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SELF-DIFFUSION FOR BROWNIAN MOTIONS WITH LOCAL INTERACTION

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We derive explicitly the asymptotic law of the tagged particle process in a system of interacting Brownian motions in the presence of a diffusive scaling in nonequilibrium. The interaction is local and interpolates between the totally independent case (noninteracting) and the totally reflecting case and can be viewed as the limiting local version of an interaction through a pair potential as its support shrinks to zero. We also prove the independence of two tagged particles in the limit.

CONTENTS

1. Introduction and results. In his thesis [2] at New York University in 1984, Guo studied an infinite system of interacting Brownian motions on the line in equilibrium. The interaction was governed by a pair potential $V(x-y)$ which is a smooth, positive, symmetric and compactly supported function. He provided a variational formula for the self-diffusion coefficient (the diffusion coefficient of the tagged particle process) in a diffusive scaling ($x \to Nx$ and $t \to N^2 t$ for some large N). Of course one may recast the dynamics on the unit circle; it turns out that an interesting problem is the study of the tagged particle process in a limiting case of the interaction which will be described below. This is the object of the present work. We shall be able to derive the asymptotic law of the tagged particle process in nonequilibrium. The diffusion coefficient as well as the drift term (present in nonequilibrium) will be computed explicitly.

Throughout this paper we make the assumption that the initial profile has a bounded density. However, this hypothesis can be eliminated and the existence and uniqueness of the tagged particle can be proven for singular measures by

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making further assumptions on the mass distribution in the initial empirical density, as it is shown in [1].

Let us give a general outline of the problems we are concerned with. In our context, an interacting particle system is defined by a dynamics associated to a given number n of particles evolving in some state space Γ, in this case the circle. Once the dynamics has been established, one has to define a scaling of the problem (e.g., the diffusive scaling in which $n = N\bar{\rho}$, $x \to Nx$ and $t \to N^2 t$ such that x^2/t is preserved).

At this point one can research the limiting behavior of the system from two points of view. One way is to look at the empirical distribution of the particles (i.e., spatial averages) in order to derive the hydrodynamic limit. In this approach particles are indistinguishable and various types of interaction may lead to the same solution in the limit. The other approach is to single out one particle (or a finite number of them), the so-called tagged particle, and follow its evolution as the scaling parameter tends to infinity. In this approach the underlying dynamics leaves its mark on the limiting behavior of the tagged particle.

The results presented in this work regard exactly this type of analysis. The dynamics of the system (described briefly below) is perhaps easier to understand as the weak limit of n Brownian motions $P^{n,\varepsilon}$ interacting through a pair potential V_ε with support in the interval $[-\varepsilon, \varepsilon]$ taken as $\varepsilon \to \infty$. $P^{n,\varepsilon}$ is the law of the n-dimensional process

$$dx_i = d\beta_i - \sum_{j \neq i} V'_\varepsilon(x_i - x_j)\, dt \qquad \forall\ i = 1, \dots, n$$

with $\beta_1(t), \dots, \beta_n(t)$ independent Wiener processes and $V_\varepsilon(x)$ a smooth, even, compactly supported potential. For a positive parameter λ, the convergence takes place under the condition $\phi_\varepsilon(x) = \exp(2V_\varepsilon(x)) - 1 \to (1/\lambda)\delta_0$ as $\varepsilon \to 0$ in the distributional sense. The resulting interaction will be described in the following.

Since the interaction is local and only two particles can collide at one time, the definition of the model can be presented by considering the case when there are only two particles.

We have two particles that perform independent Brownian motions until they collide. In the noninteracting case, the particles go through each other and in the reflected case they bounce off each other. If we do not tag the particles but consider them as a system of two indistinguishable particles, there is no difference between the two. If we now try to tag them, there is no trouble keeping track of the tags, that is, the relative labels of the two particles until a collision. After the collision, in the noninteracting case, the probability is $\frac{1}{2}$ for each of the two possible ways of labeling them, and in the reflecting case the labels are completely determined by the relative ordering of the particles prior to the collision. In our model, that in some sense interpolates between the two, we start with the reflected case. There is a canonical local time that measures the "amount of collision" in a natural scale. The switching of labels takes place as a Poisson event at a rate λ in the time scale determined by this local time.

Of course, the switching occurs only when the particles are colliding and does not always happen. This can be made rigorous and these models interpolate between the noninteracting case where $\lambda = \infty$ and the reflecting case where $\lambda = 0$. We can extend this model to the case of n particles on the circle. We will do so now and give a formal description of the model.

1.1. *The interaction model.* Consider a positive integer n and $\lambda \geq 0$. Let Γ^n be the n-dimensional torus. We define $F^{ij} = \{\xi \in \Gamma^n: \xi_i = \xi_j\}$ for any i, j in $\{1, \dots, n\}$ and $F = \bigcup_{1 \leq i < j \leq n} F_{ij}$. We shall denote by $\bar{C}(\Gamma^n, F)$ the set of functions f that are piecewise smooth (up to the boundary F) on $\Gamma_n \setminus F$.

DEFINITION 1. Let $\bar{C}(\Gamma^n, F) = \{f\colon \Gamma^n \longrightarrow R\colon f \in C^2(\Gamma^n \setminus F)$ with $f^{ij}(\xi_0)$ and $D^{ij} f(\xi_0)$ finite for any $\xi_0 \in F$ and any $(i, j)\}$, the set of smooth functions up to the boundary F, where f^{ij} and $D^{ij} f$ are defined as

$$\begin{aligned} f^{ij}&(\xi_1, \xi_2, \dots, \xi_{i-1}, \xi, \xi_{i+1}, \dots, \xi_{j-1}, \xi, \xi_{j+1}, \dots, \xi_n), \\ &= f(\xi_1, \xi_2, \dots, \xi_{i-1}, \xi + 0, \xi_{i+1}, \dots, \xi_{j-1}, \xi - 0, \xi_{j+1}, \dots, \xi_n), \end{aligned} \tag{1.1}$$

$$\begin{aligned} D^{ij} f&(\xi_1, \xi_2, \dots, \xi_{i-1}, \xi, \xi_{i+1}, \dots, \xi_{j-1}, \xi, \xi_{j+1}, \dots, \xi_n) \\ &= (\partial_i - \partial_j) f\big(\xi_1, \xi_2, \dots, \xi_{i-1}, \xi + 0, \\ &\qquad \xi_{i+1}, \dots, \xi_{j-1}, \xi - 0, \xi_{j+1}, \dots, \xi_n\big). \end{aligned} \tag{1.2}$$

We are now in a position to define the generator of the process

$$\xi^n(t) = (\xi_1(t), \dots, \xi_n(t))$$

on Γ^n. For a real $\lambda \geq 0$ we define the boundary conditions

$$\text{(BC)} \quad D^{ij} f(\xi) + \lambda(f^{ji}(\xi) - f^{ij}(\xi)) = 0 \qquad \forall\, i, j \in \{1, \dots, n\}. \tag{1.3}$$

The operator $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$ with

$$\mathscr{L} f = \tfrac{1}{2}\Delta f, \qquad \mathscr{D}(\mathscr{L}) = \big\{f \in \bar{C}(\Gamma^n, F)\text{: (BC) are satisfied}\big\} \tag{1.4}$$

is the infinitesimal generator of a process P^n_λ on Γ^n.

1.2. *The scaled model.* The considerations made up to this point regard a process P^n_λ for a given n. Let us consider a large positive N and let us blow up the space scale by a factor of N, such that the particles evolve on a circle of length N instead of 1; in the scaled version we shall look at $\xi(t)/N$. The time scale will also be amplified by N^2 to produce a diffusive scaling (ξ^2/t is invariant; i.e., the Laplacian is preserved).

Let $\bar{\rho} > 0$ and $\lambda > 0$ be fixed constants. The number of particles will be scaled to $N\bar{\rho}$; physically this implies that the average density of the system does not change.

The scaled process will be defined by (1.4) with $n := N\bar{\rho}$ and $\lambda_N := N\lambda$. It will be denoted by

$$P^{\xi^N} = P^{N\bar{\rho}}_{N\lambda}. \tag{1.5}$$

The new process evolves on the $n = N\bar{\rho}$-dimensional torus Γ^n. Each particle ξ_k, for $k = 1, \dots, n$, performs a Brownian motion on the unit circle until it collides with some other particle, where the given interaction governed by $\lambda_N = N\lambda$ takes place and then the reflected or switched pair proceeds by performing independent Brownian motions until the next collision and so on.

1.3. *The martingale form of the problem.* It is known (see [4]) from the definition of the process $\{\xi^N(t)\}_{t\geq 0}$ with the filtration $\{\mathscr{F}_t\}_{t\geq 0}$, $\mathscr{F}_t = \sigma(\xi^N(s): 0 \leq s \leq t)$ that there exist $n^2 - n$ local times $\{A^{ij}(t)\}_{t\geq 0}$ for $i \neq j$ in the set $\{1, 2, \dots, n\}$ such that for any $f \in \bar{C}(\Gamma^n, F)$,

$$\begin{aligned}\mathscr{M}_f(t) &:= f(\xi^N(t)) - f(\xi^N(0)) - \tfrac{1}{2}\int_0^t \Delta f(\xi^N(s))\,ds \\ &\quad - \sum_{i\neq j}\int_0^t \big(D^{ji} f(\xi^N(s)) + (\lambda N)[f^{ij}(\xi^N(s)) - f^{ji}(\xi^N(s))]\big)\,dA^{ji}(s)\end{aligned} \tag{1.6}$$

is a $(P^{\xi^N}, \{\mathscr{F}_t\}_{t\geq 0})$-martingale. More precisely,

$$\mathscr{M}_f(t) = \sum_{k=1}^n \int_0^t \partial_{\xi_k} f(\xi^N(s))\,d\beta_k(s) + \sum_{i\neq j}\int_0^t [f^{ij}(\xi^N(s)) - f^{ji}(\xi^N(s))]\,dM_N^{ji}(s),$$

where $\{\beta_k(t)\}$ $(k = 1, \dots, n)$ is a family of independent Brownian motions and $M_N^{ij}(t)$, $M_N^{ji}(t)$ are the jump martingales corresponding to the interaction along the boundary F^{ij} such that $[M_N^{ij}(t)]^2 - (\lambda N)A_N^{ij}(t)$ is also a martingale.

1.4. *The lifted process.*

DEFINITION 2. We shall denote by Ω_{Γ^n} the space of continuous paths from $[0, \infty)$ on the n-dimensional torus Γ^n and by Ω_{R^n} the space of continuous paths from $[0, \infty)$ on R^n.

Each continuous path on the unit circle can be lifted in a canonical way to a continuous path on the covering space R. The mapping Λ will be the Cartesian product of the n canonical mappings for each component with the given initial condition

$$\xi_k(0) = \xi_k = x_k \in [0, 1] \quad \text{with } k = 1, \dots n.$$

There is an important distinction to make between the process

$$\xi(\cdot) = (\xi_1(\cdot), \dots, \xi_n(\cdot))$$

with state space the n-dimensional torus Γ^n and the lifted process

$$x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot)) \tag{1.7}$$

with state space R^n given by $x(\cdot) = \Lambda(\xi(\cdot))$ constructed with the lift mapping Λ: $\Omega_{\Gamma^n} \to \Omega_{R^n}$ by lifting each component $\xi_1(\cdot), \dots, \xi_n(\cdot)$.

We use the following notation.

DEFINITION 3. Let Λ be the lift mapping for $n = N\bar{\rho}$. Then $P^N := P^{\xi^N} \circ \Lambda^{-1}$ and P^N is a measure on the path space $C([0, \infty), R^n)$.

DEFINITION 4. The process $\{x_1^N(\cdot)\}_{t \geq 0}$ will be called the tagged particle process.

REMARK. For any function $\Phi \in \bar{C}(T^n, F)$, $\Phi(x) = \Phi(x_1, \dots, x_n)$ periodic of period 1 in each variable, the mappings $t \to \Phi(x^N(t))$ can be identified to $t \to \Phi(\xi^N(t))$ by taking the image of $x_1^N(t)$ on Γ^n. Consequently, we may always substitute the original $\xi(\cdot)$ process with the lifted process $x(\cdot)$ as long as the test functions are periodic.

DEFINITION 5. For any $k = 1, \dots, n$ we shall write

$$A_N^{k,\,\mathrm{left}}(t) := \frac{1}{N} \sum_{j \neq k} A_N^{kj}(t), \tag{1.8}$$

$$A_N^{k,\,\mathrm{right}}(t) := \frac{1}{N} \sum_{j \neq k} A_N^{jk}(t), \tag{1.9}$$

$$A_N^k(t) := A_N^{k,\,\mathrm{left}}(t) + A_N^{k,\,\mathrm{right}}(t) \tag{1.10}$$

and

$$A_N(t) := \frac{1}{N} \Biggl(\sum_{k=1}^{n=N\bar{\rho}} A_N^k(t) \Biggr). \tag{1.11}$$

1.5. *The initial profile.* For a fixed $N > 0$, let P^{ξ^N} be the process defined by the infinitesimal generator $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$ and let $\xi^N = (\xi_1^N, \dots, \xi_n^N)$ be an $n = N\bar{\rho}$-dimensional family of vectors such that

$$P^{\xi^N}\bigl(\{(\xi_1^N(0), \dots, \xi_n^N(0)) = (\xi_1^N, \dots, \xi_n^N)\}\bigr) = 1.$$

DEFINITION 6. A macroscopic initial profile is a measure $\mu(d\xi) \in \mathscr{M}(\Gamma)$ such that the empirical densities at time $t = 0$ converge weakly to $\mu(d\xi)$, that is,

$$\frac{1}{N}\bigl(\delta_{\xi_1^N} + \cdots + \delta_{\xi_n^N}\bigr) \Longrightarrow \mu(d\xi) \tag{1.12}$$

as $N \to \infty$.

In the same spirit, $\mu(d\xi)$ has an initial density profile if there exists a function $\rho_0(\xi)$ such that $\mu(d\xi) = \rho_0(\xi)\, d\xi$ with $\int_\Gamma \rho_0(\xi)\, d\xi = \bar{\rho}$.

As far as the dynamics of the entire system of unlabeled particles is concerned, the behavior of the particles is indistinguishable from the unlabeled independent Brownian motions on the torus. As a consequence we shall show there exists a hydrodynamic limit of the empirical density

$$\frac{1}{N}\big(\delta_{\xi_1^N(t)} + \cdots + \delta_{\xi_n^N(t)}\big) \Longrightarrow \mu(t, d\xi)$$

as $N \to \infty$. The measures $\mu(t, d\xi)$ are solutions to the heat equation

$$\mu_t = \tfrac{1}{2}\mu_{\xi\xi}, \qquad \mu(0, d\xi) = \mu(d\xi) \tag{1.13}$$

in the sense of distributions. As a consequence $\mu(t, d\xi) = \rho(t, \xi)\, d\xi$ for any $t > 0$ and any initial profile $\mu(d\xi)$.

REMARK. In equilibrium the macroscopic density is constant $= \bar{\rho}$. The limiting behavior of the above process is not interesting in itself since it reduces to the simple independent case; however, by studying the particular evolution of the tagged particle we shall derive a nontrivial result.

1.6. *The hydrodynamic limit.* Let $P^N := P^N_{\bar{x}} = P^{\xi^N} \circ \Lambda^{-1}$ for $n = \bar{\rho}N$ be the lifted process (Definition 3). We assume $\bar{x} = (x_1^N, x_2^N, \ldots, x_n^N)$ is a family of $n = N\bar{\rho}$-dimensional vectors in R^n, the images of the initial configuration $(\xi_1^N, \xi_2^N, \ldots, \xi_n^N)$ and $x_1 \equiv \xi_1$ is a given point in $[0, 1]$ such that

$$P^N(\{x_1^N = x_1\}) = 1. \tag{1.14}$$

For the initial configuration $(\xi_1^N, \ldots, \xi_n^N)$, the lifting mapping Λ is simply the identity. Therefore we shall abuse the notation and write $\mu(dx)$ on $[0, 1]$ for the lifted measure corresponding to $\mu(d\xi)$ on the unit circle.

We are ready to prove a preliminary result underlying all considerations regarding our problem. It is the derivation of the hydrodynamic limit of the density profile of the process, showing rigorously that the density profile $\mu(t, dx)$ at time t satisfies the heat equation.

THEOREM 1. *For any smooth periodic $f\colon R \to R$ of period 1 and any $t > 0$,*

$$\lim_{N\to\infty} E^N \sup_{0\le s\le t}\left|\frac{1}{N}\sum_{k=1}^{n=N\bar{\rho}} f(x_k^N(s)) - \int_0^1 f(x)\mu(s, dx)\right|^2 = 0, \tag{1.15}$$

where $\mu(s, dx) = \rho(s, x)\, dx$ for $s > 0$ is the solution to the heat equation as in (1.13).

PROOF. The crucial remark is that because the function

$$(x_1, \ldots, x_n) \to \frac{1}{N}\left(\sum_{k=1}^{n} f(x_k)\right)$$

is symmetric in all variables, the boundary conditions are identically zero so the problem of the macroscopic profile of the scaled process is exactly the same as in the case of noninteracting Brownian motions (independent).

It is easy to check (Doob's inequality) that for any smooth function f on the unit circle Γ, the family of processes

$$\left\{\frac{1}{N}\left(\sum_{k=1}^{n} f(x_k^N(s))\right)\right\}_{s\geq 0}$$

is tight.

Let $\mu(s, dx)$ be the marginal at time s of a particular limiting measure on the path space. Then

$$\lim_{N\to\infty} E^N\left|\frac{1}{N}\left(\sum_{k=1}^{n} f(x_k^N(s))\right) - \int_0^1 f(x)\mu(s, dx)\right|^2 = 0$$

(i.e., the limit in the theorem only pointwise in s).

By Itô's formula, we see that μ is a solution to the heat equation (1.13) starting at $\mu(dx)$ in the sense of distributions. The solution to the PDE is unique. It is clear that for $s > 0$, $\mu(s, dx) = \rho(s, x)\,dx$ and for $s = 0$ we have (1.12). The pointwise statement of the theorem follows.

The uniform convergence in t. Let $\varepsilon > 0$ be fixed. We divide the interval $[0, t]$ in an arbitrary number $m \in Z_+$ of equal intervals and we denote by S_m the set of endpoints of these intervals.

The lim sup as $N \to \infty$ of the quantity we look at is zero for any point in S_m, and because this set is finite it is zero uniformly on S_m; hence it is enough to show that the differences between the L^2 norms of

$$Z_{s'} = \frac{1}{N}\left(\sum_{k=1}^{n} f(x_k^N(s'))\right) - \int_0^1 f(x)\rho(s', x)\,dx$$

and

$$Z_{s''} = \frac{1}{N}\left(\sum_{k=1}^{n} f(x_k^N(s''))\right) - \int_0^1 f(x)\rho(s'', x)\,dx$$

for $|s' - s''| < t/m$ can be made less than ε as $m \to \infty$.

Then $E^N|Z_{s'} - Z_{s''}|^2$ can be made arbitrarily small as $|s' - s''| \to 0$ from Itô's formula and the boundedness of f,f' and f''. Since ε is arbitrary the proof is complete. □

COROLLARY 1. *For any smooth* $\phi(t, x, y)$, *the limit as* $N \to \infty$ *of*

$$E^N \sup_{0\leq s\leq t}\left|\frac{1}{N^2}\sum_{1\leq k,\, j\leq n} \phi(s, x_k(s), x_j(s)) - \int_0^1\int_0^1 \phi(s, x, y)\rho(s, y)\rho(s, x)\,dy\,dx\right|^2 \tag{1.16}$$

is zero.

PROOF. Any smooth Φ is the uniform limit of functions $\sum_\alpha c_\alpha \phi^1_\alpha(t,x)\phi^2_\alpha(t,y)$ where $\sum_\alpha |c_\alpha|$ is bounded by a constant depending only on Φ. This translates the problem in a consequence of Theorem 1. □

The next theorem establishes the asymptotic behavior of the average local time per particle (1.10).

THEOREM 2. *For any initial profile* $\mu(dx)$:

(i) *The average interaction local time per particle* $\{A^1_N(\cdot)\}_N$ *is tight*;
(ii) $dA^1_N(t)$ *is asymptotically equal to* $\rho(t, x^N_1(t))\,dt$, *that is*, $\forall\, t \ge 0$,

$$\lim_{N\to\infty} E^N \left| A^1_N(t) - \int_0^t \rho(s, x^N_1(s))\,ds \right| = 0.$$

In Section 2 we shall prove that Theorem 2 implies the following result.

THEOREM 3. *For any initial profile the tagged particle family of processes* $\{x^N_1(\cdot)\}_N$ *is tight, that is*,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} P^N\left(\left\{ \sup_{|t-s|\le\delta} |x^N_1(t) - x^N_1(s)| \ge \varepsilon \right\}\right) = 0 \tag{1.17}$$

for any $\varepsilon > 0$.

The next theorem is our main result.

