad = \sum_{n=0}^{\infty} \frac{(6x)^{2n+1}}{(2n+1)!} \int_{w=0}^{\infty} w^{2n+5/2} \exp(-2w^2)\, dw \\
&\qquad\qquad \times \int_{s=0}^{\infty} s^{3n+1/4} \exp(-6x^2s^3)\, ds \\
&\quad = \frac{x^{1/6}\sqrt{\pi}}{6^{5/12}\cdot 2^{11/4}} \sum_{n=0}^{\infty} \frac{\Gamma(n+7/4)\Gamma(n+5/12)}{\Gamma(n+3/2)n!}(3/4)^n \\
&\quad = \frac{x^{1/6}\sqrt{\pi}}{6^{5/12}\cdot 2^{11/4}} \frac{\Gamma(5/12)\Gamma(7/4)}{\Gamma(3/2)}\, {}_2F_1(5/12, 7/4; 3/2, 3/4).
\end{aligned}$$

4. The marginal distribution of the conditioned process. In Section 2 we analyzed the behavior of $P_{(x,y)}\{\tau_0 > t\}$ and $P_{(x,y)}\{\tau_0 \in dt,\ V(\tau_0)/\sqrt{t} \in -dz\}$ for large t when $x > 0$. Now we extend those results to $x = 0$ and obtain the marginal density of the process $(\tilde{U}, \tilde{V})$ started from $(0, 0)$; recall that $(\tilde{U}, \tilde{V})$ is the process (U, V) conditioned on $U(t) \geq 0$ for all $t \geq 0$. Our main result in this section is the following theorem.

THEOREM 4.1. *The marginal density of* $(\tilde{U}, \tilde{V})$ *started at* $(0, 0)$ *is given by*

$$\begin{aligned}
f_t(u, v) &= P_{(0,0)}\big(\tilde{U}(t) \in du, \tilde{V}(t) \in dv\big)\big/du\, dv \\
&= 2^{29/4} u^{1/6} g(vu^{-1/3}) \overline{h}(t, u, -v),
\end{aligned} \tag{4.1}$$

where g is defined in Lemma 2.1 *and* $\overline{h}$ *in* (2.24).

REMARK. Writing $u = \bar{u}t^{3/2}$ and $v = \bar{v}t^{1/2}$ and using the change of variables $s \to st$ and $w \to wt^{1/2}$ in the definition of $\bar{h}$, we get

$$2^{29/4}t^{-2}\bar{u}^{1/6}g(\bar{v}\bar{u}^{-1/3})\bar{h}(1, \bar{u}, -\bar{v}), \tag{4.2}$$

showing that the joint density of $(\tilde{U}(t)t^{-3/2}, \tilde{V}(t)t^{-1/2})$ does not depend on t, which also follows from consideration of Brownian scaling.

PROOF OF THEOREM 4.1. First note that

$$f_t(u, v) = \lim_{z\downarrow 0} f_{t,z}(u, v) = \lim_{z\downarrow 0}\lim_{s\to\infty} \frac{\bar{p}_t(0, z; u, v)P_{(u,v)}\{\tau_0 > s - t\}}{P_{(0,z)}\{\tau_0^+ > s\}}. \tag{4.3}$$

Here $f_{t,z}$ is the density (at time t) of the process (U, V) started at $(0, z)$, where U is conditioned to be positive on $(0, \infty)$. By Theorem 2.1(iii), it follows that

$$\lim_{s\to\infty} s^{1/4}P_{(u,v)}\{\tau_0 > s - t\} = \frac{12\Gamma(5/4)u^{1/6}g(vu^{-1/3})}{2^{3/4}\pi\sqrt{\pi}}.$$

Moreover, using (2.3), we get for all $z > 0$,

$$\lim_{s\to\infty} s^{1/4}P_{(0,z)}\{\tau_0^+ > s\} = \frac{3\Gamma(5/4)}{\pi^{3/2}2^{11/2}}\sqrt{z}\,.$$

