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THE LIMIT POINTS IN $\overline{R^d}$ OF AVERAGES OF I.I.D. RANDOM VARIABLES

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Given any closed subset C of $\overline{R^d}$, containing a pair of antipodal points at ∞, there is a sequence of independent and identically distributed random variables $\{\mathbf{X}_i\}$ such that the set of limit points (in the topology of $\overline{R^d}$) of $\{(\mathbf{X}_1+\cdots+\mathbf{X}_t)/t\}_{t\geq 1}$ equals C. Here $\overline{R^d}$ is the compact space gotten by "adjoining the sphere, $S^{d-1}\infty$ at infinity."

1. Introduction. Let $d \geq 2$ and let $\overline{R^d}$ denote the compactification of the Euclidean space R^d obtained by "adjoining the sphere at ∞." More precisely, $\overline{R^d}$ is the compact metric space obtained by the completion of R^d with respect to the metric $\rho(\mathbf{x},\mathbf{y}) = |(1+|\mathbf{x}|)^{-1}\mathbf{x} - (1+|\mathbf{y}|)^{-1}\mathbf{y}|$ where $|\cdot|$ is the usual Euclidean norm. We use the same letter ρ to denote the extension of the metric to $\overline{R^d}$. Points in $R^d_\infty \equiv \overline{R^d}\backslash R^d$ are in a one-to-one correspondence with the points of the unit sphere, S^{d-1}, and it is convenient to write such points in the form $\mathbf{z}\cdot\infty$ for $\mathbf{z}\in S^{d-1}$.

Let $\mathbf{W}_0 = 0$, $\mathbf{W}_t = \sum_{i=1}^t \mathbf{X}_i$, $t = 1, 2, \ldots$ where $\{\mathbf{X}_i\}$ is a sequence of i.i.d. random variables with values in R^d and common distribution F. For a sequence $\{a_t\}$ of positive numbers which increase to ∞ with t, we define

$$A\big\{\mathbf{W}_t/a_t\big\} = \Big\{\mathbf{x}\in\overline{R^d}\colon \liminf_{t\to\infty} \rho(a_t^{-1}\mathbf{W}_t,\mathbf{x}) = 0 \text{ a.s.}\Big\} \tag{1.1}$$

We call the elements of this set the (extended sense) limit points of the normalized random walk $\{\mathbf{W}_t/a_t\}$. If F has at least two distinct points in its support, then $\{\mathbf{W}_t/a_t\}$ is the same thing as the set accumulation points in $\overline{R^d}$ of the random point set $\{\mathbf{W}_t/a_t\}$.

With probability 1, the set $A\{\mathbf{W}_t/a_t\}$ coincides with the *nonstochastic* closed set

$$A(F,\{a_t\}) = \big\{\mathbf{x}\in\overline{R^d}\colon P\big[\rho(a_t^{-1}\mathbf{W}_t,\mathbf{x}) < \varepsilon \text{ i.o.}\big] = 1\ \forall\ \varepsilon > 0\big\}. \tag{1.2}$$

[See Kesten (1970), Theorem 1. His proof generalizes easily.] In other words, the step distribution F and the sequence $\{a_t\}$, but not the sample paths, determine these sets. In the case $a_t = t^\beta$ one may write $A(\beta)$ or $A(F,\beta)$ for $A(F,\{a_t\})$. [Note: It is not asserted that $\{\mathbf{W}_t/a_t\}$ approaches points in $A(F,\{a_t\})$ as in Martin boundary theory. The equivalence of (1.1) and (1.2) says that there is a single null event such that *every* neighborhood of *every* point in $A(F,\{a_t\})$ (and no others) will be visited infinitely often, as $t\uparrow\infty$, by *every* sample sequence of averages $\{\mathbf{W}_t(\omega)/a_t\}$ for all ω outside of the null

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event. If a Markov process has a Martin boundary, then w.p.1, only one point, albeit random, will occur in the limit for any given sample path.]

It is easily seen that a point $\mathbf{x} \in R^d$ is a finite limit point if and only if

$$P\left[\liminf_{t\to\infty}\left|\frac{\mathbf{W}_t}{a_t} - \mathbf{x}\right| = 0\right] = 1, \tag{1.3}$$

and a point $\mathbf{z}\cdot\infty$, $\mathbf{z} \in S^{d-1}$ is an infinite limit point if and only if

$$P\left[\liminf_{t\to\infty}\left(\left|\frac{\mathbf{W}_t}{|\mathbf{W}_t|} - \mathbf{z}\right| + \frac{a_t}{|\mathbf{W}_t|}\right) = 0\right] = 1. \tag{1.4}$$

Let us write A_f for $A \cap R^d$, and A_∞ for $A \cap (\overline{R^d}\backslash R^d)$. Note that if $b_t = o(a_t)$, then $A_\infty\{\mathbf{W}_t/a_t\} \subset A_\infty\{\mathbf{W}_t/b_t\}$, and if also $A_f\{\mathbf{W}_t/a_t\} = \varnothing$ (i.e., $|\mathbf{W}_t|/a_t \to \infty$, a.s.), then $A_\infty\{\mathbf{W}_t/b_t\} = A_\infty\{\mathbf{W}_t/a_t\} = A_\infty\{\mathbf{W}_t\}$ for any increasing sequence $1 \le b_t = O(a_t)$. The role of the normalizing constants is not as critical for $A_\infty\{\mathbf{W}_t/a_t\}$ as it is for $A_f\{\mathbf{W}_t/a_t\}$ in this case.