THEOREM 4. *If the initial density profile* $\mu(dx)$ *has a bounded initial density* $\rho_0(x)$, *that is*, $\mu(dx) = \rho_0(x)\,dx$ *and* (1.14) *is satisfied, then the family of measures* $P^N_{\tilde{x}} \circ (x^N_1(\cdot))^{-1}$ *has a weak limit* Q^{x_1} *as* $N \to \infty$ *and* Q^{x_1} *is the unique solution to the martingale problem given by*

$$\mathscr{L}_t = \frac{1}{2}\left(\frac{\lambda}{\lambda + \rho(t,x)}\right)\frac{d^2}{dx^2} - \left(\frac{1}{2}\partial_x \rho(t,x)\frac{2\lambda + \rho(t,x)}{(\lambda + \rho(t,x))^2}\right)\frac{d}{dx} \tag{1.18}$$

starting at $(0, x_1)$.

We shall prove Theorem 2 in Sections 4, 5 and 6 of the paper. Sections 2 and 3 will present the proof of Theorem 4 assuming that the results of Theorem 2 are true.

One can compute the diffusion coefficient in equilibrium $\sigma^2 = \lambda/(\lambda + \bar{\rho})$ (the density is constant). It is also worth mentioning that for the nonscaled version of the process the asymptotic of $x_1(t)/\sqrt{t}$ as $t \to \infty$ is $N(0, (\lambda+1)/(\lambda+n))$.

In nonequilibrium we notice the presence of a drift term, involving a gradient factor $-\partial\rho(t,x)/\partial x$. This corresponds to the tendency of the individual particle to avoid any region of high density and to seek relatively "rarefied" environments, a consequence of the repulsive character of the interaction. The drift term is zero in equilibrium ($\rho = $ constant).

The following theorem concerns the relative correlation of two tagged particle processes. Two particles with distinct labels become independent in the limit. The definition of the interaction process being symmetric, we do not lose any generality by picking two particular distinct labels, say #1 and #2. We shall set the condition that the processes $x_1^N(\cdot)$ and $x_2^N(\cdot)$ start almost surely from two given values x_1 and x_2 on the unit circle, that is, $P^N(\{x_i^N(0) = x_i\}) = 1$ for $i = 1, 2$.

THEOREM 5. *For an initial profile $\mu(dx)$ with bounded density $\rho_0(x)$, let $x_1(\cdot)$ and $x_2(\cdot)$ be the two processes such that $x_1^N(\cdot) \Rightarrow x_1(\cdot)$ and $x_2^N(\cdot) \Rightarrow x_2(\cdot)$, that is, there exists a measure $Q^{(x_1, x_2)}$ on $\Omega_2 = C([0, \infty), R^2)$ such that*

$$P^N \circ (x_1^N(\cdot), x_2^N(\cdot))^{-1} \Rightarrow Q^{(x_1, x_2)}.$$

Then $x_1(\cdot)$ and $x_2(\cdot)$ are independent with respect to $Q^{(x_1, x_2)}$, or equivalently $Q^{(x_1, x_2)} = Q^{x_1} \otimes Q^{x_2}$.

The proof is the object of Section 7.

Similar results for other models have been obtained. Although in principle they derive the distribution of the tagged particle, in practice they involve the distribution of the tagged particle averaged over the individual particles. In other words, a law of large numbers for the empirical measure (a) $(1/n)\sum_{i=1}^{n} \delta_{x_i^N(\cdot)}$ over the path space should identify its limit as Q, the law of the tagged particle process. See in this context [3] and [5]. However, we shall be able to prove exactly that $P^N \circ x_k^N(\cdot)^{-1} \Rightarrow Q$ for any fixed k, determine Q explicitly in nonequilibrium and show that the particles become independent in the limit.

2. The tightness of the tagged particle process.

We shall define a few test functions needed for the rest of the paper.

DEFINITION 7. We define ν: $R \to R$ to be the periodic function of period 1 equal to $\nu(x) = x$ on $[0, 1)$.

In the following the martingales we mention will be considered with respect to the filtration of the process $\bar{x}^N(\cdot)$, denoted by $\{\mathscr{F}_t\}_{t \geq 0}$.

PROPOSITION 1. *Let $f_1(x_1, x_2, \ldots, x_n) = x_1$ and*

$$f_2(x_1, x_2, \ldots, x_n) = \frac{1}{\lambda + \bar{\rho}} \frac{1}{N} \sum_{k \neq 1} \nu(x_k - x_1).$$

The associated martingales $\mathscr{M}_{f_1}(\cdot)$ and $\mathscr{M}_{f_2}(\cdot)$ from the differential formula (1.6) are

$$\mathscr{M}_{f_1}(t) = x_1(t) - x_1(0) - \sum_{k \neq 1} [A^{1k}(t) - A^{k1}(t)] = \beta_1(t) \tag{2.1}$$

and

$$\mathscr{M}_{f_2}(t) = f_2(x(t)) - f_2(x(0)) + (n + \lambda N)\left(\sum_{k\neq 1}[A^{1k}(t) - A^{k1}(t)]\right) \tag{2.2}$$

with the continuous martingale part equal to $-(n-1)\beta_1(t) + \sum_{k\neq 1}\beta_k(t)$ *and the jump martingale part equal to* $-\sum_{k\neq 1}[M^{1k}(t) - M^{k1}(t)]$.

A short verification of the coefficients given in the formula from above is provided.

PROOF. If $1 \notin \{i, j\}$ then the function f_2 is continuous along $\{x_i = x_j\}$ and there is no jump. The D^{ij} term is given by $\nu'(x_i - x_1) - \nu'(x_j - x_1)$ exactly where $x_i = x_j$, and hence the dA^{ij} term has no contribution.

We look at the dA^{1k} term (the dA^{k1} term is treated identically, except for an opposite sign). There is one jump of size $f_2^{k1} - f_2^{1k} = -1$ which naturally provides a contribution of $-\lambda N$. Then $D^{1k} f_2$ has a contribution of -2 for the $\nu(x_k - x_1)$ and of -1 for each of the $n-2$ remaining terms. This adds up to a total of $-(\bar{\rho}N + \lambda N) = -N(\lambda + \bar{\rho})$.

The advantage of these formulas is that we note the presence of the difference of $A_N^{1,\text{right}}(t) = (1/N)(\sum_{k\neq 1} A_N^{1k}(t))$ and $A_N^{1,\text{left}}(t) = (1/N)(\sum_{k\neq 1} A_N^{k1}(t))$ in both of them; by creating a linear combination of the two we get a martingale free of the local times expressions, which will be computable.

DEFINITION 8. We denote the transformed process

$$z_1^N(t) := x_1^N(t) + \frac{1}{\lambda + \bar{\rho}}\frac{1}{N}\sum_{k\neq 1}\nu(x_k^N(t) - x_1^N(t)). \tag{2.3}$$

PROPOSITION 2. *Here* $z_1^N(t) - z_1^N(0)$ *is a martingale with quadratic variation equal to*

$$\sigma^2(N, \lambda)t + \frac{1}{(\lambda + \bar{\rho})^2}\frac{\lambda}{N}\sum_{k\neq 1}[A^{1k}(t) + A^{k1}(t)], \tag{2.4}$$

where

$$\sigma^2(N, \lambda) = \left[1 - \frac{n-1}{N(\lambda + \bar{\rho})}\right]^2 + \frac{1}{(\lambda + \bar{\rho})^2}\frac{(n-1)}{N^2}.$$

PROOF. It is enough to compute the sum involved in $z_1^N(t) - z_1^N(0)$. It is equal to

$$\left[1 - \frac{n-1}{N(\lambda + \bar{\rho})}\right]\beta_1(t) + \frac{1}{\lambda + \bar{\rho}}\frac{1}{N}\sum_{k\neq 1}\beta_k(t) + \frac{1}{\lambda + \bar{\rho}}\frac{1}{N}\sum_{k\neq 1}[M^{k1}(t) - M^{1k}(t)];$$

to make sure its quadratic variation is given by (2.4), we check that the martingales β_k, $k = 1, \dots, n$ and M^{1j}, M^{j1}, $j = 1, \dots, n$ are mutually orthogonal and $[M^{ij}(t)]^2 - \lambda N A^{ij}(t)$ is a martingale, $\forall\, i, j = 1, \dots, n$. $\square$

PROPOSITION 3. *The sequence* $\{z_1^N(t)\}_N$ *is tight.*

PROOF. Now $z_1^N(0)$ belongs to a bounded interval, hence at time $t=0$ the conditions for tightness are met. Since $z_1^N(t)$ is a $\mathscr{F}_{t\geq 0}$-martingale, by using Doob's inequality it is enough to check its quadratic variation. There is a Brownian part, obviously tight; we need an estimate for the average local time of collision corresponding to the tagged particle, that is, $A_N^1(t)$. But this is a consequence of the limit

$$\lim_{N\to\infty} E^N \left| A_N^1(t) - \int_0^t \rho(s, x_1^N(s))\, ds \right| = 0$$

stated in Theorem 2.

We introduce the notation $P_{|x}^N := P_{x_1(0)=x}^N$ for the probability measure associated with the condition $x_1(0) = x$.

The theorem is equivalent to showing

$$\lim_{\delta\to 0} \limsup_{N\to\infty} P_{|x}^N \left(\left\{ \sup_{0\leq t\leq \delta} |x_1^N(t) - x| \geq \varepsilon \right\} \right) = 0 \tag{2.5}$$

for any $\varepsilon > 0$. With this in mind we shall consider the stopping time

$$\tau_x := \inf\left\{t\colon |x_1^N(t) - x| \geq \varepsilon\right\} \wedge t_0 \tag{2.6}$$

and using this notation, we want to prove $\lim_{\delta\to 0} \limsup_{N\to\infty} P_{|x}^N(\{\tau_x \leq \delta\}) = 0$. For $z_1^N(t)$ defined in (2.3), we define $u_N := z_1^N(t) - x_1^N(t)$ and $\widehat{x_1^N} := x_1^N(\tau) - x$, $\widehat{u_N} := u_N(\tau) - u_N(0)$ and $\widehat{z_1^N} := z_1^N(\tau) - z_1^N(0)$.

The set $\{\tau_x \leq \delta\}$ can be written as

$$\{\tau_x \leq \delta\} = \left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon\right\} \cup \left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = -\varepsilon\right\}$$

and we shall concentrate on the first set, the second one being treated identically. For the moment let us just pick an $\alpha \in (0, 1)$. Then $\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon\}$ will be equal to

$$\left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon \text{ and } |\widehat{z_1^N}| \leq \alpha\varepsilon\right\} \cup \left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon \text{ and } |\widehat{z_1^N}| > \alpha\varepsilon\right\}.$$

Since z_1^N is tight, the sequence of limits applied to the probability of the last set is zero. We have to work out the limits for the other set

$$\left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon \text{ and } |\widehat{z_1^N}| \leq \alpha\varepsilon\right\},$$

included in

$$\left\{\tau_x \leq \delta \text{ and } \widehat{x_1^N} = \varepsilon \text{ and } \widehat{u_N} \leq -(1-\alpha)\varepsilon\right\}. \tag{2.7}$$

To simplify the facts we also denote $\nu_\varepsilon(x) := \nu(x - \varepsilon)$.

The function $(\lambda+\bar\rho)\cdot\widehat{u_N}$ is equal to

$$
\begin{aligned}
&\frac{1}{N}\sum_{k\neq 1}\nu(x_k(\tau)-x_1(\tau))-\frac{1}{N}\sum_{k\neq 1}\nu(x_k(0)-x_1(0))\\
&\qquad=\frac{1}{N}\sum_{k\neq 1}\nu(x_k(\tau)-[x_1(\tau)-x_1(0)]-x_1(0))\\
&\qquad\quad-\frac{1}{N}\sum_{k\neq 1}\nu(x_k(0)-x_1(0)),
\end{aligned}\tag{2.8}
$$

which in turn is equal to

$$
\frac{1}{N}\sum_{k\neq 1}\nu_\varepsilon(x_k(\tau)-x)-\frac{1}{N}\sum_{k\neq 1}\nu(x_k(0)-x).\tag{2.9}
$$

We shall prove at the end of this subsection the following proposition.

PROPOSITION 4. *There is a smooth* [*at least of class* $C^2(T^1)$] *function* ϕ *such that* $\nu_\varepsilon(x)\ge\phi(x)-\varepsilon$ *and* $-\nu(x)\ge-\phi(x)$ *for any* x *on the unit circle.*

Then

$$
(\lambda+\bar\rho)\widehat{u_N}\ge\left[\frac{1}{N}\sum_{k\neq 1}\phi(x_k(\tau)-x)-\frac{1}{N}\sum_{k\neq 1}\phi(x_k(0)-x)\right]-\varepsilon\frac{(n-1)}{N},
$$

implying that

$$
\widehat{u_N}\ge-\varepsilon\frac{\bar\rho-1/N}{(\lambda+\bar\rho)}+\frac{1}{(\lambda+\bar\rho)}\left[\frac{1}{N}\sum_{k\neq 1}\phi(x_k(\tau)-x)-\frac{1}{N}\sum_{k\neq 1}\phi(x_k(0)-x)\right].
$$

We remember the set

$$
\{\tau_x\le\delta\text{ and }\widehat{x_1^N}=\varepsilon\text{ and }\widehat{u_N}\le-(1-\alpha)\varepsilon\}
$$

from (2.7); the set we are interested in is included in

$$
\begin{aligned}
\Bigg\{&\tau_x\le\delta\text{ and }\widehat{x_1^N}=\varepsilon\text{ and }\frac{1}{(\lambda+\bar\rho)}\\
&\times\left[\frac{1}{N}\sum_{k\neq 1}\phi(x_k(\tau)-x)-\frac{1}{N}\sum_{k\neq 1}\phi(x_k(0)-x)\right]\\
&\qquad\qquad\le-\varepsilon\left[(1-\alpha)-\frac{\bar\rho-1/N}{(\lambda+\bar\rho)}\right]\Bigg\},
\end{aligned}
$$

which is simply included in

$$
\begin{aligned}
\Bigg\{\tau_x\le\delta\text{ and }\frac{1}{(\lambda+\bar\rho)}\Bigg[&\frac{1}{N}\sum_{k\neq 1}\phi(x_k(\tau)-x)\\
&-\frac{1}{N}\sum_{k\neq 1}\phi(x_k(0)-x)\Bigg]\le-\varepsilon l\Bigg\}
\end{aligned}\tag{2.10}
$$

for some $l > 0$. This is obtained from the fact that α is arbitrary in $(0, 1)$, that is, we can take $\alpha < \lambda/(\lambda + \bar{\rho})$.

Chebyshev's inequality implies that the limit (1.17) will be zero if we can prove that for ϕ smooth,

$$\lim_{t\to 0} \limsup_{N\to\infty} E^{P^N_{|x}} \left| \frac{1}{N} \sum_{k\neq 1} \phi(x_k(t) - x) - \frac{1}{N} \sum_{k\neq 1} \phi(x_k(0) - x) \right| = 0. \tag{2.11}$$

We shall prove a more general statement. The limit

$$\lim_{t\to 0} \limsup_{N\to\infty} E^{P^N} \left[\sup_q \left| \frac{1}{N} \sum_{k\neq 1} \phi(x_k(t) - q) - \frac{1}{N} \sum_{k\neq 1} \phi(x_k(0) - q) \right| \right] = 0. \tag{2.12}$$

For each separate average $(1/N)\sum_{k\neq 1} \phi(x_k(t) - q)$ and

$$(1/N) \sum_{k\neq 1} \phi(x_k(0) - q),$$

we can apply Theorem 1 and write

$$\lim_{N\to\infty} E^{P^N} \sup_{t\in[0, t_0]} \left| \frac{1}{N} \sum_{k\neq 1} \phi(x_k(t) - q) - \int_0^1 \phi(y)\rho(t, y + q)\, dy \right| = 0. \tag{2.13}$$

The passage to the uniform statement in q is granted as an easy application of Lemma 4 (6.3)–(6.5). The proof will be done if we show

$$\lim_{t\to 0} \left| \int_0^1 \phi(y - q)\rho(t, y)\, dy - \int_0^1 \phi(y - q)\mu(0, dy) \right| = 0. \tag{2.14}$$

We note that $\mu(dx)$ is arbitrary but ϕ is smooth. It is clear that the solution to the heat equation starting at $\phi(x)$ at time $t = 0$ satisfies the limit. We actually have the convolution of that solution (smooth) with the measure $\mu(dx)$ of finite total mass. This proves the limit.

To finish the proof of tightness we have to prove Proposition 4.

PROOF OF PROPOSITION 4.

$$\nu(x) = 1 + x, \text{ if } x \in \left[-\tfrac{1}{2}, 0\right) \text{ and } \nu(x) = x \text{ if } x \in \left[0, \tfrac{1}{2}\right]$$

and

$$\nu_\varepsilon(x) + \varepsilon = 1 + x, \text{ if } x \in \left[-\tfrac{1}{2}, \varepsilon\right) \text{ and } \nu_\varepsilon(x) + \varepsilon = x \text{ if } x \in \left[\varepsilon, \tfrac{1}{2}\right],$$

that is, one can take

$$\phi(x) = \begin{cases} \nu(x), & x \in \left[-\tfrac{1}{2}, 0\right) \cup \left[\varepsilon, \tfrac{1}{2}\right], \\ \text{a smooth function } \theta(x), & \text{if } x \in [0, \varepsilon). \end{cases} \tag{2.15}$$

It is clear that $\theta(x) - x$ can be taken to be a convolution of $\mathbf{1}_{[0, \varepsilon)}(x)$ with a smooth positive approximation of the delta function at $x = \varepsilon/2$. □

3. The asymptotic tagged particle process.

3.1. *An intermediate process.* In this section we assume that the initial profile has a bounded density $\rho_0(x)$.

DEFINITION 9. Let $F(t,x) = x + 1/(\lambda+\bar\rho)\int_0^1 \nu(y-x)\rho(t,y)\,dy$ for $t > 0$ and $F(0,x) = x + 1/(\lambda+\bar\rho)\int_0^1 \nu(y-x)\rho_0(y)\,dy$ if $t = 0$.

PROPOSITION 5. *Suppose $\rho_0(x)$ is bounded. Then the function $F(t,x)$ is a smooth function* [*of class $C^\infty((0,\infty), R)$*] *and for any fixed $t > 0$, $x \to F(t,x)$ is a strictly nondecreasing function with*

$$0 < \frac{\lambda}{\lambda+\bar\rho} \le \partial_x F(t,x) = \frac{\lambda+\rho(t,x)}{\lambda+\bar\rho} \le C' < \infty,$$

where $C' = \lambda + \|\rho_0\|/(\lambda+\bar\rho)$ and $\|\rho_0\| = \sup_x \rho_0(x)$. Moreover, for a given x, $\lim_{t\to 0} F(t,x) = F(0,x)$ and $x \to F(0,x)$ is also strictly nondecreasing.

PROOF. It is clear that $\rho(t,x) = \int_0^1 \rho_0(x-y)p(t,y)\,dy$ where $p(t,x)$ is the fundamental solution to the heat equation $\partial_t\rho = \frac{1}{2}\partial_{xx}\rho$ and as such the smoothness is established. The contents of this proposition is the computation of $\partial_x F(t,x)$,

$$\partial_x F(t,x) = 1 + \frac{1}{\lambda+\bar\rho}\partial_x\left(\int_0^1 \nu(y-x)\rho(t,y)\,dy\right).$$

We look at the derivative of the integral $\partial_x(\int_0^1 \nu(y-x)\rho(t,y)\,dy)$ equal to

$$\partial_x \int_0^1 \nu(y)\rho(t,y+x)\,dy = \int_0^1 \nu(y)\partial_x\rho(t,y+x)\,dy$$

(since the functions have period 1), further equal to

$$\nu(y)\rho(t,y+x)|_0^1 - \int_0^1 \rho(t,y+x)\,dy = \rho(t,1+x) - \bar\rho = \rho(t,x) - \bar\rho.$$

We want to prove $\lim_{t\to 0} F(t,x) = F(0,x)$. For this we have to show that

$$\int_0^1 \nu(y-x)\rho(t,x)\,dy \to \int_0^1 \nu(y-x)\rho_0(x)\,dy$$

as $t \to 0$.

$$\int_0^1 \nu(y-x)\rho(t,x)\,dy = \int_0^1\int_0^1 \nu(y-x)p(t,z)\rho_0(y-z)\,dz\,dy,$$

where $p(t,z)$ is the heat kernel for the unit circle. We notice that

$$z \to R(z,x) := \int_0^1 \nu(y-x)\rho_0(y-z)\,dy$$

is a continuous function. This is because $\nu(\cdot)$ is continuous except at one point and bounded while ρ_0 is also bounded, hence their convolution is continuous by dominated convergence. For an arbitrary $\varepsilon > 0$ we denote

$$D_\varepsilon = \left\{ z \colon |R(z, x) - R(0, x)| \le \frac{\varepsilon}{2} \right\}$$

and so

$$\begin{aligned}
&\left| \int_0^1 \nu(y - x)\rho(t, y)\, dy - \int_0^1 \nu(y - x)\rho_0(y)\, dy \right| \\
&\quad \le \int_{D_\varepsilon} p(t, z)|R(z, x) - R(0, x)|\, dz \\
&\qquad + \int_{D_\varepsilon^c} p(t, z)|R(z, x) - R(0, x)|\, dz \\
&\quad \le \frac{\varepsilon}{2} + 2\|\rho_0\| \int_{D_\varepsilon^c} p(t, z)\, dz.
\end{aligned}$$

The lim sup as $t \to 0$ is less than $\varepsilon/2$ for an arbitrary ε. □

DEFINITION 10. For any $t \ge 0$ let $G(t, \cdot) := (F(t, \cdot))^{-1}$.