Therefore,

$$f_{t,z}(u, v) = 2^{29/4}u^{1/6}g(vu^{-1/3})z^{-1/2}\bar{p}_t(0, z; u, v) \tag{4.4}$$

and we get, using Theorem 2.2(ii),

$$\begin{aligned}\lim_{z\downarrow 0} \frac{\bar{p}_t(0, z; u, v)}{z^{1/2}} &= \lim_{z\downarrow 0} \frac{P_{(u,-v)}\{\tau_0 \in dt, V(\tau_0) \in -dz\}/(dt\,dz)}{z^{3/2}} \\ &= \bar{h}(t, u, -v),\end{aligned} \tag{4.5}$$

where $\bar{h}$ is as defined in (2.24). Combining (4.3), (4.4) and (4.5), we get the expression given in (4.1). □

5. Stochastic differential equations and sample path properties. We now study the system of SDEs,

$$dU(t) = V(t)\,dt, \qquad dV(t) = c(U(t), V(t))\,dt + dW(t), \tag{5.1}$$

where the function c is defined by

$$c(x, y) = h(x, y)^{-1}\frac{\partial}{\partial y}h(x, y), \qquad x > 0,\ y \in \mathbb{R}, \tag{5.2}$$

and h is defined by (2.4). Several difficulties arise in analyzing this system.

1. The system clearly does not define a two-dimensional diffusion, since the matrix of second derivatives of the differential operator is singular.

2. The function $(x, y) \mapsto c(x, y)$ is not uniformly Lipschitz, nor is this function bounded.
3. The growth of the function $(x, y) \mapsto c(x, y)$ is faster than linear, as $y \to -\infty$.

Note that, for all $x > 0$, the function $y \mapsto h(x, y)$, $y \in \mathbb{R}$, is one-to-one, since

$$\frac{\partial}{\partial y} h(x, y) > 0, \qquad y \in \mathbb{R},\ x > 0. \tag{5.3}$$

Also note that the function $(x, y) \to c(x, y)$ is positive since $h(x, y)$ and $(\partial/\partial y)h(x, y)$ are both positive for all $x > 0$, $y \in \mathbb{R}$, as is easily seen from (2.23), (2.9) and (2.10), using the explicit representation of g in terms of the confluent hypergeometric functions. We also have

$$\lim_{x \downarrow 0} h(x, y) = 0 \text{ for all } y < 0 \quad \text{and} \quad \lim_{x \downarrow 0,\, y \uparrow 0} h(x, y) = 0. \tag{5.4}$$

Since $U(t)$ can only hit zero for values $V(t) \le 0$, we can define $h(U(t), V(t)) = 0$, if $U(t) = 0$.

In spite of the difficulties mentioned above, we have the following existence and unicity result for the system (5.1), showing that the system actually characterizes our conditioned process.

THEOREM 5.1. *The system of SDEs* (5.1) *has a unique strong solution* $(\tilde{U}, \tilde{V})$, *for any starting point* $(\tilde{U}(0), \tilde{V}(0)) = (x, y)$, *with* $x > 0$. *Furthermore, let the function* h *be defined by* (2.4) *and suppose that the process* $(\tilde{U}, \tilde{V})$ *solves* (5.1) *for a starting value* (x, y), *at time zero, with* $x > 0$. *Then*:

(i) *The transition density* $\tilde{p}_t$ *of the process* $(\tilde{U}, \tilde{V})$ *is given by*

$$\tilde{p}_t(x, y; u, v) = h(x, y)^{-1} \bar{p}_t(x, y; u, v) h(u, v), \tag{5.5}$$

that is, $(\tilde{U}, \tilde{V})$ *is distributed as the process* (U, V), *for* U *away from zero*.

(ii) *The process*

$$t \mapsto 1/h\big(\tilde{U}(t), \tilde{V}(t)\big), \qquad t \ge 0$$

is a local martingale w.r.t. the natural filtration, induced by $(\tilde{U}, \tilde{V})$.