If the sequence a_t is not linear, for example $a_t = t^\beta L(t)$ for some $0 < \beta \neq 1$ and L slowly varying, then the known results seem to support the assertion that the sets $A(F, a_t)$ cannot have a very complicated structure. In one dimension, for example, and $\beta \neq 1, 1/2$, then $A(\beta)$ is either a ray $[-\infty, b]$ or $[b, \infty]$, the entire line, a singleton, or a set containing one or more of the points $0, -\infty, +\infty$. [Also $A(1/2)$ must contain at least a ray if the finite part is nonempty.] See Kesten (1970) and Erickson and Kesten (1974). The results in higher dimensional spaces are not as complete but seem to support the assertion that the possible geometric structures for $A(\beta)$, $\beta \neq 1$ are limited.

Much more interesting is the linear case $a_t = t$ with $E|\mathbf{W}_1| = \infty$. Kesten (1970) for $d = 1$, and Erickson (1976) for $d > 1$, show that given *any* nonempty closed subset of R^d, there is a distribution F for which the set of *finite* points $A_f(F, 1)$ coincides with that given set. Moreover, given any closed subset of the sphere at infinity, it is quite easy to construct a distribution for which $A(F, 1) = A_\infty(F, 1)$ coincides with the given closed set. See the last section of this paper.

A problem arises when it is required that $A(1)$ contain both finite and infinite points. In this case there may be a restriction on the possible structures of the closed set of infinite limit points $A_\infty(1)$. What form the restriction will take is not completely clear. Kesten (1970) shows that, in one dimension, if there is more than one point in $A(1)$, then both $+\infty$ and $-\infty$ must be in $A(1)$. From this result one can show that for $d > 1$, if $E|\mathbf{W}_1| = \infty$ and if $A(1)$ has a finite point, then the limit points at infinity, identified with a set of unit vectors, is nonempty and cannot be contained in an open half-space of the form $\{\mathbf{x}\colon \mathbf{x}\cdot\mathbf{v} > 0\}$ for some nonzero vector $\mathbf{v}$. (This was first pointed out by S. Kalikov.)

Given any $C \subset \overline{R^d}$ provided only that the infinite part of C contains a pair of antipodal points $\pm\mathbf{e}\cdot\infty$, we show how to construct a random walk $\{\mathbf{W}_t\}$ such that $A(F, 1)$ coincides with C. This construction extends and greatly simplifies (and corrects a slight error in) the example of Erickson (1976).

Our method was inspired by Harry Kesten (1970) in his proof of his Theorem 7, 1196–1205.

The requirement that C contains a pair of antipodal points at infinity in the construction described here may be unnecessary. We leave the reader with the following conjecture: given any closed subset C of the extended Euclidean space which contains at least one finite point and the infinite part is not contained in a half-space, then there is a distribution F such that $A(F,1)$ coincides with C.

2. Construction of a random walk. Let $\{p_k\}$ denote a probability distribution on the nonnegative integers and $\{\vartheta_i\}$, a sequence of independent nonnegative integer valued random variables each having the distribution $\{p_k\}$. Next, let $\{r_k\}$ be an increasing sequence of positive integers and let $\{Y_s^k\}$ be a doubly indexed sequence of totally independent random variables (and independent of the sequence $\{\vartheta_n\}$) such that Y_s^k has a symmetric Bionomial distribution on the set of integers $-r_k, -r_k+1, \dots, r_k$. This distribution assigns mass $\binom{2r_k}{r_k - j}2^{-2r_k}$ to j and has characteristic function $E\exp(i\theta Y_s^k) = \cos^{2r_k}(\theta/2)$.