REMARK. F has an inverse when $t = 0$ because we have shown in Proposition 5 that $F(0, \cdot)$ is also strictly nondecreasing.

PROPOSITION 6. *$G(t, y)$ has the same properties as $F(t, x)$, that is, there are two constants c' and c'' such that $0 < c' \le \partial_y G(t, y) \le c'' < \infty$ for any $t \ge 0$ and for a given y, $\lim_{t\to 0} G(t, y) = G(0, y)$.*

The proof is immediate from Proposition 5.

The next theorem relates the limits of the processes $x_1^N(\cdot)$ and $z_1^N(\cdot)$.

THEOREM 6. *For any limit process $\{x_1(\cdot)\}_{t\ge 0}$ of the family of processes $\{x_1^N(\cdot)\}_{N>0}$ there is a limit point $\{z_1(\cdot)\}_{t\ge 0}$ of the family of processes $\{z_1^N(\cdot)\}_{N>0}$ such that if $Q^{(x_1, z_1)}$ is the limit point of $\{P^N \circ (x_1^N(\cdot), z_1^N(\cdot))^{-1}\}_N$ corresponding to their joint distribution, then*:

(i) *$(z_1(t) - z_1(0), \mathscr{F}_t)$ is a continuous martingale with respect to $Q^{(x_1, z_1)}$*;

(ii)
$$\left([z_1(t) - z_1(0)]^2 - \int_0^t \frac{\lambda(\lambda + \rho(s, x_1(s)))}{(\lambda + \bar{\rho})^2}\, ds, \mathscr{F}_t \right)$$

is also a $Q^{(x_1, z_1)}$-martingale.

PROOF. We already know that $x_1^N(\cdot)$ and $z_1^N(\cdot)$ are tight. We extract a convergent subsequence for $x_1^N(\cdot)$ and from that subsequence another subsequence such that $z_1^N(\cdot)$ also becomes convergent. The whole result follows from Theorem 2 in addition to the next proposition.

PROPOSITION 7. *Let $T > 0$ be a positive number. The mappings U_1 and U_2, $U_i\colon \Omega_T \to \Omega_T$ with $\Omega_T := C([0, T], R)$ for $i = 1, 2$ defined by $U_1\omega(\cdot) = \int_0^{\bullet} \rho(s, \omega(s))\, ds$ and $U_2\eta(\cdot) = \int_0^{\bullet} \rho(s, G(s, \eta(s)))\, ds$ are bounded continuous functionals.*

PROOF.

$$\sup_{t\in[0,T]} U_1\omega(t) \le \text{const} \sup_{t\in[0,T]} \int_0^t s^{-1/2}\, ds \le \text{const}\sqrt{T}$$

shows $U_1(\cdot)$ is bounded. For the continuity we want to prove that given a sequence of paths ω_m such that there is a path ω with the property

$$\lim_{m\to\infty} \sup_{t\in[0,T]} |\omega_m(t) - \omega(t)| = 0,$$

we can conclude that

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \rho(s, \omega_m(s))\, ds - \int_0^t \rho(s, \omega(s))\, ds \right| = 0.$$

This is bounded above by $\lim_{m\to\infty} \sup_{t\in[0,T]} \int_0^t |\rho(s, \omega_m(s)) - \rho(s, \omega(s))|\, ds$, which is less than

$$\lim_{m\to\infty} \int_0^T \big|\rho(s, \omega_m(s)) - \rho(s, \omega(s))\big|\, ds.$$

This limit is zero by dominated convergence since for each s except $s = 0$ the limit is zero (i.e., pointwise). Moreover, $\rho(t, x)$ is of order of $\sqrt{1/t}$ uniformly in x. The integrand is bounded by an integrable function.

The proof for U_2 is analogue since $G(\cdot, \cdot)$ is smooth for $t > 0$. □

3.2. *The process $y_1^N(\cdot)$.*

DEFINITION 11. Let

$$y_1^N(t) = F(t, x_1^N(t)) = x_1^N(t) + \frac{1}{\lambda + \bar\rho} \int_0^1 \nu(y - x_1^N(t))\rho(t, y)\, dy$$

for any $t \ge 0$.

LEMMA 1. *If the initial profile $\rho_0(\cdot)$ is bounded, then for any $t > 0$,*

$$\lim_{N\to\infty} E^N \sup_{0\le s\le t} |y_1^N(s) - z_1^N(s)| = 0.$$

PROOF. What really matters in the difference $|y_1^N(s) - z_1^N(s)|$ is the absolute value of

$$B^\nu(s) := \frac{1}{N} \sum_{k\ne 1} \nu(x_k(s) - x_1^N(s)) - \int_0^1 \nu(y - x_1^N(s))\rho(s, y)\, dy.$$

Let $\delta > 0$. We shall consider a smooth function $g(x)$ approximating $\nu(x)$ pointwise; we mean that $\nu \equiv g$ everywhere except a neighborhood $(-\frac{\delta}{2}, +\frac{\delta}{2})$ of the origin. Now let $\phi_\delta(x)$ be a smooth function bounded by 1 with compact support $\mathrm{supp}(\phi)$ included in $[-\frac{\delta}{2}, +\frac{\delta}{2}]$ approximating $|\phi - g|$ in the L^1 norm with respect to the Lebesgue measure as $\delta \to 0$. This is the same type of estimate we are going to use in Section 6 to show the validity of the hydrodynamic limits for a nonsmooth function with a jump at $x = 0$. Hence $|B^\nu(s)| \le |B^\nu(s) - B^g(s)| + |B^g(s)|$ and we know that the expected value of the second term tends to zero uniformly in s because g is smooth. We only have to take care of the first term which is less than or equal to $|B^{\nu-g}(s)|$, which in turn is bounded above by

$$\left|\sum_{k\neq 1}(\nu - g)(x_k(s) - x_1^N(s))\right| + \left|\int_0^1 (\nu - g)(y - x_1^N(s))\rho(s, y)\,dy\right|$$
$$\le |B^{\phi_\delta}(s)| + 2\int_0^1 \phi_\delta(y - x_1^N(s))\rho(s, y)\,dy.$$

When we take the expected value $E^N[\ ]$, the first term above tends to 0 uniformly in s as $N \to \infty$ because ϕ_δ is smooth (as in Lemma 5), while the second needs a change of variable (we do not have to forget that ρ and ν are periodic of period 1) to bring down our proof to

$$\lim_{N\to\infty} E^N \sup_{0\le s\le t}\left[\int_0^1 \phi_\delta(y)\rho(s, y + x_1^N(s))\,dy\right] = O(\delta).$$

It is essential that ρ_0 be bounded. Let the bound be C. Then

$$\int_0^1 \phi_\delta(y)\rho(s, y + x_1^N(s)) \le C \cdot \delta. \qquad \square$$

COROLLARY 2. *$\{y_1^N(\cdot)\}_N$ is tight.*

REMARK. The proof of the tightness of $x_1^N(\cdot)$ as a direct consequence of the tightness of $z_1^N(\cdot)$ (given in Section 2) is rather complicated compared to the observation that if $y_1^N(\cdot)$ is tight and the relation between $x_1(\cdot)$ and $F(\cdot, x_1(\cdot))$ is differentiable with a bounded gradient (hence one-to-one), then $x_1^N(\cdot)$ is also tight. The problem is that we cannot assume the one-to-one differential correspondence if the initial profile is not bounded. We shall explain these considerations in the following.

3.3. *The asymptotic limit for $y_1^N(\cdot)$.*

THEOREM 7. *The process $y_1^N(\cdot)$ converges weakly to the diffusion P_y on Ω with generator*

$$\mathscr{L}^y = \frac{1}{2}\frac{\lambda(\lambda + \rho(t, G(t, y)))}{(\lambda + \bar{\rho})^2}\frac{d^2}{dy^2} \tag{3.1}$$

starting at $y_1 = F(0, x_1)$.

PROOF. Suppose $\{y(\cdot)\}$ is a limit process for the tight sequence $\{y_1^N(\cdot)\}$. We also know that $\{z_1^N(\cdot)\}$ and $\{x_1^N(\cdot)\}$ are tight. From the subsequence for which $y_1^N(\cdot) \Rightarrow y(\cdot)$ we extract convergent subsequences of the other two families $\{z_1^N(\cdot)\}_N$ and $\{x_1^N(\cdot)\}_N$ such that $z_1^N(\cdot) \Rightarrow z_1(\cdot)$ and $x_1^N(\cdot) \Rightarrow x_1(\cdot)$. Moreover, Lemma 1 tells us that y_1^N and z_1^N have the same limit as $N \to \infty$. Of course, $y_1^N(t) = F(t, x_1^N(t))$ and $x_1^N(t) = G(t, y_1^N(t))$.

By construction $y_1^N(0) = F(0, x_1^N)(0) = F(0, x_1) = y_1$ almost surely for all N so $P^y(\{y(0) = y\}) = 1$.

For any limit point $x_1(\cdot)$ of the sequence $\{x_1(\cdot)\}_N$ one naturally expects from Proposition 2 that $z_1(t) - z_1(0) = y(t) - y(0)$ be a continuous martingale with quadratic variation

$$\int_0^t \frac{\lambda(\lambda + \rho(t, x_1(s)))}{(\lambda + \bar{\rho})^2}\, ds.$$

It is enough to substitute $x_1(s)$ by $G(s, y(s))$ to deduce that $y(t) - y(0)$ is a continuous martingale with the above quadratic variation. The uniqueness is established if we note the uniform ellipticity of the generator, that is,

$$0 < C' \leq \frac{\lambda(\lambda + \rho(t, G(t, y)))}{(\lambda + \bar{\rho})^2}. \qquad \square$$

Roughly speaking, we have derived the asymptotic law $y_1(\cdot)$ of the processes $\{y_1^N(\cdot)\}_{N>0}$, obtained from the tagged particle process $x_1^N(\cdot)$ through the mapping $F(t, x)$. We need a way to make sure that $x_1(\cdot)$ (a limit point of the tight family of the tagged particle processes) is unique and we can recuperate it as soon as we know $y_1(\cdot)$. For this purpose we need a few more results.

LEMMA 2. *We assume that the martingale problem is well posed for the pair* $(a(t, y), b(t, y))$, *that is, for any* $(t, y) \in [0, \infty) \times R$ *there is a measure* $P^{(s, y)}$ *on the path space* $\Omega = C([0, \infty), R)$ *such that if* $y(\cdot)$ *denotes an element of* Ω *and*

$$\mathscr{L}_t := \frac{1}{2} a(t, y) \frac{d^2}{dy^2} + b(t, y) \frac{d}{dy},$$

then:

(i) $P^{(s,y)}(\{y(s) = y\}) = 1$;
(ii) $\forall\, f(\cdot, \cdot) \in C_0^\infty([0, \infty), R)$, *the expression*

$$f(t, y(t)) - f(s, y(s)) - \int_s^t \left(\partial_u + \mathscr{L}_u\right) f(u, y(u))\, du$$

is a $(P^{(s,y)}, \mathscr{F}_t)$*-martingale, where* $\mathscr{F}_t = \sigma(\omega(s)\colon 0 \leq s \leq t)$.

Suppose $\Phi\colon [0, \infty) \times R \to R$ *is a* C^2 *mapping such that*:

(i) $\Phi(t, x) = y$;
(ii) $0 < c_1 \leq \partial_x \Phi(t, x) \leq c_2 < \infty$ *for any* (t, x).

Then $x \to \Phi(t, x)$ *has an inverse* $y \to \Psi(t, y)$ *for any fixed* $t \geq 0$ *and if we define a mapping on the path space* $\Xi\colon \Omega \to \Omega$ *by* $[\Xi(y)](t) := \Psi(t, y(t)) = x(t)$, *then* $\hat{P}^{(s,x)} := P^{(s,\Phi(s,x))} \circ \Xi^{-1}$ *solves the martingale problem* $(\hat{a}(t,x), \hat{b}(t,x))$ *with*

$$\hat{a}(t, x) = [a(\partial_y \Psi)^2] \circ (t, \Phi(t, x)) \tag{3.2}$$

and

$$\hat{b}(t, x) := \left[(\partial_t \Psi) + \tfrac{1}{2} a(\partial_{yy}\Psi) + b(\partial_y \Psi)\right] \circ (t, \Phi(t, x)). \tag{3.3}$$

In other words, if $\{y(\cdot)\}_{t\geq 0}$ is a diffusion with coefficients (a, b), then $x(t) = \Psi(t, y(t))$ is a diffusion with coefficients $(\hat{a}, \hat{b})$.

PROOF. As usual, $\mathscr{F}_t^y = \mathscr{B}(\bigcup_{t'\leq t} y(t'))$ and since Φ is measurable, invertible and $(t, x(t))$ depends exclusively on $(t, y(t))$ for any $t \geq 0$, the filtrations $\mathscr{F}_t^y$ and $\mathscr{F}_t^x$ will be the same, equal to $\mathscr{B}(\bigcup_{t'\leq t} x(t'))$. It will be enough to prove that

$$g(x(t)) - g(x(s)) - \int_s^t \hat{\mathscr{L}}_u g(x(u))\, du$$

is a $\hat{P}^{(s,x)}$-martingale $\forall\ (s, x)$ and all $g \in C_0^\infty(R)$.

Let $s \leq t_0 \leq t$. We want to show that

$$E^{\hat{P}^{(s,x)}}\left[g(x(t)) - g(x(t_0)) - \int_{t_0}^t \hat{\mathscr{L}}_u g(x(u))\, du \,\middle|\, \mathscr{F}_{t_0}^x\right] = 0.$$

Of course we defined $\hat{P}^{(s,x)} \equiv P^{(s,y)}$ for $y = \Phi(s, x)$ and so we rewrite the expression from above as

$$E^{P^{(s,y)}}\left[f(t, y(t)) - f(t_0, y(t_0)) - \int_{t_0}^t [\partial_u + \mathscr{L}_u] f(u, y(u))\, du \,\middle|\, \mathscr{F}_{t_0}^y\right]$$

for the new function $f(t, y) = g \circ \Psi(t, y)$; we only have to make sure the coefficients match with the ones prescribed by the lemma.

A little computation shows that

$$\begin{aligned}
\partial_u(g \circ \Psi) + \frac{1}{2} a(u, y(u)) \frac{d}{dy^2}(g \circ \Psi) + b(u, y(u)) \frac{d}{dy}(g \circ \Psi) &= \hat{\mathscr{L}}_u^x g(x(u)) \\
= \frac{1}{2} a(u, y(u))(g'' \circ \Psi)(\partial_y \Psi)^2& \\
+ (g' \circ \Psi)\left[\frac{1}{2} a(u, y(u))(\partial_{yy}\Psi) + (\partial_u \Psi) + b(\partial_y \Psi)\right]&.
\end{aligned}$$

□

LEMMA 3. *For* $T > 0$ *and as long as* $\rho_0(x)$ *is integrable, the function* $\Xi\colon \Omega_T \to \Omega_T$ *defined by* $[\Xi(y)](t) := \Psi(t, y(t))$ *and its inverse* Ξ^{-1} *are well defined* (*map continuous paths into continuous paths*) *and continuous.*

REMARK. The conclusion naturally holds for ρ_0 bounded.

PROOF OF LEMMA 3. We want to show that if $\omega \in \Omega_T = C[0, T]$,

$$t \to \omega(t) + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y)\rho(t, \omega(t) + y)\, dy \tag{3.4}$$

is a continuous path and that

$$\omega(\cdot) \to \omega(\cdot) + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y)\rho(t, \omega(\cdot) + y)\, dy \tag{3.5}$$

is a continuous functional.

To prove these statements we look at

$$\begin{aligned}\int_0^1 \nu(y)\rho(t, \omega(t) + y)\, dy &= \int_0^1 \nu(y) \int_0^1 \rho_0(y - z)p(t, \omega(t) + z)\, dz\, dy\\ &= \int_0^1 \int_0^1 \nu(y)\rho_0(y - z)\, dy\; p(t, \omega(t) + z)\, dz;\end{aligned}$$

we denote $R(z) := \int_0^1 \nu(y)\rho_0(y - z)\, dy$ and so our integral is

$$\int_0^1 R(z)p(t, \omega(t) + z)\, dz = \int_0^1 R(z - \omega(t))p(t, z)\, dz.$$

By a change of variable and writing down the solution to the heat equation on the real line applied to the lifted function $R(z)$-periodic on R, we get

$$= \int_R R(\sqrt{t}w - \omega(t)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw.$$

At this point we notice that $R(z)$ is continuous since it is the convolution of ν, a bounded function, with only one discontinuity and ρ_0, an integrable function (this is also implied by dominated convergence). Clearly R is periodic, so by looking at it as a continuous function on $[0, 1]$, we see it is uniformly continuous. Finally, if $\omega_m(t) \to \omega(t)$ in the supremum norm as $m \to \infty$, then the arguments $\|[\sqrt{t}w - \omega_m(t)] - [\sqrt{t}w - \omega(t)]\|$ also tend to zero; once again we take advantage of dominated convergence theorem and conclude that Ξ is well defined and continuous.

The inverse is continuous because Ω_T is a compact set in the uniform convergence topology. □

PROOF OF THEOREM 4. We assume that Theorem 2 is true. Its proof is given in Sections 4, 5 and 6. We have seen in Section 2 that Theorem 3 is a consequence of Theorem 2.

Hence $\{x_1^N(\cdot)\}_{t\geq 0}$ is tight. The one-to-one correspondence between $x = \Psi(t, y)$ and $y = \Phi(t, x)$ guarantees that we can safely define a measure $\tilde{P}^{(0, x)}$ for any measure $P^{(0, y)}$ on the path space $\Omega = C([0, \infty), R)$ by inversion,

$$\tilde{P}^{(0, x)} := P^{(0, \Phi(s, x))} \circ \Xi^{-1}, \tag{3.6}$$

where $\Xi\colon \Omega \to \Omega$ is defined as above, $[\Xi(y)] = \Psi(t, y(t))$. Naturally, the mapping Ξ is continuous and one-to-one.

This shows the process $x_1(\cdot)$ is well defined as a measure on Ω as soon as $y(\cdot)$ is well defined. As it was proved, $y(\cdot)$ is a diffusion with bounded coefficients and $a(t, y) \geq c > 0$. What is not clear is the explicit form of the limiting process $x_1(\cdot)$; if the coefficients were smooth, the problems would vanish. However, even when $\rho_0(x)$ is bounded, one can show that $x_1(\cdot)$ solves the martingale problem for $\hat{\mathscr{L}}$. We already know by construction that $P_{x_1} = P_{y_1} \circ \Xi^{-1}$, when viewed as measures on Ω. The plan is to show that (i), P_{x_1} solves the martingale problem for $(\hat{a}, \hat{b})$ (3.2), (3.3) and (ii), P_{x_1} is the unique solution.

(i) *The existence.* For $f \in C_0^\infty(R)$ we have the expression (yet to be proved to be a martingale)

$$\mathscr{M}_f(t) = f(x(t)) - f(x(0)) - \int_0^t \hat{\mathscr{L}}_u f(x(u))\, du$$

and for $s > 0$ we know (Lemma 2) that $\mathscr{M}_f(t) - \mathscr{M}_f(s)$ is a martingale with respect to P_{x_1} and $\{\mathscr{F}_u^x\}_{u \geq 0}$. We only want to check that for any $t \geq 0$, $E^{P_{x_1}}[\mathscr{M}_f(t)] = 0$. Let us pick a $\delta < t$ and naturally the problem is reduced to proving the limit $\lim_{\delta\to 0} E^{P_{x_1}}[\mathscr{M}_f(\delta)] = 0$. Now $\lim_{\delta\to 0} E^{P_{x_1}}|f(x(\delta)) - f(x(0))| = 0$ since P_{x_1} is concentrated on the set of continuous paths (from tightness), $f \in C_0^\infty(R)$ and Chebyshev's inequality.
The actual form of the generator $\hat{\mathscr{L}_u}$ is

$$\hat{\mathscr{L}_u} = \frac{1}{2}\left(\frac{\lambda}{\lambda + \rho(t, x)}\right)\frac{d^2}{dx^2} - \left(\frac{1}{2}\partial_x \rho(t, x)\frac{2\lambda + \rho(t, x)}{(\lambda + \rho(t, x))^2}\right)\frac{d}{dx}$$

and so the only part causing some trouble as $\delta \to 0$ is $\partial_x \rho(t, x)$; all the others are bounded.

We remember that $p(t, x)$ is the heat kernel on the unit circle $\rho = \rho_0 * p$. Let us write down the solution to the heat equation on the line, assuming ρ_0 is the periodic extension on the line of the function on the circle (we keep the same notation for simplicity). Then

$$\begin{aligned}
\int_0^\delta |\partial_x \rho(u, x(u))|\, du &\leq \int_0^\delta \int_0^1 \rho_0(z) |\partial_x p(u, x(u) - z)|\, dz\, du \\
&= \int_0^\delta \int_R \rho_0(z - x(u)) \left| \frac{z}{\sqrt{2\pi u^3}} \exp\left(-\frac{z^2}{2u}\right)\right| dz\, du \\
&\leq 2\|\rho_0\| \int_0^\delta \int_0^\infty \partial_z p(u, z)\, dz\, du \\
&\leq 2\|\rho_0\| \int_0^\delta \frac{1}{\sqrt{2\pi u}}\, du \leq \text{const}\sqrt{\delta}.
\end{aligned}$$

This proves that $E^{P_{x_1}}[\mathscr{M}_f(\delta)]$ goes to zero as $\delta \to 0$; since δ was arbitrary we conclude that $E^{P_{x_1}}[\mathscr{M}_f(t)] = 0$.