(iii) *With probability* 1, $\tilde{U}$ *never hits* 0,

$$P_{(x,y)}\big\{\tilde{U}(t) > 0 \text{ for all } t > 0\big\} = 1.$$

(iv) *The process* $\tilde{U}$ *is transient*, *that is*,

$$P_{(x,y)}\Big\{\lim_{t \to \infty} \tilde{U}(t) = \infty\Big\} = 1.$$

PROOF. We prove the existence of a unique strong solution to (5.1) by a localization argument. Let, for $N > 0$, the function c_N be defined by

$$c_N(x, y) = c(x \vee 1/N, y \vee (-N)).$$

Then c_N is globally Lipschitz. Hence it follows from Theorem 3.1, page 164, Chapter IV of [3] that the system

$$dU(t) = V(t)\,dt, \qquad dV(t) = c_N(U(t), V(t))\,dt + dW(t)$$

has a unique strong solution (U_N, V_N) for each $N > 0$. Moreover, (U_N, V_N) is a solution of the original system up to time $T_N = \inf\{t > 0\colon U_N(t) < 1/N$ or $V_N(t) < -N\}$. Pasting these solutions yields a solution $(\tilde{U}, \tilde{V})$ to the system (5.1) up to time $T = \sup_N T_N$. Below we show that $T = \infty$.

Itô's formula shows that the process $t \mapsto 1/h(\tilde{U}(t), \tilde{V}(t))$ is a nonnegative local martingale and hence a supermartingale. Hence $t \mapsto 1/h(\tilde{U}(t), \tilde{V}(t))$ satisfies Doob's supermartingale theorem; see [15], Theorem 49.1 and Corollary 49.2, page 147. By the Fatou lemma, 14.3, [14], page 22, we then get that

$$\lim_{t \to T} 1/h\big(\tilde{U}(t), \tilde{V}(t)\big) \tag{5.6}$$

exists almost surely and is finite, for any starting point $(x, y) \in (0, \infty) \times \mathbb{R}$ of the process $(\tilde{U}, \tilde{V})$. By (5.4) this implies that $\tilde{U}$ does not hit zero up to (and including) time T.

Since, by the second equation of the system (5.1), any solution $(\tilde{U}, \tilde{V})$ of the system (5.1) has to satisfy $\tilde{V}(t) \geq W(t)$, for all $t \geq 0$ for which the solution is defined, we cannot have $\tilde{V}(t) = -\infty$ ("explosion to $-\infty$") at a finite time t. Also, by (5.6), $\tilde{U}$ cannot hit zero (see above). So the only way in which explosion could occur is when $\tilde{V}(t) = \infty$ at a finite time t. However, this possibility is actually excluded by the growth condition on the function $(x, y) \mapsto c(x, y)$, as $y \to \infty$, using that $\min_{t \in [0, M]} \tilde{U}(t) > 0$ for each time interval $[0, M]$. Thus $T = \infty$, implying that we have a unique strong solution to the system (5.1). We now also have proved (ii) and (iv).

Now note that the infinitesimal generator of the process $(\tilde{U}(t), \tilde{V}(t))$ is given for any test function φ by

$$\overline{\mathscr{D}}_{(x, y)}\varphi(x, y) = h(x, y)^{-1}\mathscr{D}_{(x, y)}[h(x, y)\varphi(x, y)].$$

This corresponds to the transition density (5.5). Since $1/h$ is harmonic for $\overline{\mathscr{D}}_{(x, y)}$ we now have (i). In fact, (ii) follows from the harmonicity of $1/h$ for $\overline{\mathscr{D}}_{(x, y)}$, as was seen above by applying Itô's formula.

Part (iii) would follow from

$$\lim_{t \to \infty} E\big\{1/h\big(\tilde{U}(t), \tilde{V}(t)\big)\big\} = 0 \tag{5.7}$$

and the fact that $1/h(\tilde{U}(t), \tilde{V}(t))$ has, almost surely, a finite limit, as $t \to \infty$, since almost sure convergence to a finite limit implies convergence in probability to the same limit, and since (5.7) implies that $1/h(\tilde{U}(t), \tilde{V}(t))$ converges to zero in probability. However, by (5.5) for the transition density of the process $(\tilde{U}, \tilde{V})$, it follows that