(ii) *The uniqueness.* As soon as P_{x_1} is the solution to the martingale problem starting at $(0, x_1)$, the uniqueness of the solution for any (s, x) with $s > 0$ (a consequence of the smoothness of the coefficients for $s > 0$) implies that for any bounded continuous f,

$$E^{P_{x_1}}\left[f(x(t))\middle|\Sigma^s_{x_1}\right] = E^{P_{(s,x_1)}}[f(x(t))] \tag{3.7}$$

where $\Sigma^s_{x_1} = \sigma(\bigcup_{0\le u\le s}\{\omega(u)\colon \omega(0) = x_1\})$.

Let us suppose that Q is another solution to the martingale problem. Let $0 < s < t$ and

$$E^Q[f(x(t))] = E^Q\left[E^Q\left[f(x(t))\middle|\Sigma^s_{x_1}\right]\right] = E^Q\left[E^{P_{(s,x_1)}}[f(x(t))]\right]$$

and this is true for an arbitrary $s < t$. We plan to apply the dominated convergence theorem to this last expression. From the one-to-one correspondence between the measures $\hat{P}_{(s,x)}$ and $P_{(s,y)}$ through Ξ, one can see that since $\lim_{s\to 0} P_{(s,y_1)} = P_{y_1}$ (the tight sequence $P_{(s,y_1)}$ has only one limit point) the same has to be true for $\hat{P}_{(s,x)}$, that is, $\lim_{s\to 0} P_{(s,x_1)} = P_{x_1}$ for $y_1 = \Phi(0, x_1)$.

This implies that for any bounded continuous f,

$$\lim_{s\to 0} E^{P_{(s,x_1)}}[f(x(t))] = E^{P_{x_1}}[f(x(t))];$$

the expected values in this limit are bounded because f is bounded and the limit itself is nonrandom, implying by the dominated convergence theorem (as announced) that both Q and P_{x_1} are solutions to the martingale problem and $E^{P_{x_1}}[f(x(t))] = E^Q[f(x(t))]$ for all f and $t > 0$. We only have to notice that by definition $P_{x_1}(\{x(0) = x_1\}) = 1$ and similarly $Q(\{x(0) = x_1\}) = 1$. This concludes the proof of the theorem. □

4. The preliminary estimates.

4.1. *A general test function.* A general test function is needed in several proofs in this work. For any $l > 1$ there exists a smooth positive function $\phi(x) = \phi_l(x)$ with compact support in the interval $[0, 1]$, with integral normlaized to one and such that $\sup_x \phi(x) = l$.

We now choose $l = \frac{3}{2}$ and denote $\phi_l := \phi$. For a given $1 > \varepsilon > 0$ and a given $c \ge 1$, we define $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon} - \frac{1}{2})$ and $(\gamma^c_\varepsilon)''(x) := -\phi_\varepsilon(x) - \phi_\varepsilon(c - x)$. Defined this way, the function

$$\gamma^c_\varepsilon(x) = x \text{ on } \left[0, \frac{\varepsilon}{2}\right], \quad c - x \text{ on } \left[c - \frac{\varepsilon}{2}, c\right] \quad \text{and} \quad \frac{3}{2}\varepsilon \text{ on } \left[\frac{3}{2}\varepsilon, c - \frac{3}{2}\varepsilon\right] \tag{4.1}$$

will be smooth and concave.

REMARK. $\sup_x \phi_\varepsilon(x) = 3/2\varepsilon$ and $\operatorname{supp}(\phi'_\varepsilon) \subset [0, 3\varepsilon/2] \cup [c - 3\varepsilon/2, c]$.

For simplification we shall give the following notations.

DEFINITION 12.

$$\alpha_\varepsilon(\cdot) := \gamma_\varepsilon^1(\cdot) \quad \text{for the case } c = 1 \text{ and}$$
$$\gamma_\varepsilon(\cdot) := \gamma_\varepsilon^{\lambda+\bar\rho}(\cdot) \text{ for the case } c = \lambda + \bar\rho. \tag{4.2}$$

4.2. *The four estimates.* For any $k = 1, \dots, n$, we shall recall the definitions of the average local times of interaction from Definition 5 and (1.8) through (1.11).

We can state the following proposition.

PROPOSITION 8. *There exist constants $c_1, \dots, c_5$ depending only on the endpoint $T > 0$ of the time interval $[0, T]$ such that for any $k = 1, \dots, n$ and any $t \in [0, T]$, we have*:

(4.3) *Estimate* 1. $$E^N[A_N^k(t)] \le c_1 + c_2 t,$$

(4.4) *Estimate* 2. $$E^N[A_N^k(t)]^2 \le c_3,$$

(4.5) *Estimate* 3. $$E^N[A_N(t)] \le c_4 t^{1/2},$$

(4.6) *Estimate* 4. $$E^N[A_N(t)]^2 \le c_5.$$

We shall need the test function defined in (4.2), α_ε ($= \alpha$ for simplicity) and we take

$$G_k^{N,\varepsilon}(t) = G_k(t) = \frac{1}{N}\sum_{j\neq k} \alpha(x_j(t) - x_k(t)). \tag{4.7}$$

We start writing the differential formulas,

$$\begin{aligned} dG_k(t) = {} & \frac{1}{N}\sum_{j\neq k} \alpha''(x_j(t) - x_k(t))\,dt \\ & + \frac{1}{N}\sum_i dA^{ik}(t)\Big[\sum_{j\neq k} \alpha'(x_j(t) - x_k(t)) + 2\alpha'(0+)\Big] \\ & + \frac{1}{N}\sum_i dA^{ki}(t)\Big[-\sum_{j\neq k} \alpha'(x_j(t) - x_k(t)) - 2\alpha'(1-)\Big] \\ & + d\mathscr{M}_{G_k}(t). \end{aligned}$$

One can isolate the total local time for the particle "k",

$$2 \cdot A_N^k(t) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}),$$

where

$$\begin{aligned} (\mathrm{I}) &= dG_k(t), \\ (\mathrm{II}) &= -\frac{1}{N}\sum_{j\neq k} \alpha''(x_j(t) - x_k(t))\,dt, \end{aligned}$$

$$(\mathrm{III}) = \sum_i (dA^{ik}(t) - dA^{ki}(t)) \sum_{j \neq k} \alpha'(x_j(t) - x_k(t)),$$

$$(\mathrm{IV}) = d\mathscr{M}_{G_k}(t)$$

with i, j, k distinct.

We first prove Estimate 3 and Estimate 4.

PROOF OF ESTIMATE 3. To prove Estimate 3, that is, $E^N[A_N(t)] \leq c_4 t^{1/2}$, we shall take the expected value of the four expressions listed above as (I) to (IV) and show that each satisfies the bound in the estimate.

Clearly the martingale disappears; (I) is bounded by a constant $c\varepsilon$ independent of t and N, for arbitrary ε. In the same way (II) $\leq c't$ and the only remarkable fact is that

$$(4.8)\qquad (\mathrm{III}) = \int_0^t d\left[\sum_i [dA^{ik}(s) - dA^{ki}(s)]\right]\left[\sum_{j \neq k} \alpha'(x_j(s) - x_k(s))\right] \equiv 0.$$

It is significant to note that we deal with an *identity*; it is evidently zero by taking the expected value, hence it will be omitted from future calculations. This fact will be shown in the following.

Let us suppress the "t" temporarily; it doesn't matter in the algebra below. We denote by α'_{jk} the expression $\alpha(x_j(t) - x_k(t))$. It is clear that $\sum_{i,\,j,\,k} \alpha'_{jk}\, dA^{ki} = \sum_{i,\,j,\,k} \alpha'_{ji}\, dA^{ik}$ by changing the order of summation (this computation is valid for a fixed j) and $\sum_{i,\,j,\,k} \alpha'_{ji}\, dA^{ik} = \sum_{i,\,j,\,k} \alpha'_{jk}\, dA^{ik}$ because we integrate against dA^{ik}, which is nonzero only where $x_k(s) = x_i(s)$, hence (III) is identically 0. The only thing we still have to prove is that $c\varepsilon + c't$ can be made of order $t^{1/2}$; since ε is arbitrary, we pick $\varepsilon = t^{1/2}$ and we notice that c_4 in the estimate is independent of N. □

We need to prove Estimate 4, that is, that $E^N[A_N(t)]^2 \leq c_5$.

PROOF OF ESTIMATE 4. From the identity given above we will be done as soon as the expected values of the squares of (I), (II) and (IV) are bounded uniformly in N. (I) is a bounded function, hence the bound is valid uniformly in t. (II) is bounded by $\bar{\rho}\|\alpha''\|t$ independently of N. The martingale term needs more attention.

The martingale term is

$$d\mathscr{M}_{G_k}(s) = \frac{1}{N} \sum_{j,\, j \neq k} [\alpha'(x_j(s) - x_k(s))\, d\beta_j(s)] - \left[\frac{1}{N} \sum_{j,\, j \neq k} \alpha'(x_j(s) - x_k(s))\right] d\beta_k(s),$$

and the coefficients of β_l, $l = 1, \dots, n$ are

$$B_l^N(s) = \frac{1}{N^2} \sum_k \alpha'(x_l(s) - x_k(s)) - \frac{1}{N^2} \sum_k \alpha'(x_k(s) - x_l(s)).$$

Then $\alpha(x) = \alpha(1-x)$ by construction, and since it is also periodic of period 1, $\alpha(x) = \alpha(-x)$, implying that α' is odd. We rewrite the martingale as

$$B_l^N(s) = 2\frac{1}{N^2}\sum_k \alpha'(x_l(s) - x_k(s)).$$

Here β_l are mutually orthogonal, hence the the expected value of the square is less than $\sum_l |\sup B_l^N(s)|^2\, dt$ which is clearly $O(1/N)\, dt$. □

We want to prove Estimate 1 and 2 for an arbitrary $1 \le l \le n$. For Estimate 1,

$$E^N[A_N^l(t)] \le c_1 + c_2 t. \tag{4.9}$$

For Estimate 2,

$$E^N[A_N^l(t)]^2 \le c_3. \tag{4.10}$$

The proof is identical for all l, hence we concentrate on the case $l = 1$. A construction is needed.

DEFINITION 13. We define $\sigma(k)$ as the rank of the particle k counted in positive trigonometric sense from x_1 or

$$\sigma(k) := \sum_{j=1}^{n} \mathbf{1}_{[0,\, x_k - x_1]}(x_j - x_1). \tag{4.11}$$

Let us define the test functions

$$T_f(x) = \sum_{k\neq 1} c_k(x_k - x_1) \quad \text{with } c_k := \left(\frac{n+2}{2} - \sigma(k)\right) \tag{4.12}$$

and

$$T_g(x) = \sum_{k\neq 1} g(x_k - x_1) \quad \text{where } g(x) := x(1-x). \tag{4.13}$$

$$dT_g(t) = \sum_{k\neq 1} g''(x_k(t) - x_1(t))\, dt \tag{4.14}$$

$$+ \sum_{k\neq 1} dA^{1k}(t)\left[2 - \sum_{j\notin\{1,\,k\}} g'(x_j(t) - x_1(t))\right]$$

$$+ \sum_{k\neq 1} dA^{k1}(t)\left[2 + \sum_{j\notin\{1,\,k\}} g'(x_j(t) - x_1(t))\right] + d\mathscr{M}_f(t). \tag{4.15}$$

The martingale $\mathscr{M}_g(t)$ is

$$\begin{aligned} d\mathscr{M}_g(t) &= \sum_{k\neq 1}[g'(x_k(t) - x_1(t))\, d\beta_k(t)] \\ &\quad - \left[\sum_{k\neq 1} g'(x_k(t) - x_1(t))\right] d\beta_1(t). \end{aligned} \tag{4.16}$$

At the same time,

$$dT_f(t) = \sum_{k\neq 1} dA^{1k}(t)\Big[-\sum_{j\neq 1} c_j - c_n\Big] + \sum_{k\neq 1} dA^{k1}(t)\Big[+\sum_{j\neq 1} c_j + c_2\Big] \tag{4.17}$$

$$+ (\lambda N)\sum_{k\neq 1} dA^{1k}(t)\Big[\sum_{j\neq 1} c'_j(x_j(t)-x_1(t)) - \sum_{j\neq 1} c_j(x_j(t)-x_1(t))\Big] \tag{4.18}$$

$$+ (\lambda N)\sum_{k\neq 1} dA^{k1}(t)\Big[\sum_{j\neq 1} c''_j(x_j(t)-x_1(t)) - \sum_{j\neq 1} c_j(x_j(t)-x_1(t))\Big] \tag{4.19}$$

$$+ \sum_{i,\, j\neq 1} dA^{ij}(t)(c_i - c_j) + d\mathscr{M}_f(t), \tag{4.20}$$

given that

$$\mathscr{M}_f(t) = \mathscr{M}_f^{\text{cont}}(t) + \mathscr{M}_f^{\text{jump}}(t);$$

$$\mathscr{M}_f^{\text{cont}}(t) = \sum_{k\neq 1} c_k\, d\beta_k(t) - \Big(\sum_{k\neq 1} c_k\Big)\, d\beta_1(t) \tag{4.21}$$

and

$$\mathscr{M}_f^{\text{jump}}(t) = \sum_{k\neq 1}\Big[\sum_{j\notin\{1,\, k\}} (x_j(t)-x_1(t)) + \frac{2-n}{2}\Big][M^{1k}(t) - M^{k1}(t)]. \tag{4.22}$$

One can see that $c'_j \to c_{j+1}$ if $j \neq k$ and c_k is exactly the #n particle before the jump and exactly the #2 particle after the jump. This makes the coefficient of $(\lambda N)A^{1k}(t)$ equal to

$$-\Big[\sum_{j\notin\{1,\, k\}} (x_j(t)-x_1(t)) + \frac{2-n}{2}\Big] \tag{4.23}$$

and similarly the coefficient of $(\lambda N)A^{k1}(t)$ equal to the opposite.

The $c_i - c_j$ factor is equal to -1 because integrated against A^{ij} we have $\sigma(i) = \sigma(j)+1$, simply because when the particle #i collides with the particle #j the trigonometric order of the two is naturally such that #i is exactly ahead of #j and since they collide they are consecutive to each other.

As for the martingales, one can note that the coefficient present in the jump martingale is the one computed for $(\lambda N)A^{1k}(t)$ and $(\lambda N)A^{k1}(t)$ (4.23); finally, one can see that $\sum_{k\neq 1} c_k = 0$.

The two test functions presented will be combined into a new test function T and the integral form of the differential formula of T provides a clear-cut bound of both $E^N[A_N^1]$ and $E^N[A_N^1]^2$.

Let $T(t) := (\lambda N/2)T_g(t) + T_f(t)$. Just for the calculations below, we denote $a := (\lambda N/2)$. It is now clear that

$$dT(t) = -2a(n-1)\,dt + \left[\sum_{k\neq 1} dA^{1k}(t)\right]\left[a(4-n) - c_n - (\lambda N)\frac{2-n}{2}\right] + \left[\sum_{k\neq 1} dA^{k1}(t)\right]\left[an + c_2 + (\lambda N)\frac{2-n}{2}\right] - \sum_{i,\, j\neq 1} A^{ij}(t) + d\mathscr{M}_T(t);$$

we divide everything by N^2 and a few calculations lead to

$$\frac{1}{N^2} dT(t) + \left(\lambda\frac{n-1}{N}\right)dt + \frac{1}{N^2}\left[\sum_{i,\, j\neq 1} A^{ij}(t)\right] - \frac{1}{N^2}\, d\mathscr{M}_T(t) \tag{4.24}$$

$$= \left(\frac{1}{N}\sum_{k\neq 1} dA^{1k}(t)\right)\left[\lambda - \frac{1}{N} + \frac{1}{2}\bar{\rho}\right] + \left(\frac{1}{N}\sum_{k\neq 1} dA^{k1}(t)\right)\left[\lambda - \frac{1}{N} + \frac{1}{2}\bar{\rho}\right]. \tag{4.25}$$

A simpler writing of this relation (in integral form) is

$$\frac{1}{N^2}[T(t) - T(0)] + \lambda\frac{n-1}{N}t + A_N^{\mathrm{total}}(t) - \mathscr{M}_T(t), \tag{4.26}$$

equal to $[\lambda - \frac{1}{N} + \frac{1}{2}\bar{\rho}]A_N^1(t)$.

Proof of Estimate 1. We only have to note that $(1/N^2)T(s) \le \frac{1}{2}\bar{\rho}(\lambda + \bar{\rho}) =: c_T$ uniformly in N, that $s \in [0, T]$ and $\lambda((n-1)/N) \le \lambda\bar{\rho}$ and take the expected value of the expression (4.26) to obtain

$$\left[\lambda - \frac{1}{N} + \frac{1}{2}\bar{\rho}\right]E^N[A_N^1(t)] \le 2c_T + \lambda\bar{\rho}t + E^N[A_N^{\mathrm{total}}(t)] \tag{4.27}$$

proving Estimate 1. □

Proof of Estimate 2. We can bound below the coefficient $[\lambda - \frac{1}{N} + \frac{1}{2}\bar{\rho}]$ by $[\lambda + \frac{1}{4}\bar{\rho}]$ since we don't need an optimal inequality. The problem boils down to proving $E^N(1/N^2)[\mathscr{M}_T(t)]^2 \le$ const with a const independent of N.

To do this we have to calculate explicitly the martingale term. Again, we shall break down the martingale into $(\lambda/2N)\mathscr{M}_g(t)$ and $(1/N^2)\mathscr{M}_f(t)$.

Let us consider $\mathscr{M}_f(t)$ first: $\mathscr{M}_f(t) = \mathscr{M}_f^{\mathrm{cont}}(t) + \mathscr{M}_f^{\mathrm{jump}}(t)$ and the two martingales are mutually orthogonal. The square of the continuous martingale is bounded by $(1/N^4)[\sum_{k\neq 1} c_k^2]t$ (the Brownian martingales $\beta_j(t)$ are also mutually orthogonal); the coefficients $|c_k|$ are bounded by n and so a bound for the summation given above is $O(1/N)$.

We now have to take care of the jump martingale. We recall that the coefficients of the martingales $M^{1k}(t)$ and $M^{k1}(t)$ were equal to (4.22),

$$m_k(t) := \left[\sum_{j \notin \{1,k\}} (x_j(t) - x_1(t)) + \frac{2-n}{2} \right].$$

Since $|m_k(t)| \le 2n$ and given that $M^{1k}(t)$ and $M^{k1}(t)$ are mutually orthogonal, the square of the jump martingale for f is

$$\frac{1}{N^4} \sum_{k \neq} m_k^2(t)(\lambda N)[A^{1k}(t) + A^{k1}(t)]$$

bounded by $(2\lambda)^2 \bar{\rho} A_N^{\text{total}}(t)$, a quantity proved to be bounded when we take the expected value.

The last step of the proof is to show that the quadratic variation of $\mathscr{M}_g(t)$ is bounded uniformly in N,

$$\begin{aligned} \mathscr{M}_g(t) = \int_0^t & \sum_{k \neq 1} \big(g'(x_k(s) - x_1(s))\big)\, d\beta_k(s)\big) \\ & - \left(\textstyle\sum_{k \neq 1} g'(x_k(s) - x_1(s)) \right) d\beta_1(s) \end{aligned} \tag{4.28}$$

[we recall (4.16)] and since $\|g'\| \le 3$ and we actually need to get a bound for the quadratic variation of $\lambda(1/2N)\mathscr{M}_g(t)$, we basically look at terms as $\text{const}(1/N^2)[9(n-1)t + [3(n-1)]^2 t]$, evidently of order $O(1)$. □

5. The collision time for the tagged particle.

5.1. *A differential formula.* Some further notation is given.

DEFINITION 14. For any $g: \Gamma \times \Gamma \to R$ we shall write

$$r_k^g(t) := \frac{1}{N} \sum_{j=1}^{n} g(x_k(t) - x_1(t), x_j(t) - x_1(t)). \tag{5.1}$$

DEFINITION 15. For $\sigma(k)$ defined in (4.11),

$$q_k^v(t) = \frac{\sigma(k)}{N} + \lambda(x_k(t) - x_1(t)). \tag{5.2}$$

We can write $r_k^v(t) = q_k^v(t) + 1/N$ for the function

$$v(x, y) = \frac{\lambda}{\bar{\rho}} x + \mathbf{1}_{[0,x]}(y) \tag{5.3}$$

with intervals taken in trigonometric sense.

We are now in a position to define the test function used to isolate the average local time of collision per particle $A_N^1(t)$.

DEFINITION 16.

$$F_\varepsilon^N(t) = \frac{1}{N} \sum_{2 \le k \le n} \gamma_\varepsilon\left(\frac{\sigma(k)}{N} + (x_k(t) - x_1(t))\right), \tag{5.4}$$

briefly denoted by $F(t)$. The subscript ε will be dropped in the following calculation.

We want to write the differential formula for $F(t)$, that is,

$$dF(t) = \sum_{l=1}^{7} (D(l)) + d\mathscr{M}_F^N(t) \sim (D). \tag{5.5}$$

It is helpful to mention the following.

REMARK. The jump $\gamma_{1,n-1} = (\gamma(1/N) - \gamma((n-1)/N + \lambda \cdot 1)) = 0$ because γ is symmetric.

We shall proceed by describing the seven terms in (D):

$$(D1) = \int \frac{\lambda^2}{N} \sum_k \gamma''\left(\frac{\sigma(k)}{N} + \lambda(x_k(t) - x_1(t))\right) dt, \tag{5.6}$$

$$(D2) = \frac{\lambda}{N} \sum_k \left\{ dA^{1k}(t) \left[- \sum_{j \ne k} \gamma'(q_j(t)) - 2\gamma'\left(\frac{n-1}{N} + \lambda \cdot 1\right)\right]\right\}, \tag{5.7}$$

$$(D3) = \frac{\lambda}{N} \sum_k \left\{ dA^{k1}(t) \left[+ \sum_{j \ne k} \gamma'(q_j(t)) + 2\gamma'\left(\frac{1}{N}\right)\right]\right\}, \tag{5.8}$$

$$(D4) = \frac{(\lambda N)}{N} \sum_k \left\{ dA^{1k}(t) \left[\sum_{j \ne k} \gamma(q'_j(t)) - \gamma(q_j(t)) + \gamma_{1,n-1}\right]\right\}, \tag{5.9}$$

$$(D5) = \frac{(\lambda N)}{N} \sum_k \left\{ dA^{k1}(t) \left[\sum_{j \ne k} \gamma(q''_j(t)) - \gamma(q_j(t)) - \gamma_{1,n-1}\right]\right\} \tag{5.10}$$

and the last term will be broken down in two naturally equal parts, only to simplify a future calculation (see Section 6, proof of Proposition 22),

$$(D6) = \frac{\lambda}{2N} \sum_{i,\, j \ne 1} dA^{ij} [\gamma'(q_i(t)) - \gamma'(q_j(t))], \tag{5.11}$$

$$(D7) = \frac{\lambda}{2N} \sum_{i,\, j \ne 1} dA^{ji} \Big[\gamma'(q_j(t)) - \gamma'(q_i(t))\Big]. \tag{5.12}$$

$\mathscr{M}_F^N(t)$ is

$$\mathscr{M}_F^N(t) = \mathscr{M}_F^{N,\,\mathrm{cont}}(t) + \mathscr{M}_F^{N,\,\mathrm{jump}}(t) \tag{5.13}$$

with

$$d\mathscr{M}_F^{N,\,\mathrm{cont}}(t) = -\lambda\frac{1}{N}\bigg[\sum_{2\le k\le n}\gamma'_\varepsilon\bigg(\frac{\sigma(k)}{N}+(x_k(t)-x_1(t))\bigg)\bigg]\,d\beta_1(t)$$
$$(5.14)\qquad\qquad + \lambda\frac{1}{N}\bigg[\sum_{2\le k\le n}\gamma'_\varepsilon\bigg(\frac{\sigma(k)}{N}+(x_k(t)-x_1(t))\bigg)\,d\beta_k(t)\bigg]$$

and

$$d\mathscr{M}_F^{N,\,\mathrm{jump}}(t) = \sum_k\bigg\{\bigg[\frac{1}{N}\sum_{j\ne k}\gamma(q'_j(t))-\gamma(q_j(t))+\gamma_{1,\,n-1}\bigg]\,dM^{1k}(t)\bigg\}$$
$$(5.15)\qquad\qquad + \sum_k\bigg\{\bigg[\frac{1}{N}\sum_{j\ne k}\gamma(q''_j(t))-\gamma(q_j(t))-\gamma_{1,\,n-1}\bigg]\,dM^{k1}(t)\bigg\},$$

where $\gamma_{1,\,n-1} = (\gamma(1/N) - \gamma((n-1)/N+\lambda)) = 0$.

Here $\{M_N^{ij}(t)\}$ are the jump martingales of the collision (i, j) when $i > j$, that is, if $N^{ij}(t)$ is the number of change of labels between i and j when $i > j$ up to time t, $N^{ij}(t) - (\lambda N)A_N^{ij}(t) =: M_N^{ij}(t)$ and $[M_N^{ij}(t)]^2 - (\lambda N)A_N^{ij}(t)$ are martingales, $\forall\, i, j \in \{1, 2, \dots, n\}$.

5.2. *The tightness of* $A_N^1(t)$. The description of the terms given before will be used to derive two types of results: one is an estimate of the growth of $A_N^1(t)$ (the following proposition) which also implies the tightness of the process $\{A_N^1(t)\}_N$ and the other is the fundamental Theorem 2 giving the asymptotic law of $A_N^1(t)$

$$\lim_{N\to\infty}\bigg|A_N^1(t) - \int_0^t \rho(s, x_1(s))\,ds\bigg| = 0.$$

PROPOSITION 9. *For any* $a \in (0, \frac{1}{2})$ *and for any* s, t *with* $0 < s < t < T$ *there is a constant* $C_T > 0$ *independent of* N *such that* $E^N[A_N^1(t) - A_N^1(s)] \le C_T(t-s)^a$.

For the first result we only need the expected value of the terms in (D), hence the martingale term is not actively used in the proof. However, in the second we need the absolute value estimate and we shall rely on the fact that the quadratic variations of the martingales are negligible as $N \to \infty$.

Notation. For $j \ne k$,

$$(5.16)\qquad q'_j = \frac{\sigma(j)+1}{N} + \lambda(x_j(t) - x_1(t))$$

and

$$(5.17)\qquad q''_j = \frac{\sigma(j)-1}{N} + \lambda(x_j(t) - x_1(t)).$$

In the formulas $(D6)$ and $(D7)$, $\sigma(i) = \sigma(j) + 1$ whenever we integrate against dA^{ji}, and $\sigma(i) = \sigma(j) - 1$ whenever we integrate against dA^{ij}.

We look at the differential formula (D) in integral form. It is clear that the left-hand side terms are bounded by $\frac{3}{2}\bar{\rho}\varepsilon$. We shall consider a term as "negligible" if $\limsup_{N\to\infty}\cdots = 0$.

The *first step* is to show that in $(D3)$ we can replace $\gamma'(1/N)$ by $\gamma'(0) = 1$ and in $(D4)$ we can replace $\gamma'((n-1)/N + \lambda)$ by $\gamma'(\bar{\rho}+\lambda) = -1$. They work out in the same fashion; we do the first one.

The term $\gamma'(1/N) - \gamma'(0)$ is of order $1/N$ and the local time $\sum_k A^{k1}(t)$ already comes with a coefficient of $1/N$, such that

$$\left(\gamma'\left(\frac{1}{N}\right) - \gamma'(0)\right)\left(\frac{1}{N}\sum_k A^{k1}(t)\right)$$

is of order $1/N$ [Estimate (4.3)].

The *second step* is to notice the simple fact that

$$\left|\left(\gamma\left(\frac{1}{N}\right) - \gamma\left(\frac{n-1}{N} + \lambda\cdot 1\right)\right)\left(\sum_k dA^{k1}(t) - \sum_k dA^{1k}(t)\right)\right|$$

is identically 0 as well because $(\gamma(1/N) - \gamma((n-1)/N + \lambda\cdot 1)) = 0$ from the symmetry of γ.

The *third step* is to look at the pairs of terms $(D2)$, $(D4)$ and $(D3)$, $(D5)$ and show that we can approximate $(D2)+(D4)$ by

$$\lambda\int_0^t\left(\frac{1}{N}\sum_k dA^{1k}(s)\right)\left(2 + \frac{1}{2N}\sum_{j\neq k}\gamma''(q_j(s))\right) \tag{5.18}$$

and $(D3)+(D5)$ by

$$\lambda\int_0^t\left(\frac{1}{N}\sum_k dA^{k1}(s)\right)\left(2 + \frac{1}{2N}\sum_{j\neq k}\gamma''(q_j(s))\right). \tag{5.19}$$

These approximations are consequences of the Taylor expansions

$$-\gamma(q_j) + \left(\gamma\left(q_j + \frac{1}{N}\right) - \gamma(q_j)\right) = \frac{1}{2N}\gamma''(q_j) + O\left(\frac{1}{N^2}\right)$$

and

$$\gamma(q_j) + \left(\gamma\left(q_j - \frac{1}{N}\right) - \gamma(q_j)\right) = \frac{1}{2N}\gamma''(q_j) + O\left(\frac{1}{N^2}\right).$$

The expected value of the whole formula (D) in integral form shows that the term

$$\lambda\limsup_{N\to\infty} E^N\int_0^t\left(2 + \frac{1}{2N}\sum_j\gamma''(q_j(s))\right)dA_N^1(s) \tag{5.20}$$

is less than

$$(5.21)\qquad \begin{aligned} &2\frac{3}{2}\bar\rho\varepsilon + \lambda^2 \limsup_{N\to\infty} E^N \int_0^t \frac{1}{N}\sum_j \gamma''(q_j(s))\,ds \\ &\quad + \lambda \limsup_{N\to\infty} E^N \int_0^t \frac{1}{N}\sum_j \gamma''(q_j(s))\,dA_N^j(s) \end{aligned}$$

and since $\gamma'' \le 3/2\varepsilon$ and $E^N(1/N)\sum_j A_N^j(t) \le c_1 t^{1/2}$ less than $3\bar\rho\varepsilon + \lambda^2\bar\rho(1/\varepsilon)t + \lambda\bar\rho c_1 t^{1/2}$.

Because ε is arbitrary and λ, c_1 and $\bar\rho$ are independent of N, we deduce that all the expression above is bounded above by const $\cdot\, t^a$ where $a \in (0, \frac{1}{2})$. (We plug in $\varepsilon = t^a$ and note that $1 - a > a$.) A lower bound for (5.20) is $\lambda(2 - l)E^N A^1(t)$ with $l =$ the maximum value of ϕ_l from definition (4.2) $= \frac{3}{2}$. Consequently the tightness of $A_N^1(t)$ is proved. □

6. The asymptotic behavior of $A_N^1(t)$. The goal of this section is to prove Theorem 2.

6.1. *A brief outlook of our plan.* The differential formula $dF_\varepsilon^N(t) \sim (D)$ from Section 5 depends on both N and ε. Our goal is to find its asymptotic value as $\lim_{\varepsilon\to 0}\limsup_{N\to\infty}$. Propositions 12 through 14 (stated in the next subsection) and their pairs for smooth functions (Propositions 18, 19 and 20) are intermediary steps for establishing the limits

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}(D(l))$$

for all $l = 1, \dots, 7$. (5.6)–(5.12). After this we shall be able to write down for $\rho = \rho(s, x_1^N(s))$ and $A^1 = A_N^1(s)$ the asymptotic identity

$$\lambda^2(-2)\left(\frac{\rho}{\rho+\lambda}\right)ds + \frac{\lambda}{2}(-2)\left(\frac{\rho^2}{\rho+\lambda}\right)ds + \lambda\left[2 + \frac{1}{2}(-2)\left(\frac{\rho}{\rho+\lambda}\right)\right]dA^1 = 0.$$

After some algebra, this implies that

$$\lambda\rho\left[\frac{\rho+2\lambda}{\rho+\lambda}\right]ds = \lambda\left[\frac{\rho+2\lambda}{\rho+\lambda}\right]dA^1(s),$$

providing the formal identity $A^1(t) = \int_0^t \rho(s, x_1(s))\,ds$, which is exactly the result we need.

In the following we shall exploit the integral formula (D) (5.5) to derive rigorously the identities from above.

DEFINITION 17. We regard as negligible an expression $H_N^\varepsilon(t, \omega)$ if

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} E^N\left|H_N^\varepsilon(t, \omega)\right| = 0.$$

PROPOSITION 10. *The integral of the sum of the differential terms* $(D1),\dots,(D7)$ (5.6) *through* (5.12) *is negligible, that is,*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\left|\int_0^t d[(D1)+\cdots+(D7)](s)\right| = 0.$$

PROOF. It is enough to show that $\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N \int_0^t d|(D1)+\cdots+(D7)|^2 = 0$ and, since $\sup_u |F_\varepsilon^N(u)| \le \frac{3}{2}\bar{\rho}\varepsilon$ (5.4), it is enough to show that $|\mathscr{M}_F^N(t)|^2$ (5.13) is negligible.

We have to review the expressions of (5.14) and (5.15), the continuous and the jump martingales. Again, it is sufficient to prove that both $|\mathscr{M}_F^{N,\mathrm{cont}}(t)|^2$ and $|\mathscr{M}_F^{N,\mathrm{jump}}(t)|^2$ are negligible. The Brownian motions $\{\beta_j(t)\}$ for $j = 1,\dots,n$ are independent; the quadratic variation of $\mathscr{M}_F^{N,\mathrm{cont}}(t)$ is equal to

$$\int_0^t\left[\lambda\frac{1}{N}\sum_k \gamma'(q_k(s))\right]^2 ds + \left(\lambda\frac{1}{N}\right)^2\int_0^t\left[\sum_k(\gamma')^2(q_k(s))\right] ds. \tag{6.1}$$

The support of γ' is equal to the support of (4.2), that is, it is at most 2ε in length. This implies by a crude approximation that there can be only $n_\varepsilon = 2\varepsilon N$ particles which fall in the support of γ', $|\gamma'|\le 1$. Finally, the first integrand above is less than $\lambda^2(2\varepsilon)^2$, an obviously negligible quantity as $\varepsilon\to 0$.

The other integrands in the summation are less than $\lambda^2 N\cdot N^{-2}$, clearly negligible.

As for the jump martingale, we note that $M_N^{ij}(t)$ given for all pairs such that $i\ne j$ are mutually orthogonal and $[M_N^{ij}(t)]^2-(\lambda N)A_N^{ij}(t)$ is a martingale. Consequently the quadratic variation of $\mathscr{M}_F^{N,\mathrm{jump}}(t)$ is

$$(\lambda N)\int_0^t\sum_k[V_k^{\mathrm{left}}(s)]^2\,dA_N^{1k}(s) + (\lambda N)\int_0^t\sum_k[V_k^{\mathrm{right}}(s)]^2\,dA_N^{k1}(s) \tag{6.2}$$

with

$$V_k^{\mathrm{left}}(s) := \left[\frac{1}{N}\left(\sum_{j\ne k}\gamma(q'_j(t)) - \gamma(q_j(t)) + \gamma_{1,n-1}\right)\right]$$

and

$$V_k^{\mathrm{right}}(s) := \left[\frac{1}{N}\left(\sum_{j\ne k}\gamma(q''_j(t)) - \gamma(q_j(t)) - \gamma_{1,n-1}\right)\right].$$

We recall that $(\gamma(1/N)-\gamma((n-1)/N+\lambda\cdot 1))\equiv 0$, implying that

$$\left|\gamma(q'_j(t))-\gamma(q_j(t))\right| = \frac{1}{N}\left|\gamma'(q_j(t))\right| + O\left(\frac{1}{N^2}\right)$$

and

$$\left|\gamma(q''_j(t))-\gamma(q_j(t))\right| = \left|\gamma'(q_j(t))\right| + O\left(\frac{1}{N^2}\right).$$

We are primarily interested in $[NV_k^{\text{left}}(s)]^2$ and $[NV_k^{\text{right}}(s)]^2$; they are bounded by $2(2\varepsilon)^2 + (O(1/N))^2$ and this implies

$$E^N\big[\mathscr{M}_F^{N,\text{jump}}(t)\big]^2 \le 2E^N\left[(2\varepsilon)^2 + O\left(\frac{1}{N^2}\right)\right][A_N^1(t)],$$

a negligible quantity as $N \to \infty$ and $\varepsilon \to 0$. □

6.2. *Three estimates in the case of smooth functions.* In the following propositions we obtain the hydrodynamic limit of three expressions containing the tagged particle $x_1^N(t)$ and we show that the limit is not affected by its presence, as if the limits would take place uniformly in x_1. We shall need the next results, with proofs in the Appendix.

LEMMA 4 (Uniform convergence). *Suppose* $\{u_N(z,\omega)\}_N$ *is a sequence of positive random variables which satisfy*

(6.3)(i)
$$\lim_{N\to\infty} E^N[u_N(z,\omega)] = 0.$$

(ii) *There exists a positive random Lipschitz constant* $L_N(\omega)$ *such that*

$$\big|u_N(z',\omega) - u_N(z'',\omega)\big| \le L_N(\omega)\big|z' - z''\big| \tag{6.4}$$

with $\sup_N E^N(L_N(\omega)) \le l < \infty$ *and*

(6.5)(iii) $z \in K$ *with* K *a compact space.*

Then $\lim_N E^N[\sup_z u_N(z,\omega)] = 0$.

The proof is given in the Appendix.

LEMMA 5. *For any smooth function* f *on the unit circle*

$$E^N \sup_{0\le s\le t}\left|\frac{1}{N}\sum_{k\ne 1} f(x_k^N(s) - x_1^N(s)) - \int_0^1 f(y - x_1^N(s))\rho(s,y)\,dy\right|$$

converges to zero as $N \to \infty$.

For the proof, we apply the preceding lemma and Theorem 1 from the Introduction.

LEMMA 6.

$$\begin{aligned} E^N \sup_{0\le s\le t}\Bigg|\frac{1}{N}\sum_k f\left(\frac{1}{N}\sum_j g(x_k(s), x_j(s))\right) \\ - \int_0^1 f\left(\int_0^1 g(x,y)\rho(s,y)\,dy\right)\rho(s,x)\,dx\Bigg|^2 \to 0 \end{aligned} \tag{6.6}$$

as $N \to \infty$.

The proof is given in the Appendix.

The following proposition gives an L^q estimate for the heat kernel.

PROPOSITION 11. *If $p(s,x)$ is the heat kernel for the unit circle, that is, for any $f \in L^1(T^1)$,*

$$\rho(s,x) := f * p(s,x) = \int_0^1 f(y)p(s,x-y)\,dy$$

is the solution to the Cauchy problem for the heat equation

$$\partial_t \rho = \tfrac{1}{2}\rho_{xx}, \qquad \rho(0,x) = f(x), \tag{6.7}$$

then for any $q \in [1,\infty]$ $\limsup_{s\to 0} s^{+1/2(1-1/q)}\|p(s,\cdot)\|_{L^p} < \infty$. For $q=\infty$ the statement is

$$\limsup_{s\to 0} s^{+1/2} \sup_{x\in T^1} |p(s,\cdot)| < \infty. \tag{6.8}$$

The proof is given in the Appendix.

Let v be the function $v(x,y) = (\lambda/\bar{\rho})x + \mathbf{1}_{[0,x]}(y)$ (5.3). Then we may state the following proposition.

PROPOSITION 12. *For $f \in C_0^\infty(R)$ (smooth and with compact support) we have*

$$\begin{aligned} E^N\bigg| &\int_0^t \frac{1}{N}\sum_k f\bigg(\frac{1}{N}\sum_j v(x_k^N(s)-x_1^N(s), x_j^N(s)-x_1^N(s))\bigg)\,ds \\ &- \int_0^t\int_0^1 f\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg)\rho(s,x)\,dx\,ds\bigg| \to 0 \end{aligned} \tag{6.9}$$

as $N\to\infty$.

PROPOSITION 13. *For $f \in C_0^\infty(R)$ (smooth and with compact support) we have*

$$\begin{aligned} E^N\bigg| &\int_0^t \frac{1}{N}\sum_k f\bigg(\frac{1}{N}\sum_j v(x_k^N(s)-x_1^N(s), x_j^N(s)-x_1^N(s))\bigg)\,dA_N^1(s) \\ &- \int_0^t\int_0^1 f\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg) \\ &\qquad\qquad \times \rho(s,x)\,dx\,dA_N^1(s)\bigg| \to 0 \end{aligned} \tag{6.10}$$

as $N\to\infty$.

PROPOSITION 14. *For $f \in C_0^\infty(R)$ (smooth and with compact support) we have*

$$E^N\bigg|\int_0^t \frac{1}{N}\sum_k\bigg[f\bigg(\frac{1}{N}\sum_j v(x_k^N(s)-x_1^N(s), x_j^N(s)-x_1^N(s))\bigg)\,dA_N^k(s)\bigg] - \int_0^t\int_0^1 f\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg) \times \rho^2(s,x)\,dx\,ds\bigg| \to 0 \tag{6.11}$$

as $N\to\infty$.

6.3. *The estimates as $\varepsilon \to 0$.* Our actual computation involves $f=\gamma_\varepsilon''$ from the test function $F_\varepsilon^N(t)$ (5.4). The next propositions give us the limits as $\varepsilon\to 0$. The proof is in Section 6.6.

PROPOSITION 15. *The $\lim_{\varepsilon\to 0}\limsup_{N\to\infty}$ of*

$$E^N\bigg|\int_0^t\int_0^1 \gamma_\varepsilon''\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg)\rho(s,x)\,dx\,ds + 2\int_0^t \frac{\rho(s,x_1^N(s))}{\lambda+\rho(s,x_1^N(s))}\,ds\bigg| \tag{6.12}$$

is zero.