$$\begin{aligned} E_{(x, y)} 1/h\big(\tilde{U}(t), \tilde{V}(t)\big) &= h(x, y)^{-1} \int_{u=0}^{\infty} \int_{v=-\infty}^{\infty} \bar{p}_t(x, y; u, v)\,du\,dv \\ &= h(x, y)^{-1} P_{(x, y)}\{\tau_0 > t\} \to 0 \quad \text{as } t \to \infty, \end{aligned}$$

for any starting point (x, y) such that $x > 0$. So we get

$$\lim_{t\to\infty} E_{(x,y)}\left\{1/h\big(\tilde{U}(t),\tilde{V}(t)\big)\right\} = 0, \tag{5.8}$$

and hence

$$\lim_{t\to\infty} 1/h\big(\tilde{U}(t),\tilde{V}(t)\big) = 0, \tag{5.9}$$

with probability 1. Now (5.9) implies

$$\lim_{t\to\infty} h\big(\tilde{U}(t),\tilde{V}(t)\big) = \infty \tag{5.10}$$

with probability 1. If $\tilde{V}(t)$ tends to ∞, then also $\tilde{U}(t)$ tends to ∞, since $\tilde{U}$ is the integral of $\tilde{V}$. On the other hand, if $\tilde{V}(t)$ does not tend to infinity, (5.10) can only happen if $\tilde{U}(t)$ tends to infinity, using (5.3) (the monotonicity of h in the second argument). So we obtain in all cases that $\tilde{U}(t)$ tends to infinity with probability 1. □

6. A critical curve. Our investigation was originally motivated by the question whether the expected amount of time that the process $\tilde{U}$ (i.e., integrated Brownian motion, conditioned to be positive) spends below any line of positive slope is finite. The following result answers this question negatively.

THEOREM 6.1. *Suppose that $k > 0$ and $0 < \alpha < 3/2$ and let the constant $c' > 0$ be given by*

$$c' = \frac{9\cdot 2^{35/4}}{5\pi\sqrt{\pi}}\int_{v=-\infty}^{\infty} g(v)g(-v)\,dv. \tag{6.1}$$

Then

$$P_{(0,0)}\big(\tilde{U}(t) < kt^{\alpha}\big) \sim c'k^{5/3}t^{5\alpha/3-5/2} \quad \text{as } t\to\infty. \tag{6.2}$$

Hence, if T_α denotes the amount of time $\tilde{U}(t)$ spends below the curve $u(t) = kt^{\alpha}$,

$$E_{(0,0)}T_{\alpha}\begin{cases} < \infty, & \text{if } \alpha < 9/10,\\ = \infty, & \text{if } \alpha \ge 9/10.\end{cases} \tag{6.3}$$

PROOF. Let $\alpha \in (0, 3/2)$. Using (4.1), (4.2) and (2.24) and denoting the constant $2^{29/4}$ by c, we get for any $k > 0$,

$$\begin{aligned}
&P\big\{\tilde{U}(t) < kt^{\alpha}\big\}\\
&\quad = c\int_{u=0}^{kt^{\alpha-3/2}}\int_{v=-\infty}^{\infty} u^{1/6}g(vu^{-1/3})\bar{h}(1,u,-v)\,dv\,du\\
&\quad = \frac{4c\sqrt{3}}{\sqrt{2\pi}}\int_{u=0}^{kt^{\alpha-3/2}}\int_{v=-\infty}^{\infty}\int_{s=0}^{1}\int_{w=0}^{\infty} u^{1/6}w^{3/2}g(vu^{-1/3})(1-s)^{-1/2}\\
&\qquad\qquad \times p_{1-s}(0,w;0,0)q_s(u,-v;0,-w)\,ds\,dw\,dv\,du.
\end{aligned}$$

By the change of variables $v \to vu^{1/3}$, $s \to su^{2/3}$ and $w \to wu^{1/3}$, we get

$$
\begin{aligned}
&P\{\tilde U(t) < kt^\alpha\}\\
&\quad = \frac{4c\sqrt{3}}{\sqrt{2\pi}} \int_{u=0}^{kt^{\alpha-3/2}} \int_{v=-\infty}^{\infty} \int_{s=0}^{u^{-2/3}} \int_{w=0}^{\infty} u^{2/3} w^{3/2} g(v)\left(1-su^{2/3}\right)^{-1/2}\\
&\qquad\qquad \times p_{1-su^{2/3}}(0, wu^{1/3}; 0, 0) q_s(1, -v; 0, -w)\, ds\, dw\, dv\, du\\
&\quad \sim \frac{4c\sqrt{3}}{\sqrt{2\pi}} \frac{\sqrt{3}}{\pi} \int_{u=0}^{kt^{\alpha-3/2}} \int_{v=-\infty}^{\infty} \int_{s=0}^{\infty} \int_{w=0}^{\infty} u^{2/3} w^{3/2} g(v) \qquad (6.4)\\
&\qquad\qquad \times q_s(1, -v; 0, -w)\, ds\, dw\, dv\, du\\
&\quad = \frac{36c}{5\pi\sqrt{2\pi}} k^{5/3} t^{(\alpha-3/2)(5/3)} \int_{v=-\infty}^{\infty} g(v)\\
&\qquad\qquad \times \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} q_s(1, -v; 0, -w)\, ds\, dw\, dv,
\end{aligned}
$$

as $t \to \infty$, yielding (6.2).

The amount of time T_α that $\tilde U$ spends below $u(t) = kt^\alpha$ can be written as $T_\alpha = \int_0^\infty 1_{\{\tilde U(t) < kt^\alpha\}}\, dt$. It now follows that the expected amount of time spent below the curve $y = kt^\alpha$ is

$$
E_{(0,0)} T_\alpha = \int_0^\infty P_{(0,0)}\left(\tilde U(t) < kt^\alpha\right) dt.
$$

By (6.2) this is finite when $(5/3)\alpha - 5/2 < -1$, and infinite when $(5/3)\alpha - 5/2 \geq -1$. Hence we get the conclusion that the expected amount of time spent below the curve $y = kt^\alpha$ is finite when $\alpha < 9/10$, and infinite when $\alpha \geq 9/10$. □

APPENDIX

Proof of Theorem 2.1(i). For the first term of (2.1) we get, if $z > 0$,

$$
\begin{aligned}
&tzq_t(x, y; 0, -z\sqrt{t})\\
&\quad = \frac{2z\sqrt{3}}{\pi t} \exp\left\{-\frac{6x^2}{t^3} - \frac{6xy}{t^2} - \frac{2y^2}{t} - 2z^2\right\} \sinh\left(\frac{6xz}{t^{3/2}} + \frac{2yz}{t^{1/2}}\right)\\
&\quad \sim \frac{2z^2 \exp(-2z^2)\sqrt{3}}{\pi t^{3/2}} \left(\frac{6x}{t} + 2y\right), \qquad t \to \infty.
\end{aligned}
$$

For the second term we get

$$
\begin{aligned}
&-tz \int_{s=0}^{t} \int_{w=0}^{\infty} q_{t-s}(x, y; 0, w) P_{(0, z\sqrt{t})}\{\tau_0 \in ds, V(\tau_0) \in dw\}\\
&\qquad = \frac{6zt\sqrt{3}}{\pi^2\sqrt{2\pi}} \int_{s=0}^{t} \int_{w=0}^{\infty} ws^{-2}(t-s)^{-2} \qquad (A.1)
\end{aligned}
$$

$$\times \exp\left\{-\frac{6x^2}{(t-s)^3} - \frac{6xy}{(t-s)^2} - \frac{2(y^2+w^2)}{t-s}\right\}$$
$$\times \sinh\left(\frac{6xw}{(t-s)^2} + \frac{2yw}{t-s}\right)\exp\left\{-\frac{2(z^2t - zw\sqrt{t} + w^2)}{s}\right\}$$
$$\times \int_0^{4zw\sqrt{t}/s} \xi^{-1/2} \exp\left\{-\frac{3}{2}\xi\right\} d\xi\, ds\, dw.$$