PROPOSITION 16. *The $\lim_{\varepsilon\to 0}\limsup_{N\to\infty}$ of*

$$E^N\bigg|\int_0^t\int_0^1 \gamma_\varepsilon''\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg)\rho(s,x)\,dx\,dA_N^1(s) + 2\int_0^t \frac{\rho(s,x_1^N(s))}{\lambda+\rho(s,x_1^N(s))}\,dA_N^1(s)\bigg| \tag{6.13}$$

is zero.

PROPOSITION 17. *The $\lim_{\varepsilon\to 0}\limsup_{N\to\infty}$ of*

$$E^N\bigg|\int_0^t\int_0^1 \gamma_\varepsilon''\bigg(\int_0^1 v(x-x_1^N(s), y-x_1^N(s))\rho(s,y)\,dy\bigg)\rho^2(s,x)\,dx\,ds + 2\int_0^t \frac{\rho^2(s,x_1^N(s))}{\lambda+\rho(s,x_1^N(s))}\,ds\bigg| \tag{6.14}$$

is zero.

6.4. *Three intermediary lemmas.* To prove the three propositions for $v = \mathbf{1}_{[y \le x]}(x, y) + (\lambda/\bar{\rho})x$ we shall first prove three easier results which are restatements of the above for $v \to g$ with g smooth. It is worth mentioning that we only need a g with compact support, since we need it to approximate a function periodic in both variables x and y.

PROPOSITION 18. *We make the same statement as Proposition* 12 *with* $g \in C_0^\infty(R^2)$ (*smooth and with compact support*) *replacing* v.

PROPOSITION 19. *We make the same statement as Proposition* 13 *with* $g \in C_0^\infty(R^2)$ (*smooth and with compact support*) *replacing* v.

PROPOSITION 20. *We make the same statement as Proposition* 14 *with* $g \in C_0^\infty(R^2)$ (*smooth and with compact support*) *replacing* v.

PROOF OF PROPOSITION 18 AND 19. We first reduce the proposition to the uniform statement in x_1, that is, with the help of Lemma 4 we only have to prove that $u_N(z, \omega)$, equal to

$$\begin{aligned}\Bigg| \int_0^t \frac{1}{N} \sum_k f\bigg(\frac{1}{N} \sum_j g(x_k(s) - z, x_j(s) - z) \bigg) \, ds \\ - \int_0^t \int_0^1 f\bigg(\int_0^1 g(x - z, y - z) \rho(s, y) \, dy \bigg) \rho(s, x) \, dx \, ds \Bigg|\end{aligned} \tag{6.15}$$

is Lipschitz in z and Lemma 6 takes care of the rest. One may easily see that here the constant $L_N(\omega)$ is nonrandom and independent from N, namely is equal to $2\|f'\| \cdot \|\nabla g\| \cdot \|\rho_0\|^2$.

For the case of Proposition 19: again we first reduce the proof to its uniform version in x_1. We denote by $u_N(z, \omega)$ the quantity

$$\begin{aligned}\Bigg| \int_0^t \frac{1}{N} \sum_k f\bigg(\frac{1}{N} \sum_j g(x_k(s) - z, x_j(s) - z) \bigg) \, dA_N^1(s) \\ - \int_0^t \int_0^1 f\bigg(\int_0^1 g(x - z, y - z) \rho(s, y) \, dy \bigg) \rho(s, x) \, dx \, dA_N^1(s) \Bigg|\end{aligned} \tag{6.16}$$

and we need to show that $|u_N(z', \omega) - u_N(z'', \omega)| \le L_N(\omega)|z' - z''|$, with $L_N(\omega)$ satisfying (iii) in Lemma 4. We take the z derivatives of the smooth functions f and g and see that we are done if $E^N A_N^1(t)$ is finite for t finite, which is one of the preliminary estimates.

We shall proceed to the proofs of Proposition 18 and Proposition 19.

Let us use the notation

$$\begin{aligned}C_N(s, z) =: \Bigg| \frac{1}{N} \sum_k f\bigg(\frac{1}{N} \sum_j g(x_k(s) - z, x_j(s) - z) \bigg) \\ - \int_0^1 f\bigg(\int_0^1 g(x - z, y - z) \rho(s, y) \, dy \bigg) \rho(s, x) \, dx \Bigg|\end{aligned} \tag{6.17}$$

The next step is to look at the two integrals

$$u_N(z, \omega) := \int_0^t C_N(s, z)\, ds$$

and

$$u'_N(z, \omega) := \int_0^t C_N(s, z)\, dA_N^1(s).$$

The first is bounded by $E^N \sup_{0 \le s \le t} |C_N(s)| t$ and the second by $E^N \sup_{0 \le s \le t} |C_N(s)| A_N^1(t)$. The Schwarz inequality shows that Lemma 6 and the estimate $E^N[A_N^1(t)]^2$ uniformly bounded in N conclude the proof. □

PROOF OF PROPOSITION 20. The uniform estimate in

$$N \text{ of } E^N\left[\left(\frac{1}{N}\right) \sum_k A_N^k(t)\right]$$

and the smoothness of f and g provide the Lipschitz condition needed to reduce the limit in Proposition 20 to an expression independent from x_1. Hence we have to show that

$$E^N \left| \int_0^t \frac{1}{N} \sum_k \left[f\left(\frac{1}{N} \sum_j g(x_k(s), x_j(s)) \right) dA_N^k(s) \right] - \int_0^t \int_0^1 f\left(\int_0^1 g(x, y) \rho(s, y)\, dy \right) \rho^2(s, x)\, dx\, ds \right|$$

tends to 0 as $N \to \infty$ for any smooth and compactly supported functions f and g.

The expression in the limit will be split in two,

$$\left| \int_0^t \frac{1}{N} \sum_k \left[f\left(\frac{1}{N} \sum_j g(x_k(s), x_j(s)) \right) dA_N^k(s) \right] - \int_0^t \frac{1}{N} \sum_k f\left(\int_0^1 g(x_k(s), y) \rho(s, y)\, dy \right) dA_N^k(s) \right| \tag{6.18}$$

and

$$\left| \int_0^t \frac{1}{N} \sum_k f\left(\int_0^1 g(x_k(s), y) \rho(s, y)\, dy \right) dA_N^k(s) - \int_0^t \int_0^1 f\left(\int_0^1 g(x, y) \rho(s, y)\, dy \right) \rho^2(s, x)\, dx\, ds \right|. \tag{6.19}$$

PROOF OF (6.18). Let us define the quantity

$$D(N, s) := \sup_x \left| f\left(\frac{1}{N} \sum_j g(x, x_j(s)) \right) - f\left(\int_0^1 g(x, y) \rho(s, y)\, dy \right) \right|$$

and then we can have a bound

$$(6.18) \le E^N \int_0^t D(N,s)d\biggl(\sum_k \frac{1}{N} A_N^k(s)\biggr).$$

By Schwarz's inequality we deduce that it is enough to show that $E^N[\sup_{s\le t} D(N,s)]^2$ tends to ∞ as $N \to 0$ and

$$E^N\biggl(\sum_k \frac{1}{N} A_N^k(s)\biggr)^2$$

is uniformly bounded in N.

The second bound is provided by the basic estimates established before.

The first limit can be proved by using Lemma 4 once again. One has to denote

$$u_N(x,\omega) := \sup_{s\le t}\left| f\biggl(\frac{1}{N}\sum_j g(x, x_j(s))\biggr) - f\biggl(\int_0^1 g(x,y)\rho(s,y)\,dy\biggr)\right|^2,$$

where x stands for the z in the lemma. This expression is less than

$$\left[\|f'\|\bar{\rho} \sup_{0\le s\le t}\left|\frac{1}{N}\sum_j g(x,x_j(s)) - \int_0^1 g(x,y)\rho(s,y)\,dy\right|\right]^2$$

and for each fixed x we see that this expression tends to 0 as N tends to ∞ (Lemma 6). To check the Lipschitz condition in Lemma 4 we note that f and g are smooth and as such they have bounded derivatives, implying that

$$\bigl|u_N(x',\omega) - u_N(x'',\omega)\bigr| \le 2\|f'\| \cdot \|\nabla g\|(\sup \rho_0)|x'-x''|.$$

PROOF OF (6.19). (6.19) is the content of Lemma 10, which will be proven in the Appendix. This concludes the proof of Proposition 20. □

6.5. *Proof of Propositions* 12, 13 *and* 14. We have shown that the three limits prescribed by Propositions 12, 13 and 14 are zero if we replace the function v which has jumps along the diagonal $x=y$ and along the line $x=0$ (mod 1) with a smooth g (at least of class C^1).

The next step is to prove that the passage from g to $v = \mathbf{1}_{[y\le x]}(x,y)$ is possible with no further restrictions on the initial profile: $\mu(dx) := \mu(0,dx)$. Then μ may be any finite measure on the unit circle with total mass $\bar{\rho}$.

We shall treat Propositions 12 and 13 together and Proposition 14 separately.

Our point is to compare

$$\begin{aligned} C_N^v(s) &= \frac{1}{N}\sum_k f\biggl(\frac{1}{N}\sum_j v(x_k(s)-x_1(s), x_j(s)-x_1(s))\biggr) \\ &\quad - \int_0^1 f\biggl(\int_0^1 v(x-x_1(s), y-x_1(s))\rho(s'y)\,dy\biggr)\rho(s,x)\,dx \end{aligned}$$

and the analogue for g,

$$C_N^g(s) = \frac{1}{N}\sum_k f\biggl(\frac{1}{N}\sum_j g(x_k(s) - x_1(s), x_j(s) - x_1(s))\biggr) - \int_0^1 f\biggl(\int_0^1 g(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\biggr)\rho(s, x)\,dx.$$

$$\begin{aligned}\bigl|C_N^v(s) - C_N^g(s)\bigr| &\le \|f'\|\biggl[\biggl|\frac{1}{N^2}\sum_{k, j}|v - g|(x_k(s) - x_1(s), x_j(s) - x_1(s)\biggr| \\ &\quad + \biggl|\int_0^1\int_0^1 |v - g|(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\rho(s, x)\,dx\biggr|\biggr].\end{aligned}$$

Suppose there is a $\phi(x, y)$ smooth with compact support such that $\phi \ge |v - g|$.

We can find a bound for the difference from above by $\|f'\|$ times

$$\biggl[\biggl|\frac{1}{N^2}\sum_{k, j}\phi(x_k(s) - x_1(s), x_j(s) - x_1(s))\biggr| + \biggl|\int_0^1\int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\rho(s, x)\,dx\biggr|\biggr],$$

which is less than or equal to

$$\begin{aligned}&\biggl[\biggl|\frac{1}{N^2}\sum_{k, j}\phi(x_k(s) - x_1(s), x_j(s) - x_1(s)) \\ &\quad - \int_0^1\int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\rho(s, x)\,dx\biggr| \\ &\quad + 2\biggl|\int_0^1\int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\rho(s, x)\,dx\biggr|\biggr];\end{aligned}$$

$$\sup_{0\le s\le t}\biggl[\biggl|\frac{1}{N^2}\sum_{k, j}\phi(x_k(s) - x_1(s), x_j(s) - x_1(s)) \tag{6.20}$$

$$- \int_0^1\int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\,dy\rho(s, x)\,dx\biggr|\biggr]^2 \tag{6.21}$$

is such that its expected value with respect to P^N goes to zero as N goes to 0.

The only conditions which have to be met are

$$\lim_{|\mathrm{supp}\{\phi\}|\to 0}\ \limsup_N E^N(E1) = 0 \tag{6.22}$$

with

$$(E1)=\int_0^t\left|\int_0^1\int_0^1\phi(x-x_1(s),\,y-x_1(s))\rho(s,y)\,dy\,\rho(s,x)\,dx\right|ds$$

for Proposition 12 and

$$\lim_{|\mathrm{supp}\{\phi\}|\to 0}\limsup_N E^N(E2)=0 \tag{6.23}$$

with

$$(E2)=\int_0^t\left|\int_0^1\int_0^1\phi(x-x_1(s),\,y-x_1(s))\rho(s,y)\,dy\,\rho(s,x)\,dx\right|dA_N^1(s)$$

for Proposition 13.

We shall leave these limits at the end of the proof. Assuming (6.22), (6.23) and (6.24) are true, the rest of the argument flows along the same type of estimates as before. We get the uniform version of the expression (6.21) that is, without $x_1(s)$ using the boundedness of $\triangledown\phi$ and we prove the limit as $N\to\infty$ (Theorem 1).

We want to bring down Proposition 14 to some estimate of the type (6.22) and (6.23). To do that we need to write

$$\begin{aligned}D_N^v:=\Bigg|\int_0^t\frac{1}{N}\sum_k\Bigg[f\Bigg(\frac{1}{N}\sum_j v(x_k(s)-x_1(s),\,x_j(s)-x_1(s))\Bigg)\,dA_N^k(s)\Bigg]\\-\int_0^t\int_0^1 f\Bigg(\int_0^1 v(x-x_1(s),\,y-x_1(s))\rho(s,y)\,dy\Bigg)\rho^2(s,x)\,dx\,ds\Bigg|\end{aligned}$$

and the analogue for g, D_N^g and consider their difference bounded by $\|f'\|$ times,

$$\begin{aligned}\Bigg|\int_0^t\frac{1}{N}\sum_k\Bigg[\frac{1}{N}\sum_j|v-g|(x_k(s)-x_1(s),\,x_j(s)-x_1(s))\,dA_N^k(s)\Bigg]\Bigg|\\+\Bigg|\int_0^t\int_0^1\int_0^1 v(x-x_1(s),\,y-x_1(s))\rho(s,y)\,dy\,\rho^2(s,x)\,dx\,ds\Bigg|\end{aligned}$$

and again bounded by the same expressions with $|v-g|\to\phi$.

We are mostly interested in the first expression. From Proposition 14 we already know that for ϕ smooth,

$$\begin{aligned}\lim_{N\to\infty}E^N\Bigg|\int_0^t\frac{1}{N}\sum_k\Bigg[\frac{1}{N}\sum_j\phi(x_k(s)-x_1(s),\,x_j(s)-x_1(s))\,dA_N^k(s)\Bigg]\\-\int_0^t\int_0^1\int_0^1\phi(x-x_1(s),\,y-x_1(s))\rho(s,y)\rho^2(s,x)\,dy\,dx\,ds\Bigg|=0.\end{aligned}$$

(This is exactly Proposition 20 with $f\to$ identity and $g\to\phi$.)

Given this fact we only have to prove the remainder of this approximation,

$$\lim_{|\mathrm{supp}\{\phi\}|\to 0}\limsup_N E^N(E3)=0 \tag{6.24}$$

with

$$(E3) = \int_0^t \left| \int_0^1 \int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\, dy \rho^2(s, x)\, dx \right| ds = 0.$$

Before integrating with respect to the time-variable in any of the three limits (6.22), (6.23) and (6.24), we concentrate on the xy-integral and choose a $p > 1$ yet to be determined and a corresponding $q = 1/(1 - 1/p)$ and then write down the Hölder inequality for each integral.

We obtain

$$\int_0^1 \int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\, dy \rho(s, x)\, dx \le |\mathrm{supp}\{\phi\}|^{1/q} \|\rho\|_{L^p}^2 \tag{6.25}$$

for (6.22), while

$$\int_0^1 \int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\, dy \rho^2(s, x)\, dx \tag{6.26}$$

is bounded above by less than or equal to $|\mathrm{supp}\{\phi\}|^{1/q} \|\rho\|_{L^p} \|\rho\|_{L^{2p}}^2$ for (6.24) and

$$\int_0^t \int_0^1 \int_0^1 \phi(x - x_1(s), y - x_1(s))\rho(s, y)\, dy \rho^2(s, x)\, dx\, dA_N^1(s) \tag{6.27}$$

by less than or equal to $|\mathrm{supp}\{\phi\}|^{1/q} \int_0^t \|\rho\|_{L^p}^2\, dA_N^1(s)$ for (6.23), as long as

$$\int_0^1 \int_0^1 \phi^q(x - x_1(s), y - x_1(s))\, dy\, dx \le |\mathrm{supp}\phi|$$

does not depend on N.

It is clear that if for $p \ge 1$ we denote $\zeta(s, p) := \|\rho(s, \cdot)\|_{L^p}$ the proof is concluded by the following estimates on the L^p norm of $\rho(s, x)$.

PROPOSITION 21.

$$\text{(6.28)(I)} \qquad \int_0^t \zeta^2(s, p)\, ds \le c_1(t)$$

corresponding to (6.22),

$$\text{(6.29)(II)} \qquad E^N \int_0^t \zeta^2(s, p)\, dA_N^1(s) \le c_2(t)$$

corresponding to (6.23) *and*

$$\text{(6.30)(III)} \qquad \int_0^t \zeta(s, p)\zeta^2(s, 2p)\, ds \le c_3(t)$$

for (6.24), *with all constants* $c_1(t)$, $c_2(t)$ *and* $c_3(t)$ *independent of* N.

PROOF. For any fixed $\delta > 0$, $g(s,x)$ is in $C^\infty([\delta,\infty), T^1)$, hence Proposition 21 can be restated as saying that for t fixed, on any interval $s \in [0,t]$, $\exists\, c > 0$ s.t.

$$s^{+(1/2)(1-1/p)}\|g(s,\cdot)\| \le c. \tag{6.31}$$

Now (I) and (III) are bounded by

$$\int_0^t \zeta^2(s,p)\,ds \le c^2\int_0^t s^{-(1-1/p)}\,ds \le \mathrm{const}\cdot t^{1/p} < \infty$$

and, respectively,

$$\begin{aligned}\int_0^t \zeta(s,p)\zeta^2(s,2p)\,ds &\le c^3\int_0^t s^{-[(1/2)(1-1/p)+(1-1/2p)]}\,ds\\ &\le \mathrm{const}\cdot t^{1/p-1/2} < \infty\end{aligned} \tag{6.32}$$

if $p < 2$. We shall see that $1 < p < 2$ is the only condition needed to ensure all limits.

As for (II), we have

$$E^N\int_0^t \zeta^2(s,p)\,dA_N^1(s) \le \mathrm{const}\cdot E^N\int_0^t s^{-(1-1/p)}\,dA_N^1(s).$$

We have to show that $E^N\int_0^t s^{-(1-1/p)}\,dA_N^1(s)$ is finite and independent of N,

$$\begin{aligned}\left|E^N\int_0^t s^{-(1-1/p)}\,dA_N^1(s)\right| &= \left|t^{-(1-1/p)}E^N[A_N^1(t)] - \lim_{\delta\to 0}\delta^{-(1-1/p)}E^N[A_N^1(\delta)]\right|\\ &\quad + \left|\left(1-\frac{1}{p}\right)\int_0^t E^N[A_N^1(s)]s^{-(1-1/p)-1}\,ds\right|\\ &\le K_1\left[t^{a+1/p-1} + \lim_{\delta\to 0}\delta^{a+1/p-1}\right] + K_2 t^{a+1/p-1},\end{aligned}$$

which is finite iff $a + 1/p - 1 > 0$, that is, $p < 1/(1-a)$ where $a \in (0,\frac{1}{2})$. We have the freedom to pick $a = \frac{1}{4}$ and hence we get $p < \frac{4}{3}$ which is comfortable enough. This shows that $p \in (1, 4/3)$.

Since we always can find a sequence of positive smooth functions ϕ, bounded by 1, with compact support shrinking to 0 and approximating $|g - v|$, Propositions 12, 13 and 14 are proved. □

The goal of proving Propositions 12, 13 and 14 was to establish the corresponding limits for $f = \gamma_\varepsilon''$. Even though γ_ε is not smooth, its second derivative may be assumed to be zero on a neighborhood of the origin, hence smooth everywhere. The terms corresponding to the values $k = 1$ either as $0+$ or $0-$ appear distinctly written in the formulas $(D2)$ (5.7) and $(D3)$ (5.8).

6.6. *Proof of Propositions* 15, 16 *and* 17. The limits we are interested in are clarified by the remark that $\lim_{\varepsilon\to 0}\gamma_\varepsilon'' = -2\delta_0$ (in the sense of distributions) and the change of variable $u = u(x)$, with u defined as follows.

DEFINITION 18. For each $s \geq 0$, let

$$u_s(x) = \lambda x + \int_0^x \rho(s, y + x_1^N(s))\,dy. \tag{6.33}$$

The function $v(x, y) = \mathbf{1}_{[y\leq x]}(x, y) + (\lambda/\bar\rho)x$ appears in the argument of γ and the integration

$$\int_0^1 v(x - x_1^N(s), y - x_1^N(s))\rho(s, y)\,dy = \lambda x + \int_0^x \rho(s, y + x_1^N(s))\,dy$$

makes it clear that one can perform the change of variable $u_s = u_s(x)$ since $u_s'(x) = \lambda + \rho(s, x + x_1^N(s)) > 0$.

In the following proofs we shall omit the superscript N on top of x_1.

DEFINITION 19. Let $\pi_s := u_s^{-1}$, that is, $\pi_s(u_s) = x$.