By the change of variables $w \to w\sqrt{t}$ and $s \to st$, we get

$$\frac{6z\sqrt{3}}{t\pi^2\sqrt{2\pi}} \int_{s=0}^{1} \int_{w=0}^{\infty} ws^{-2}(1-s)^{-2}$$
$$\times \exp\left\{-\frac{6x^2}{t^3(1-s)^3} - \frac{6xy}{t^2(1-s)^2} - \frac{2y^2}{t(1-s)} - \frac{2w^2}{1-s}\right\}$$
$$\times \sinh\left(\frac{6xw}{(1-s)^2 t^{3/2}} + \frac{2yw}{(1-s)t^{1/2}}\right)\exp\left\{-\frac{2(z^2 - zw + w^2)}{s}\right\}$$
$$\times \int_0^{4zw/s} \xi^{-1/2} \exp\left\{-\frac{3}{2}\xi\right\} d\xi\, ds\, dw.$$

As will become clear in the sequel, the dominating behavior of this multiple integral, as $t \to \infty$, will come from a region of integration for s in a neighborhood of 1. We therefore first consider the region of integration $s \in [1/2, 1]$. We define

$$\psi(u) = \int_0^u \xi^{-1/2} \exp\{-\tfrac{3}{2}\xi\}\, d\xi, \tag{A.2}$$

Using the change of variables $s \to 1 - 1/(st)$ and $w \to w/\sqrt{t}$, and using the notation (A.2), we can write the integral over this region as

$$\begin{aligned}
&\frac{6z\sqrt{3}}{t\pi^2\sqrt{2\pi}} \int_{s=2/t}^{\infty} \int_{w=0}^{\infty} w(1 - 1/(st))^{-2} \\
&\qquad \times \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\}\sinh(6xws^2 + 2yws) \\
&\qquad \times \psi\big(4zwt^{-1/2}/(1 - 1/(st))\big) \\
&\qquad \times \exp\left\{-\frac{2(z^2 - zwt^{-1/2} + w^2t^{-1})}{1 - 1/(st)}\right\} ds\, dw \qquad \text{(A.3)}\\
&\sim \frac{24z^{3/2}\exp(\sim 2z^2)\sqrt{3}}{t^{5/4}\pi^2\sqrt{2\pi}} \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} \\
&\qquad \times \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \\
&\qquad \times \sinh(6xws^2 + 2yws)\, ds\, dw, \qquad t \to \infty.
\end{aligned}$$

The asymptotic equivalence of the last step can be proved in the following way. Restricting the region of integration for s to $[\varepsilon, \infty)$ for a fixed $\varepsilon > 0$, we get, using Lebesgue's dominated convergence theorem,

$$\begin{aligned}
&\frac{6z\sqrt{3}}{t\pi^2\sqrt{2\pi}} \int_{s=\varepsilon}^{\infty} \int_{w=0}^{\infty} w\bigl(1 - 1/(st)\bigr)^{-2} \\
&\qquad \times \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \sinh(6xws^2 + 2yws) \\
&\qquad \times \psi\bigl(4zwt^{-1/2}/(1 - 1/(st))\bigr) \exp\left\{-\frac{2(z^2 - zwt^{-1/2} + w^2t^{-1})}{1 - 1/(st)}\right\} ds\, dw \\
&\sim \frac{24\sqrt{3}}{t^{5/4}\pi^2\sqrt{2\pi}} \int_{s=\varepsilon}^{\infty} \int_{w=0}^{\infty} (zw)^{3/2} \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \\
&\qquad\qquad \times \sinh(6xws^2 + 2yws) \exp\{-2z^2\}\, ds\, dw, \qquad t \to \infty.
\end{aligned}$$