It is clear that $u_s \in [0, \lambda + \bar\rho]$ and $\pi_s(0) = 0$. We shall omit the subscript "s" from u and π while we carry out the integration with respect to y. By changing the variable $u = u(x)$ we shall write Propositions 15, 16, 17 as

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\left|\int_0^t\int_0^{\lambda+\bar\rho}\gamma_\varepsilon''(u)[a_i(x + x_1(s)) - a_i(x_1(s))]\,du\,ds\right| = 0 \tag{6.34}$$

(for $i = 1, 2$) and

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\bigg|\int_0^t\int_0^{\lambda+\bar\rho}&\gamma_\varepsilon''(u)\\ &\times\big[a_1(x + x_1(s)) - a_1(x_1(s))\big]\,du\,dA^1(s)\bigg| = 0,\end{aligned} \tag{6.35}$$

where $a_1(s, x) = \rho(s, x)/(\lambda + \rho(s, x))$ and $a_2(s, x) = \rho^2(s, x)/(\lambda + \rho(s, x))$.

We shall state (6.34) and (6.35) with the help of the functions

$$b_i(s, u, x_1) := a_i(s, \pi(u) + x_1)$$

for $i = 1, 2$, just to emphasize the presence of the argument u. They become

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\bigg|\int_0^t\int_0^{\lambda+\bar\rho}&\gamma_\varepsilon''(u)\\ &\times[b_i(s, u, x_1(s)) - b_i(s, 0, x_1(s))]\,du\,ds\bigg| = 0\end{aligned} \tag{6.36}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\left|\int_0^t\int_0^{\lambda+\bar\rho}\gamma_\varepsilon''(u)\right. \tag{6.37}$$
$$\left.\times[b_1(s,u,x_1(s))-b_1(s,0,x_1(s))]\,du\,dA^1(s)\right|=0,$$

for $i=1,2$.

The idea of the proof is to pick an arbitrary $\delta\in(0,t)$ and separate the integrals $\int_0^t H_i(s)\,dr(s)=\int_0^\delta H_i(s)\,dr(s)+\int_\delta^t H_i(s)\,dr(s)$ for $i=1,2$ with any one of the choices for $r(t)$ as either identical to t or equal to $A_N^1(t)$ and

$$H_i(s):=\left|\int_0^{\lambda+\bar\rho}\gamma_\varepsilon''(u)[b_i(s,u,x_1(s))-b_i(s,0,x_1(s))]\,du\right|.$$

A change of variable $w:=u/\varepsilon$, remembering that $\gamma_\varepsilon(u)=(1/\varepsilon)\gamma(u/\varepsilon)$ and $\sup_u|\gamma''(u)|\le 3/2$ makes us able to rely on the fact that as long as the integrand b_i is of class C^1 in u the difference

$$\int_0^{\lambda+\bar\rho}\gamma_\varepsilon''(u)\big|b_1(s,u,x_1(s))-b_1(s,0,x_1(s))\big|\,du$$
$$=\int_0^{\lambda+\bar\rho}\gamma''(w)\big|b_1(s,\varepsilon w,x_1(s))-b_1(s,0,x_1(s))\big|\,dw$$

is uniformly bounded in N by a constant $C(\delta)$ depending on δ times ε. It is known that $E^N[r(t)]$ is uniformly bounded in N; these facts take care of the the limit as $s\ge\delta$.

We only have to prove the uniform boundedness of

$$\sup_{s,u,x_1}\left|\left(\frac{\partial b_i}{\partial u}\right)(s,u,x_1)\right|;$$

this is implied by

$$\frac{\partial b_i}{\partial u}(s,u,x_1)=\frac{\partial b_i}{\partial x}(s,u,x_1)\left[\frac{\partial u}{\partial x}(s,x,x_1)\right]^{-1}$$

since $|(\partial b_i/\partial x)(s,u,x_1)|\le C_2(\delta)$ and $|(\partial u/\partial x)(s,x,x_1)|\ge\lambda$.

The other term (when $s\le\delta$) can be bounded in a more crude way by taking advantage of the fact that (1) $|b_1|\le 1$ and (2) $\sup_{s,u,x_1}|b_2(s,u,x_1)|\le \text{const}\cdot t^{-1/2}$ as a consequence of Proposition 11 for $p=\infty$. It will be less than $2\cdot\text{const}E^N\int_0^\delta ds$ for (6.22), than $2\cdot\text{const}E^N\int_0^\delta dA_N^1(s)$ for (6.23), and than $2\cdot\text{const}E^N\int_0^\delta s^{-1/2}\,ds$ for (6.24), all bounded by constant $\cdot\,\delta^a$, $a\in(0,\frac12)$. As $\lim_{\delta\to0}\lim_{\varepsilon\to0}\limsup_{N\to\infty}\text{constant}\cdot\delta^a=0$ the proof of (6.22), (6.23) and (6.24) is complete. □

Before proving Theorem 2 we need to prove an intermediary result.

DEFINITION 20. Let

$$B_N(s, \omega) := B_N(s) = \lambda\left(\frac{2\lambda + \rho(s, x_1^N(s))}{\lambda + \rho(s, x_1^N(s))}\right).$$

PROPOSITION 22. *The limit of*

$$E^N\left|\int_0^t [B_N(s)]\, dA_N^1(s) - \int_0^t \rho(s, x_1^N(s))[B_N(s)]\, ds\right| \tag{6.38}$$

is zero as $N \to \infty$.

PROOF. The proof is a consequence of all the estimates shown in the preceding section. We shall rewrite the differential formulas from Section 5 for the slightly modified test function $F_\varepsilon^N(t) = (1/N)\sum_{2\le k\le n} \gamma_\varepsilon(r_k^g(t))$, where

$$r_k^g(t) := \frac{1}{N}\sum_{j=1}^{n} g(x_k(t) - x_1(t), x_j(t) - x_1(t)) = q_k^g(t) + \frac{1}{N}$$

for some function $g(x)$. This will provide a symmetrized version of the calculations obtained in Section 5. The modification will not change our estimates because for g smooth one has

$$\left|\frac{1}{N}\sum_{2\le k\le n} g(q_k) - \frac{1}{N}\sum_{2\le k\le n} g(r_k)\right| \le \sup_{x\in[0,1]} |g'(x)|\frac{n-1}{N^2}$$

clearly $O(1/N)$.

Proposition 22 is a consequence of several estimates based mainly on Propositions 12, 13 and 14.

Let us recall the expressions $(D1), (D2), \dots, (D7)$ from the differential formula calculated in (5.5) through (5.12) as well as the notations (5.16) and (5.17). In the formulas $(D1)$ to $(D7)$, whenever we integrate against dA^{ji}, $\sigma(i) = \sigma(j) + 1$ and whenever we integrate against dA^{ij}, $\sigma(i) = \sigma(j) - 1$ as long as i and j are different from 1.

We also recall the four estimates from Section 4, namely, Estimates 1–4 [(4.3) through (4.6)]. In this proof we shall say that a quantity $Z_N^1(t, \omega)$ is asymptotically equal to $Z_N^2(t, \omega)$ if $\lim_{N\to\infty} E^N|Z_N^1(t, \omega) - Z_N^2(t, \omega)| = 0$.

STEP 1. Proposition 10 shows that $\int_0^t [(D1) + (D2) + \cdots + (D7)](s)$ is negligible.

STEP 2. We shall separate the expression $(D1) + (D2) + \cdots + (D7)$ into three parts,

$$(\text{Part } 1) := \left[\frac{\lambda^2}{N}\sum_k \gamma''(q_k(t))\right] dt, \tag{6.39}$$

$$(\text{Part } 2) = (\text{Part } 2)' + (\text{Part } 2)'' \tag{6.40}$$

with the notation

$$(\text{Part } 2)' = \frac{\lambda}{N}\sum_k \Big\{ dA^{1k}(t)\Big[-2\gamma'\Big(\frac{n-1}{N}+\lambda\Big) + \sum_{j\neq k}(-\gamma'(q_j(t)) + N[\gamma(q'_j)-\gamma(q_j)])\Big]\Big\}$$

and

$$(\text{Part } 2)'' = \frac{\lambda}{N}\sum_k \Big\{ dA^{k1}(t)\Big[+2\gamma'\Big(\frac{1}{N}\Big) + \sum_{j\neq k}(+\gamma'(q_j(t)) + N[\gamma(q''_j)-\gamma(q_j)])\Big]\Big\}.$$

The last part is

$$(\text{Part } 3) := \frac{\lambda}{2N}\sum_{i,\, j\neq 1} dA^{ij}[\gamma'(q_i(t)) - \gamma'(q_j(t))] + \frac{\lambda}{2N}\sum_{i,\, j\neq 1} dA^{ji}[\gamma'(q_j(t)) - \gamma'(q_i(t))].$$

STEP 3. We can substitute the integral form of (Part 1) with

$$\int_0^t \frac{\lambda^2}{N}\Big[\sum_k \gamma''(r_k(s))\Big]\, ds$$

because their difference is clearly $O(1/N)$.

STEP 4. At this point we shall write down the Taylor expansion of the function γ about the point $q_j(t)$. We mention that at any such point in the summation γ is smooth. The critical point $0 = \lambda + \bar{\rho}$ (on the circle of radius $\lambda+\bar{\rho}$) is avoided, since exactly the terms achieving the endpoints of the interval are computed separately,

$$-\gamma'(q_j) + N(\gamma(q'_j) - \gamma(q_j)) = \frac{1}{2N}\gamma''(q_j) + \frac{1}{6N^2}\gamma'''(\tilde{q}_j)$$

and

$$+\gamma'(q_j) - N(\gamma(q''_j) - \gamma(q_j)) = \frac{1}{2N}\gamma''(q_j) - \frac{1}{6N^2}\gamma'''(\tilde{\tilde{q}}_j),$$

where $\tilde{q}_j$ and $\tilde{\tilde{q}}_j$ are points in $[0, \lambda+\bar{\rho}]$. The errors are clearly negligible as $N\to\infty$; here we recall the estimates (4.3) to (4.6).

This implies that (Part 2) (in integral form) can be replaced by the sum of

$$\int_0^t \frac{\lambda}{2}\sum_k \Big\{ d\Big[\frac{1}{N}A^{k1}(s)\Big]\Big[2 + \frac{1}{N}\sum_{j\neq k}\gamma''(q_j(s))\Big]\Big\}$$

and

$$\int_0^t \frac{\lambda}{2}\sum_k\left\{d\left[\frac{1}{N}A^{1k}(s)\right]\left[2+\frac{1}{N}\sum_{j\neq k}\gamma''(q_j(s))\right]\right\},$$

since their difference is also $O(1/N)$. Once again we can substitute all this by

$$\int_0^t \frac{\lambda}{2}\left\{dA_N^1(s)\left[2+\frac{1}{N}\sum_{j\neq k}\gamma''(q_j(s))\right]\right\}.$$

Finally, this last term is asymptotically equal to

$$\int_0^t \frac{\lambda}{2}\left\{dA_N^1(t)\left[2+\frac{1}{N}\sum_{j\neq k}\gamma''(r_j(t))\right]\right\}$$

because γ'' is smooth and $\gamma''(q_k)-\gamma''(r_k)$ is $O(1/N)$ by Taylor's formula. This makes the error of the same order as $(1/N)E^N[A_N^1(t)]$, a negligible quantity (as well as its square) in agreement with the estimates obtained in Section 4.

STEP 5. Here we check that

$$\int_0^t \frac{\lambda}{2N}\sum_{i,\,j\neq 1} dA^{ij}(s)[\gamma'(q_i(s))-\gamma'(q_j(s))]$$

may be substituted with

$$\int_0^t \frac{\lambda}{2N}\sum_{i,\,j\neq 1} d\left[\frac{1}{N}A^{ij}(s)\right][\gamma''(q_j(s))],$$

also asymptotically equal to

$$\int_0^t \frac{\lambda}{2N}\sum_j \gamma''(q_j(s))\,dA_N^{j,\,\mathrm{left}}(s).$$

In the same way,

$$\int_0^t \frac{\lambda}{2N}\sum_{i,\,j\neq 1} dA^{ji}(s)[\gamma'(q_j(s))-\gamma'(q_i(s))]$$

may be substituted with

$$\int_0^t \frac{\lambda}{2N}\sum_{i,\,j\neq 1} d\left[\frac{1}{N}A^{ji}(s)\right][\gamma''(q_i(s))],$$

once again asymptotically equal to

$$\int_0^t \frac{\lambda}{2N}\sum_i \gamma''(q_i(s))dA_N^{i,\,\mathrm{right}}(s).$$

Clearly, the indices i and j are dummy variables implying that the integral of $(D6)+(D7)$ is asymptotically equal to

$$\int_0^t \frac{\lambda}{2N}\sum_i \gamma''(q_i(s))\,dA_N^i(s).$$

To conclude the proof we only need to remember that the three expressions we have reduced the time integral of $(D1) + \cdots + (D7)$ to are the object of Propositions 15, 16 and 17. □

6.7. *Proof of Theorem* 2. It has been shown that $\{P^N \circ A_N^1(t)^{-1}\}_N$ is tight. Moreover, this implies that $\{P^N \circ x_1(t)^{-1}\}_N$ is also tight. Let's suppose $x_1(\cdot)$ is a limit point of $\{x_1^N(\cdot)\}_{N>0}$ and $A^1(\cdot)$ is a limit point of $\{A_N^1(\cdot)\}_{N>0}$ such that they are limit points of the two tight sequences over the same subsequence (still denoted by N for simplicity); that is, there is a measure $Q^{(x_1, A^1)}$ over Ω such that

$$P^N \circ (x_1(t), A_N^1(t))^{-1} \Longrightarrow Q^{(x_1, A^1)}. \tag{6.41}$$

Our strategy is to prove that any limit point $Q^{(x_1, A^1)}$ has some properties which will determine it uniquely.

We want to prove that if $Q^{(x_1, A^1)}$ is a limit distribution of $(x_1^N(\cdot), A_N^1(\cdot))$ then $A^1(t) = \int_0^t \rho(s, x_1(s))\, ds$ $Q^{(x_1, A^1)}$-almost surely, or equivalently,

$$\lim_{N\to\infty} E^N \left| A_N^1(t) - \int_0^t \rho(s, x_1^N(s))\, ds \right| = 0. \tag{6.42}$$

First we define the measure space we are concerned with, denoted by $X = \Omega_T \times V_T$, where we set $\Omega_T = C([0, T], R)$ and the V_T space defined by

$$\big\{L\colon [0, T] \to R\colon \text{nondecreasing, continuous at } 0 \\ \text{with } L(0) = 0 \text{ and } L(T) < \infty\big\}$$

with the product norm of the uniform norm on Ω_T and the total variation for V_T. The object we study is a real functional U on X,

$$U(\omega, L) := \int_0^T B(s, \omega(s))\, dL(s) - \int_0^T \rho(s, \omega(s)) B(s, \omega(s))\, ds.$$

We want to show that

$$\lim_{N\to\infty} E^N \big|U(x_1^N(\cdot), A_N^1(\cdot))\big| = E^{Q^{(x_1, A^1)}} |U|. \tag{6.43}$$

The measures $\Gamma^N := P^N \circ (x_1^N(\cdot), A_N^1(\cdot))^{-1}$ are concentrated on X from Proposition 9. The proof consists in showing that U is continuous and that uniform integrability holds for U and the measures Γ^N in the sense that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \int_{L(T)>M} U\, d\Gamma^N = 0.$$

This property is needed because the functional U is not bounded. It is warranted by the uniform bound on the L^2 norm of $A_N^1(T)$ given in the estimate (4.4).

The only fact to prove is the continuity of U. Let us set an arbitrary division of the interval $[0, T]$ in two parts $[0, \delta]$ and $[\delta, T]$. On the second interval, the functions $\rho(s, x)$ and $B(s, x)$ are smooth and bounded, so we are done. On the interval $[0, \delta]$,

$$\rho(s, \omega(s))B(s, \omega(s)) < 2\lambda\rho(s, \omega(s)) < \text{const}\frac{1}{\sqrt{s}},$$

which yields a term $O(\sqrt{\delta})$ by integration. In the same time $B(s, \omega(s)) \leq 2\lambda$, so the whole variation of the first integral in U is bounded by const $\cdot\, L(\delta)$, once again negligible as $\delta \to 0$.

Then $B(s)$ is bounded above and below, that is, $0 < \lambda \leq B(s) \leq 2\lambda$; because of the presence of the absolute value in (6.43) it is clear that

$$\int_0^t B(s)\, dA^1(s) = \int_0^t \rho(s, x_1(s))B(s)\, ds$$

$Q^{(x_1, A^1)}$-almost surely. Hence for almost all ω in the probability space the positive and finite measures $B(s)\, dA^1(s) \equiv \rho(s, x_1(s)B(s)\, ds$. Since $B(s) > 0$, this implies that the positive and finite measures $dA^1(s)$ and $\rho(s, x_1)(s)\, ds$ are equivalent $Q^{(x_1, A^1)}$-almost everywhere. □

7. The asymptotic independence. This section is dedicated to the proof of Theorem 5. We shall assume that the initial profile has bounded density $\rho_0(x)$.

The proof requires a few results. We have to recall the two pairs of processes $z_i^N(\cdot)$ and $y_i^N(\cdot)$ for $i = 1, 2$ defined in the Definitions 8 and 11 in Section 3.1. So

$$z_i^N(t) = x_i^N(t) + \frac{1}{\lambda + \bar{\rho}} \frac{1}{N} \sum_{k \neq i} \nu(x_k^N(t) - x_i^N(t))$$

and

$$y_i^N(t) = x_i^N(t) + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y - x_i^N(t))\rho(t, y)\, dy,$$

that is, $y_i^N(t) = F(t, x_i^N(t))$ with

$$F(t, x) = x + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y - x)\rho(t, y)\, dy, \tag{7.1}$$

for $i = 1, 2$.

Proposition 2 tells us that $z_i^N(t) - z_i^N(0)$ is a martingale w.r.t. the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ of the interaction process $\{x^N(\cdot)\}_{t \geq 0}$ and the probability measure P^N. We actually know that

$$\begin{aligned}(\text{Mart}) \sim z_i^N(t) - z_i^N(0) &= \left[1 - \frac{n-1}{N(\lambda + \bar{\rho})}\right]\beta_i(t) \\ &\quad + \frac{1}{\lambda + \bar{\rho}} \frac{1}{N} \sum_{k \neq i} \beta_k(t) + \frac{1}{\lambda + \bar{\rho}} \frac{1}{N} \sum_{k \neq 1} [M^{ki}(t) - M^{ik}(t)]\end{aligned} \tag{7.2}$$

[see (2.4)].

We now plan to check the cross-variation process [6] of $z_1^N(\cdot)$ and $z_2^N(\cdot)$. Let us finally recall that these two martingales are tight and Theorem 6 implies the existence of two martingales $z_1(\cdot)$ and $z_2(\cdot)$, such that $z_1^N(\cdot) \Rightarrow z_1(\cdot)$ and $z_2^N(\cdot) \Rightarrow z_2(\cdot)$.

LEMMA 7. *The two martingales* $\{z_1^N(t) - z_1^N(0)\}_{t\geq 0}$ *and* $\{z_2^N(t) - z_2^N(0)\}_{t\geq 0}$ *are orthogonal, or equivalently,*

$$\lim_{N\to\infty} E^N\big[(z_1^N(t) - z_1^N(0))(z_2^N(t) - z_2^N(0))\big] = 0.$$

PROOF. All martingales in formula (Mart) (7.2) are mutually orthogonal, hence

$$E^N\big[(z_1^N(t) - z_1^N(0))(z_2^N(t) - z_2^N(0))\big] = C_\beta t + C_A\big[A_N^{12}(t) + A_N^{21}(t)\big],$$

where

$$C_\beta = \left[1 - \frac{n-1}{N(\lambda+\bar\rho)}\right]\frac{1}{\lambda+\bar\rho}\frac{1}{N}$$

and

$$C_A = \left(\frac{1}{\lambda+\bar\rho}\right)^2 \frac{1}{N^2}(\lambda N).$$

The proof reduces to the lemma.

LEMMA 8.

$$\lim_{N\to\infty} E^N\left(\frac{1}{N}[A_N^{12}(t) + A_N^{21}(t)]\right) = 0. \tag{7.3}$$

PROOF. We shall use once again the test function $\gamma_\varepsilon(\cdot)$ (4.1) from Section 4, simply denoted by $\gamma(\cdot)$. The differential formula for

$$F_N(t) := \frac{1}{N}\gamma\left(\frac{\sigma(2)}{N} + \lambda(x_2(t) - x_1(t))\right) \tag{7.4}$$

will be split into nine parts described below. To simplify things we are going to denote $(\sigma(2)/N + \lambda(x_2(t) - x_1(t)))$ by q and use the old notation $\gamma_{1,n-1}$ for $(\gamma(1/N) - \gamma((n-1)/N))$. It is true that $\gamma_{1,n-1} = 0$ but I will write it down in the formulas temporarily for clarity purposes:

$$(d1) = \frac{1}{N}\lambda^2\gamma''(q)\,dt,$$

$$(d2) = \frac{1}{N}\left(-2\lambda\gamma'\left(\frac{n-1}{N} + \lambda\cdot 1\right) + (\lambda N)\gamma_{1,n-1}\right) dA_N^{12}(t),$$

$$(d3) = \frac{1}{N}\left(+2\lambda\gamma'\left(\frac{1}{N}\right) + (\lambda N)[-\gamma_{1,n-1}]\right) dA_N^{21}(t),$$

$$(d4) = \sum_{k \neq 1,2} d\left[\frac{1}{N} A_N^{1k}(t)\right]\left\{-\lambda\gamma'(q) + (\lambda N)\left[\gamma\left(q + \frac{1}{N}\right) - \gamma(q)\right]\right\},$$

$$(d5) = \sum_{k \neq 1,2} d\left[\frac{1}{N} A_N^{k1}(t)\right]\left\{+\lambda\gamma'(q) + (\lambda N)\left[\gamma\left(q - \frac{1}{N}\right) - \gamma(q)\right]\right\},$$

$$(d6) = \sum_{k \neq 1,2} d\left[\frac{1}{N} A_N^{2k}(t)\right]\left\{-\lambda\gamma'(q) + (\lambda N)\left[\gamma\left(q + \frac{1}{N}\right) - \gamma(q)\right]\right\},$$

$$(d7) = \sum_{k \neq 1,2} d\left[\frac{1}{N} A_N^{k2}(t)\right]\left\{+\lambda\gamma'(q) + (\lambda N)\left[\gamma\left(q - \frac{1}{N}\right) - \gamma(q)\right]\right\},$$

$$(d8) = \frac{1}{N} \sum_{i,\, j \neq 1,2} dA_N^{ij}(t)[\text{identically equal to } 0]$$

$$(d9) = \{\text{Martingale term}\}.$$

It is noticeable that since γ is a smooth function all over the place except the endpoints, which are treated separately in $(d2)$ and $(d3)$, we can assume that $\gamma^{(l)}$, $l = 0, 1, 2$ are bounded by a constant M.

Another remark is that our estimates are given for a fixed T, $t \leq T$ and we actually operate *in integral form*. The left-hand side term $F_N(q(t)) - F_N(q(0))$ is $O(1/N)$, hence negligible as $N \to \infty$. The same is true about $(d1)$.

The pairs $(d4)$, $(d5)$ and $(d6)$, $(d7)$ can be treated in perfect analogy. We shall only estimate one of them: $(d8)$ is identically zero and the martingale term is irrelevant since we consider the expected value.

The Taylor formula for γ about q shows that

$$-\gamma(q) + N\left(\gamma\left(q + \frac{1}{N}\right) - \gamma(q)\right) = \frac{1}{2N}\gamma(\tilde{q}),$$

where $\tilde{q}$ belongs to $(0, \lambda + \bar{\rho})$, hence is less than or equal to M. This proves that $E^N(d4)$ is $O(1/N)$ in accordance with (4.5). Similarly $E^N(d5)$ is $O(1/N)$.

$(d2)$ is in fact equal to $\lambda[(1/N)A_N^{12}(t)]$ + Error $\cdot$ $[(1/N)A_N^{12}(t)]$ while the Error is essentially obtained by the difference $\gamma'(\lambda + \bar{\rho} - 1/N) - (-1)$, naturally, $O(1/N)$. [Again (4.5)] This proves that $E^N[(1/N)A_N^{12}(t)] \to 0$ as $N \to \infty$ and almost identically $E^N[(1/N)A_N^{21}(t)] \to 0$ as $N \to \infty$. □

At this point we see it is sufficient to show the following.

LEMMA 9. *$\{z_1(\cdot)\}$ and $\{z_2(\cdot)\}$ are independent.*

PROOF. Theorem 6 and Theorem 7 from Section 3.1 show that for $i = 1, 2$ the process $z_i^N(\cdot)$ converges weakly to a diffusion $z_i(\cdot)$ with generator

$$\mathscr{L}^z = \frac{1}{2} \frac{\lambda(\lambda + \rho(t, G(t, z)))}{(\lambda + \bar{\rho})^2} \frac{d^2}{dz^2}. \tag{7.5}$$

We can restate this by saying that there exist two Brownian motions $W_1(\cdot)$ and $W_2(\cdot)$, adapted to $\{\mathscr{F}_t\}_{t\geq 0}$ such that if we denote by

$$\sigma(s, z) := \left(\frac{\lambda(\lambda + \rho(t, G(t, z)))}{(\lambda + \bar{\rho})^2} \right)^{1/2},$$

then

$$z_i(t) - z_i(0) = \int_0^t \sigma(s, z_i(s))\, dW_i(s), \qquad i = 1, 2. \tag{7.6}$$

Since the diffusion coefficient is bounded below, that is, $\sigma(s, z) \geq (\lambda/(\lambda + \bar{\rho})) > 0$, we may express the two Brownian motions $W_i(\cdot)$ in terms of $z_i(\cdot)$ and $\tilde{\sigma}(s, z) := [\sigma(s, z)]^{-1}$ as

$$W_i(t) - W_i(0) = \int_0^t \tilde{\sigma}^i(s, z_i(s))\, dz_i(s),$$

which implies that $W_1(\cdot)$ and $W_2(\cdot)$ are orthogonal. Two orthogonal Brownian motions are independent. It is easy to see that, applying (7.6), we may conclude that $z_1(\cdot)$ and $z_2(\cdot)$ are independent. □

PROOF OF THEOREM 5. There is a natural relation between the measures P^{z_i} and the measures Q^{x_i} where $z_i = F(0, x_i)$, $i = 1, 2$.

We have proved that $z_1(\cdot)$ and $z_2(\cdot)$ are independent. On the pairs of sets

$$\Omega^{z_i} = \{\eta \in C([0, T], R) \text{ with } \eta(0) = z_i\}$$

and

$$\Omega^{x_i} = \{\omega \in C([0, T], R) \text{ with } \omega(0) = x_i\},$$

the mapping $\Theta\colon \Omega^{x_i} \to \Omega^{z_i}$ defined by $(\Theta\omega)(t) = F(t, \omega(t))$ is one-to-one and onto. Lemmas 2 and 3 from Section 3 imply that if Q^{x_i} is the tagged particle process starting at x_i, then $Q^{x_i} = P^{z_i} \circ \Theta$, $i = 1, 2$. It is easy to see from here that $x_1(\cdot)$ and $x_2(\cdot)$ are also independent. □

APPENDIX

A.1. *Proof of Lemma* 10.

LEMMA 10. *For $f(t, x)$ smooth and $A_N^k(t)$ defined in* (1.10) *we have the limit*

$$\lim_{N\to\infty} E^N \left| \int_0^t \frac{1}{N} \sum_k f(s, x_k(s))\, dA_N^k(s) - \int_0^t \int_0^1 f(s, y)\rho^2(s, y)\, dy\, ds \right| = 0. \tag{A.1}$$

PROOF. We recall the test function (4.2) $\alpha_\varepsilon = \alpha$ for simplicity and we take

$$G_k^{N,\varepsilon}(t) = G_k(t) = \frac{1}{N}\sum_{j\neq k}\alpha(x_j(t) - x_k(t)). \tag{A.2}$$

We start writing the differential formulas,

$$\begin{aligned} dG_k(t) &= \frac{1}{N}\sum_{j\neq k}\alpha''(x_j(t) - x_k(t))\,dt \\ &\quad + \frac{1}{N}\sum_i dA^{ik}(t)\bigg[\sum_{j\neq k}\alpha'(x_j(t) - x_k(t)) + 2\alpha'(0+)\bigg] \\ &\quad + \frac{1}{N}\sum_i dA^{ki}(t)\bigg[-\sum_{j\neq k}\alpha'(x_j(t) - x_k(t)) - 2\alpha'(1-)\bigg] + d\mathscr{M}_{G_k}(t). \end{aligned}$$

We can isolate the total local time for the particle #k,

$$2\frac{1}{N}d\sum_i[A^{ki}(t) + A^{ik}(t)] = 2\cdot A_N^k(t) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}),$$

where

$$\begin{aligned} (\mathrm{I}) &= dG_k(t), \\ (\mathrm{II}) &= -\frac{1}{N}\sum_{j\neq k}\alpha''(x_j(t) - x_k(t))\,dt, \\ (\mathrm{III}) &= \sum_i(dA^{ik}(t) - dA^{ki}(t))\sum_{j\neq k}\alpha'(x_j(t) - x_k(t)), \\ (\mathrm{IV}) &= d\mathscr{M}_{G_k}(t) \end{aligned}$$

with i, j, k distinct.

To write down the summation in Lemma 10, we multiply each

$$2\frac{1}{N}d\sum_i[A^{ki}(t) + A^{ik}(t)] = 2\cdot dA_N^k(t)$$

by $f(t, x_k(t))$, we sum over all k and divide by N.

The result is the sum of four terms corresponding to the formulas given above written now in integral form,

$$(\mathrm{a}) = \int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\,dG_k(s), \tag{A.3}$$

$$(\mathrm{b}) = \int_0^t \frac{1}{N^2}\sum_{k,j} f(s, x_k(s))\alpha''(x_j(s) - x_k(s))\,ds, \tag{A.4}$$

$$(\mathrm{c}) = \int_0^t \frac{1}{N^2}\sum_{i,k,j} f(s, x_k(s))\alpha'(x_j(s) - x_k(s))\,d(A^{ki}(s) - A^{ik}(s)), \tag{A.5}$$

$$(\mathrm{d}) = \int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\,d\mathscr{M}_{G_k}(s). \tag{A.6}$$

In order to show that

$$\lim_{N\to\infty} E^N\left|\int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\, dA^k(s) - \int_0^t\int_0^1 f(s, y)\rho^2(s, y)\, dy\, ds\right| = 0,$$

we shall prove the two limits

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\left|\int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\, dA^k(s) + \frac{1}{2}(\mathrm{b})\right| = 0 \tag{A.7}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N|(\mathrm{a}) + (\mathrm{c}) + (\mathrm{d})| = 0. \tag{A.8}$$

We can do a little bit more: the iterated limits for $E^N|(\mathrm{a})|$, $E^N|(\mathrm{c})|$ and $E^N|(\mathrm{d})|$ are zero. The proof will be complete by showing that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\left|\frac{1}{2}(\mathrm{b}) + \int_0^t\int_0^1 f(s, y)\rho^2(s, y)\, dy\, ds\right| = 0. \tag{A.9}$$

Proof of the limit (c). We shall suppress the "s" temporarily; it does not matter in the algebra below. We denote $f_k = f(s, x_k(s))$ and $\alpha'_{jk} = \alpha(x_j(s) - x_k(s))$,

$$\sum_{i,j,k} f_k\alpha'_{jk}\, dA^{ki} = \sum_{i,j,k} f_i\alpha'_{ji}\, dA^{ik}$$

by changing the order of summation (this computation is valid for a fixed j) and

$$\sum_{i,j,k} f_i\alpha'_{ji}\, dA^{ik} = \sum_{i,j,k} f_k\alpha'_{jk}\, dA^{ik}$$

because we integrate against dA^{ik} which is nonzero only where $x_k(s) = x_i(s)$, hence (c) is identically 0.

Proof of the limit (d). The martingale term is

$$\int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\, d\mathscr{M}_{G_k}(s)$$

and

$$\begin{aligned} d\mathscr{M}_{G_k}(s) &= \frac{1}{N}\sum_{j,\, j\neq k}[\alpha'(x_j(s) - x_k(s))\, d\beta_j(s)] \\ &\quad - \left[\frac{1}{N}\sum_{j,\, j\neq k}\alpha'(x_j(s) - x_k(s))\right] d\beta_k(s); \end{aligned}$$

the coefficients of β_l, $l = 1, \dots, n$ are

$$B_l^N(s) = \frac{1}{N^2}\sum_k f(s, x_k(s))\alpha'(x_l(s) - x_k(s)) - \frac{1}{N^2}\sum_k f(s, x_l(s))\alpha'(x_k(s) - x_l(s)).$$

Here $\beta_l(\cdot)$ are mutually orthogonal hence the expected value of the square is less than $\sum_l |\sup B_l^N(s)|^2\, dt$ which is clearly $O(1/N)$.

PROOF OF THE LIMIT (a).

$$\begin{aligned}\int_0^t \frac{1}{N}\sum_k f(s, x_k(s))\, dG_k(s) = {}& \frac{1}{N}\sum_k f(t, x_k(t))G_k(t) \\ &- \frac{1}{N}\sum_k f(0, x_k(0))G_k(0) \\ &- \int_0^t \frac{1}{N}\sum_k G_k(s)\, df(s, x_k(s))\end{aligned} \tag{A.10}$$

by integration by parts. The first two terms vanish as $\varepsilon \to 0$ uniformly in N.

The other term can be computed by writing

$$\begin{aligned}df(t, x_k(t)) = {}& \left(\partial_t f(t, x_k(t)) + \tfrac{1}{2} f''(t, x_k(t))\right) dt \\ &+ f'(t, x_k(t))\sum_i (dA^{ik}(t) - dA^{ki}(t)) + f'(s, x_k(s))\, d\beta_k(s).\end{aligned}$$

The summation over all k produces a dt term, a martingale term,

$$\frac{1}{N}\sum_k f'(s, x_k(s))G_k(s)\beta_k(s)$$

and the term

$$\sum_{i,k} f'(t, x_k(t))G_k(t)\, dA^{ik}(t) - \sum_{i,k} f'(t, x_k(t))G_k(t)\, dA^{ki}(t).$$

The last part is identically zero for similar reasons as in the proof of (c). The conclusion is proved since the integrand is less than or equal to ε and the total variation of

$$\frac{1}{N}\sum_k \left|\partial_t f(t, x_k(t)) + \tfrac{1}{2} f''(t, x_k(t))\right| dt$$

is uniformly bounded in N (for the dt integral) while the quadratic variation of the martingale part is $O(1/N)$.

PROOF OF THE LIMIT (A.9). The function α''_ε is smooth and we know from Theorem 2 that

$$\lim_{N\to\infty} E^N\bigg|\int_0^t \frac{1}{N^2}\sum_{k,j} f(s,x_k(s))\alpha''_\varepsilon(x_j(s)-x_k(s))\,ds - \int_0^t\int_0^1\int_0^1 f(s,y)\alpha''_\varepsilon(x-y)\rho(s,x)\rho(s,y)\,dy\,dx\,ds\bigg| = 0;$$

therefore, we only need to prove that

$$\lim_{\varepsilon\to 0}\bigg|\int_0^t\int_0^1\int_0^1 f(s,y)\alpha''_\varepsilon(x-y)\rho(s,x)\rho(s,y)\,dy\,dx\,ds + 2\cdot\int_0^t\int_0^1 f(s,y)\rho^2(s,y)\,dy\,ds\bigg| = 0.$$

We change the variable $z = x - y$ and $y = y$. We remember again the construction of $\alpha''_\varepsilon(x) = \varepsilon^{-1}\alpha''(\varepsilon^{-1}x)$ and change the variable to $w = \varepsilon^{-1}z$ and $y = y$. The result is the integral

$$\int_0^1\bigg|f(s,y)\rho(s,y)\bigg[\int_0^t\int_0^1 \alpha''(w)[\rho(s,y+\varepsilon w)-\rho(s,y)]\,dw\,ds\bigg]\bigg|\,dy,$$

which converges to 0 as $\varepsilon\to 0$ by dominated convergence. □

A.2. *Proof of Lemma* 4. Let $m \in Z_+$ be a fixed integer. Then there is a finite set $S_m \subseteq K$ such that $d(z, S_m) \le 1/m$ for any $z \in K$.

Let $z \in K$. There is a $z^* \in S_m$ such that

$$|u_N(z,\omega)| \le |u_N(z^*,\omega)| + |u_N(z,\omega) - u_N(z^*,\omega)|$$

so

$$|u_N(z,\omega)| \le \max_{u\in S_m}|u_N(u,\omega)| + \frac{1}{m}L_N(\omega)$$

and hence

$$\sup_{z\in K}|u_N(z,\omega)| \le \sum_{u\in S_m}|u_N(u,\omega)| + \frac{1}{m}L_N(\omega).$$

We take the expected value and we get

$$E^N\Big[\sup_{z\in K}|u_N(z,\omega)|\Big] \le \sum_{u\in S_m} E^N[|u_N(u,\omega)|] + \frac{1}{m}E^N[L_N(\omega)],$$

hence

$$\limsup_{N\to\infty} E^N\Big[\sup_{z\in K}|u_N(z,\omega)|\Big] \le \frac{1}{m}l.$$

Since m is arbitrary, the lemma is proved. □

A.3. *Proof of Lemma* 6. Before proving Lemma 6 we can split its expression in two,

$$E^N \sup_{0\le s\le t}\bigg|\frac{1}{N}\sum_k f\bigg(\frac{1}{N}\sum_j g(x_k(s), x_j(s))\bigg) - \frac{1}{N}\sum_k f\bigg(\int_0^1 g(x_k(s), y)\rho(s,y)\,dy\bigg)\bigg|^2 \tag{A.11}$$

and

$$E^N \sup_{0\le s\le t}\bigg|\frac{1}{N}\sum_k f\bigg(\int_0^1 g(x_k(s), y)\rho(s,y)\,dy\bigg) - \int_0^1 f\bigg(\int_0^1 g(x,y)\rho(s,y)\,dy\bigg)\rho(s,x)\,dx\bigg|^2, \tag{A.12}$$

both converging to 0 as $N\to\infty$.

PROOF OF (A.11). A bound for (A.11) is

$$c\cdot\sup_x\bigg|\frac{1}{N}\sum_j g(x, x_j(s)) - \int_0^1 g(x,y)\rho(s,y)\,dy\bigg|^2$$

with $c=\bar\rho^2\|f'\|_\infty^2$.

We define the function

$$u_N(x,\omega) := \sup_{0\le s\le t}\bigg|\frac{1}{N}\sum_j g(x, x_j(s)) - \int_0^1 g(x,y)\rho(s,y)\,dy\bigg|^2.$$

Using Lemma 4 we reach the conclusion as a consequence of Theorem 1 applied to the smooth function $\phi(\cdot)=g(x,\cdot)$.

To prove (A.12) we only need to use Corollary 1 for the new smooth function $\phi(s,x)=f(\int_0^1 g(x,y)\rho(s,y)\,dy)$. □

A.4. *Proof of Proposition* 11. Let f be an integrable function on the unit circle and let F be its periodic extension to the real line. We shall denote by $\tilde p(t,z)$, the heat kernel on the real line and, as before by $p(t,z)$, the heat kernel on the unit circle.

The periodic extension on the line of the solution $\rho(t,x)$ to the Cauchy problem $\lim_{t\to 0}\rho(t,x)=f(x)$ for the heat equation on the unit circle is equal to the solution to the problem on the line with initial condition $F(x)$,

$$\tilde{\rho}(t, x) = \int_R F(y)\tilde{p}(t, x-y)\,dy = \sum_{n\in Z}\int_n^{n+1} F(y)\tilde{p}(t, x-y)\,dy$$

by changing the variable $z = y - n$

$$\begin{aligned}
&= \sum_{n\in Z}\int_0^1 F(z)\tilde{p}(t, x-z+n)\,dz \\
&= \int_0^1 F(z)\sum_{n\in Z}\tilde{p}(t, x-z+n)\,dz \\
&= \int_0^1 f(z)\sum_{n\in Z}\tilde{p}(t, x-z+n)\,dz \\
&= \int_0^1 f(z)p(t, x-z)\,dz
\end{aligned}$$

from Jacobi's theta function formula (or simply by an uniqueness argument).

We are interested in the $\|\cdot\|_{L^p}$-norm of the function $\rho(t, x) = f * g\,(t, x)$,

$$\|\rho\|_{L^p[0,1]} = \left(\int_0^1\left|\int_0^1 f(z)p(t, x-z)\,dz\right|^p dx\right)^{1/p}.$$

We have

$$\begin{aligned}
p(t, x) &= \sum_{n\in Z}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x+n)^2}{2t}\right) \\
&= \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x)^2}{2t}\right) \\
&\quad + \sum_{n\in Z_+}\frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(x+n)^2}{2t}\right) + \exp\left(-\frac{(x-n)^2}{2t}\right)\right) \\
&= \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x)^2}{2t}\right) + 2\sum_{n\in Z_+}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2+n^2}{2t}\right)\cos\frac{xn}{t},
\end{aligned}$$

hence has absolute value

$$\leq \left[\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x)^2}{2t}\right)\right]\left[1 + 2\sum_{n\in Z_+}\exp\left(-\frac{n^2}{2t}\right)\right].$$

The last factor is independent of x and as t approaches 0 it grows smaller and smaller, hence we may assume it has a bound $C > 0$ independent of x, n and p, say $C = 1 + 2\cdot\sum\exp(-n^2)$. Naturally, the first term is exactly the heat kernel on the real line and by applying Hölder's inequality we obtain all the estimates. □

PROPOSITION 23. *If $\mu(dx)$ is a finite measure on the unit circle [let us assume $\mu(T^1) = 1$ without loss of generality], $p(t, x)$ is the heat kernel for the unit circle and*

$$\rho(s, x) = \int_0^1 p(t, x - y)\mu(dy)$$

then

$$\|\rho(t, \cdot)\|_{L^p} \le \|p(t, \cdot)\|_{L^p}$$

for any $p \ge 1$.

PROOF.

$$\|\rho(t, \cdot)\|_{L^p} = \left[\int_0^1 \rho^p(s, x)\, dx\right]^{1/p} = \left[\int_0^1 \left[\int_0^1 p(t, x - y)\mu(dy)\right]^p dx\right]^{1/p}$$

and we apply the Hölder inequality for the $\mu(dy)$ integral to the functions identical to 1 and $p(t, x - \cdot)$. $\square$

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