For the region $s \in [2/t, \varepsilon]$, we get

$$\begin{aligned}
&\frac{6z\sqrt{3}}{t\pi^2\sqrt{2\pi}} \int_{s=2/t}^{\varepsilon} \int_{w=0}^{\infty} w\bigl(1 - 1/(st)\bigr)^{-2} \\
&\qquad \times \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \bigl|\sinh(6xws^2 + 2yws)\bigr| \\
&\qquad \times \psi\bigl(4zwt^{-1/2}/(1 - 1/(st))\bigr) \exp\left\{-\frac{2(z^2 - zwt^{-1/2} + w^2t^{-1})}{1 - 1/(st)}\right\} ds\, dw \\
&\le \frac{24z^{3/2}e^{-2z^2}\sqrt{3}}{t^{5/4}\pi^2\sqrt{2\pi}} \int_{s=0}^{\varepsilon} \int_{w=0}^{\infty} w^{3/2} \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \\
&\qquad\qquad \times \bigl|\sinh(6xws^2 + 2yws)\bigr|\, ds\, dw \\
&\le \frac{24z^{3/2}\exp(-z^2)\sqrt{3}}{t^{5/4}\pi^2\sqrt{2\pi}} \int_{s=0}^{\varepsilon} \int_{w=0}^{\infty} w^{3/2} \\
&\qquad\qquad \times \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2s\} \\
&\qquad\qquad \times ws|6xs + 2y| \cosh(6xws^2 + 2yws)\, ds\, dw
\end{aligned}$$

due to the inequality

$$|\sinh(u)| \le |u| \cosh(u). \tag{A.4}$$

Now note that, by the change of variables $w \to ws^{-1/2}$, the last displayed integral is less than

$$\begin{aligned}
&\int_{s=0}^{\varepsilon} \int_{w=0}^{\infty} w^{5/2}s^{-3/4} \exp\{-6x^2s^3 - 6xys^2 - 2y^2s - 2w^2\} \\
&\qquad \times |6xs + 2y| \cosh\bigl(6xws^{3/2} + 2yws^{1/2}\bigr)\, ds\, dw \\
&= \mathcal{O}(\varepsilon), \qquad \varepsilon \downarrow 0.
\end{aligned}$$

This proves (A.3).

We next consider the region of integration $s \in [0, 1/2]$. The corresponding integral can be written by making the change of variables $w \to w\sqrt{s}$ and next $s \to z^2/s$, as

$$\begin{aligned}
&\frac{6z\sqrt{3}}{t\pi^2\sqrt{2\pi}} \int_{s=2z^2}^{\infty} \int_{w=0}^{\infty} w s^{-1} (1 - z^2/s)^{-2} \\
&\quad \times \exp\left\{ -\frac{6x^2}{t^3(1 - z^2/s)^3} - \frac{6xy}{t^2(1 - z^2/s)^2} - \frac{2y^2}{t(1 - z^2/s)} - \frac{2w^2}{1 - z^2/s} \right\} \\
&\quad \times \sinh\left(\frac{6xw}{(1 - z^2/s)^2 t^{3/2}} + \frac{2yw}{(1 - z^2/s) t^{1/2}} \right) \exp\{-2(s - w s^{1/2} + w^2)\} \\
&\quad \times \int_0^{4w\sqrt{s}} \xi^{-1/2} \exp\left\{ -\frac{3}{2}\xi \right\} d\xi \, ds \, dw.
\end{aligned}$$

Using inequalities (A.4) and

$$s^{-1} \int_0^{4w\sqrt{s}} \xi^{-1/2} \exp\{-\tfrac{3}{2}\xi\} \, d\xi \le 4 s^{-3/4} w^{1/2}, \tag{A.5}$$

it is easily seen that this term is $\mathcal{O}(t^{-3/2})$, as $t \to \infty$.

Concluding, we get, as $t \to \infty$,

$$\begin{aligned}
&P_{(u,v)}\left\{ \tau_0 \in dt, V(\tau_0)/\sqrt{t} \in -dz \right\} / (dt\,dz) \\
&\quad \sim \frac{24 z^{3/2} \exp\{-2z^2\} \sqrt{3}}{t^{5/4} \pi^2 \sqrt{2\pi}} \\
&\qquad \times \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} \exp\{-6x^2 s^3 - 6xys^2 - 2y^2 s - 2w^2 s\} \\
&\qquad\qquad \times \sinh(6xws^2 + 2yws) \, ds \, dw,
\end{aligned} \tag{A.6}$$

which yields the result (2.19). $\square$

Proof of Theorem 2.1(ii). The proof of this property proceeds along similar lines. For the first term of (2.1) we get

$$\begin{aligned}
&z q_t(x, y; 0, -z) \\
&\quad = \frac{2z\sqrt{3}}{\pi t^2} \exp\left\{ -\frac{6x^2}{t^3} - \frac{6xy}{t^2} - \frac{2y^2}{t} - \frac{2z^2}{t} \right\} \sinh\left(\frac{6xz}{t^2} + \frac{2yz}{t} \right) \\
&\quad \sim \frac{2z^2\sqrt{3}}{\pi t^3} \left(\frac{6x}{t} + 2y \right) \exp\left\{ -\frac{6x^2}{t^3} - \frac{6xy}{t^2} - \frac{2y^2}{t} \right\} \\
&\quad = \mathcal{O}(z^2), \qquad z \downarrow 0.
\end{aligned}$$

For the second term we get

$$
\begin{aligned}
&-z\int_{s=0}^{t}\int_{w=0}^{\infty} q_{t-s}(x,y;0,w)P_{(0,-z)}\{\tau_0^+\in ds, V(\tau_0^+)\in dw\}\\
&\quad=\frac{6z\sqrt{3}}{\pi^2\sqrt{2\pi}}\int_{s=0}^{t}\int_{w=0}^{\infty} ws^{-2}(t-s)^{-2}\\
&\qquad\times\exp\left\{-\frac{6x^2}{(t-s)^3}-\frac{6xy}{(t-s)^2}-\frac{2(y^2+w^2)}{t-s}\right\} \qquad\text{(A.7)}\\
&\qquad\times\sinh\left(\frac{6xw}{(t-s)^2}+\frac{2yw}{t-s}\right)\exp\left\{-\frac{2(z^2-zw+w^2)}{s}\right\}\\
&\qquad\times\int_0^{4zw/s}\xi^{-1/2}\exp\left\{-\frac{3}{2}\xi\right\}d\xi\,ds\,dw.
\end{aligned}
$$

We first consider the region of integration $s\in[\varepsilon,t]$, for $\varepsilon\in(0,t)$. Using Lebesgue's dominated convergence theorem, we get that this integral is asymptotically equivalent to

$$
\begin{aligned}
&\frac{24z^{3/2}\sqrt{3}}{\pi^2\sqrt{2\pi}}\int_{s=\varepsilon}^{t}\int_{w=0}^{\infty} w^{3/2}s^{-5/2}(t-s)^{-2}\\
&\qquad\times\exp\left\{-\frac{6x^2}{(t-s)^3}-\frac{6xy}{(t-s)^2}-\frac{2y^2}{t-s}-\frac{2w^2}{t-s}\right\}\\
&\qquad\times\sinh\left(\frac{6xw}{(t-s)^2}+\frac{2yw}{t-s}\right)\exp\left\{-\frac{2w^2}{s}\right\}ds\,dw,
\end{aligned}
$$

as $z\downarrow 0$. We next consider the region of integration $s\in[0,\varepsilon]$. The corresponding integral can be written, by making the change of variables $w\to w\sqrt{s}$ and next $s\to z^2/s$, as

$$
\begin{aligned}
&\frac{6z\sqrt{3}}{\pi^2\sqrt{2\pi}}\int_{s=z^2/\varepsilon}^{\infty}\int_{w=0}^{\infty} ws^{-1}(t-z^2/s)^{-2}\\
&\qquad\times\exp\left\{-\frac{6x^2}{(t-z^2/s)^3}-\frac{6xy}{(t-z^2/s)^2}-\frac{2y^2}{t-z^2/s}-\frac{2w^2z^2/s}{t-z^2/s}\right\}\\
&\qquad\times\sinh\left(\frac{6xwz/\sqrt{s}}{(t-z^2/s)^2}+\frac{2ywz/\sqrt{s}}{t-z^2/s}\right)\exp\{-2(s-ws^{1/2}+w^2)\}\\
&\qquad\times\int_0^{4w\sqrt{s}}\xi^{-1/2}\exp\left\{-\frac{3}{2}\xi\right\}d\xi\,ds\,dw.
\end{aligned}
$$

Again, using the inequalities (A.4) and (A.5), it is easily seen that this term is $\mathcal{O}(\varepsilon^{1/4}z^{3/2})$. Since $\varepsilon>0$ can be chosen arbitrarily small, we get the result. □

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