Let $\{\mathbf{b}_k\}$ be a sequence of vectors in R^d such that $\mathbf{b}_0 = \mathbf{0}$, but otherwise arbitrary, and let $\{a_k\}$ be any rapidly increasing positive numbers. Put $\mathbf{X}_s^k = a_k Y_s^k \mathbf{e} + \mathbf{b}_k$, where $\mathbf{e} = (1, 0, \dots, 0)$ and

$$\mathbf{X}_s = \mathbf{X}_s^{\vartheta_s} = \sum_{k=0}^{\infty} J_s^k \mathbf{X}_s^k, \qquad J_s^k = I(\vartheta_s = k) = \begin{cases} 1, & \text{if } \vartheta_s = k, \\ 0, & \text{if } \vartheta_s \neq k. \end{cases}$$

Finally, the random walk we seek is

$$\mathbf{W}_t = \sum_{s \le t} \mathbf{X}_s \quad \text{for } t \ge 1, \ \mathbf{W}_0 = 0.$$

Note that the step distribution of the walk (the common distribution of the $\mathbf{X}_s$) has the form

$$F = \sum_{k=0}^{\infty} p_k \big[F_k \times \delta_{\mathbf{b}_k(2)} \times \delta_{\mathbf{b}_k(3)} \times \cdots \times \delta_{\mathbf{b}_k(d)}\big], \tag{2.1}$$

where F_k is the distribution of $a_k Y_s^k + \mathbf{b}_k(1)$ and $\delta_{\mathbf{b}_k(\cdot)}$ is the distribution which puts unit mass at the point $b_k(\cdot)$ on the real axis. The notation $\mathbf{b}(i)$ denotes the ith coordinate of the vector $\mathbf{b}$. (Note that F is a weighted sum of product distributions; it is *not* a product distribution unless all of the vectors $\mathbf{b}_k$ are the same).

To better understand the structure of the walk, consider the following auxiliary random walks in R^d and R^1, respectively:

$$\mathbf{Z}_t^k = \sum_{s \le t} \Bigg(\mathbf{b}_k J_s^k + \sum_{j \le k-1} J_s^j \mathbf{X}_s^j \Bigg), \tag{2.2}$$

$$S_t^k = \sum_{s \le t} J_s^k Y_s^k, \qquad t \ge 1, \tag{2.3}$$

and $\mathbf{Z}_0^k = \mathbf{0}$, $S_0^k = 0$. Note that $\mathbf{c}_0 = \mathbf{0}$,

$$\mathbf{c}_k \equiv (1/t)E(\mathbf{Z}_t^k) = E(\mathbf{Z}_1^k) = \sum_{j=0}^{k} p_j \mathbf{b}_j, \tag{2.4}$$
$$\mathbf{b}_k = p_k^{-1}(\mathbf{c}_k - \mathbf{c}_{k-1}).$$

For each k and $m \geq 2$ define

$$T_k = T_k^1 = \min\{t: \vartheta_t = k\}.$$

If the p_k decrease to 0 fast enough, then, w.p.1,

$$T_k < T_{k+1} \quad \text{for all } k \text{ sufficiently large}$$

and when this occurs we get

$$t^{-1}\mathbf{W}_t - \mathbf{c}_k = \left(t^{-1}\mathbf{Z}_t^k - \mathbf{c}_k\right) + a_k t^{-1} S_t^k \mathbf{e}, \qquad t \in [T_k, T_{k+1}). \tag{2.5}$$

This decomposition is crucial to the proof of the main result.

It is time to choose the parameters. Let us suppose now that $\{\mathbf{c}_k\}$ is a given sequence of points in R^d such that

$$\mathbf{c}_0 = 0 < |\mathbf{c}_k| \leq k \quad \text{for all } k \geq 1. \tag{2.6}$$

We will define the parameters of $\mathbf{W}$ inductively. We also introduce some auxiliary parameters $\{m_k\}$, $\{\pi_k\}$ and $\{q_k\}$ which are useful in the proofs and help reduce the notational clutter.

Let $a_0 = a_1 = 1$, $r_0 = r_1 = 1$, $m_0 = m_1 = 1$, $p_1 = 1/8$, $p_2 = 1/16$. Let $\mathbf{b}_0 = \mathbf{0}$, $\mathbf{b}_1 = p_1^{-1}(\mathbf{c}_1 - \mathbf{c}_0)$ as at (2.4). For any j put

$$\pi_j = p_j\left(r_j a_j^2 + |\mathbf{b}_j|^2\right). \tag{2.7}$$

Suppose that for some $k \geq 2$, the parameters $\mathbf{b}_j, m_j, r_j, a_j$ and p_{j+1} have been defined for j up to $k-1$. Then we define $\mathbf{b}_k$ by (2.4) and

$$m_k = [k^3 p_k \pi_{k-1}] + 1, \tag{2.8a}$$

$$q_k = m_k + k^{12} m_k^3, \tag{2.8b}$$

$$r_k = k^4 m_k, \tag{2.8c}$$

$$p_{k+1} = \frac{p_k}{(k+1)^2 q_k}\left(\sum_{j=1}^{k-1} \pi_j\right)^{-1}, \tag{2.8d}$$

$$a_k = \frac{4k^3}{p_{k+1}}. \tag{2.8e}$$

Finally we set $p_0 = 1 - \sum p_k$. (That p_0 is positive follows from the inductively verified estimate: $p_k \leq 4^{-k}$, for all $k \geq 1$.) One can easily establish that these inductive formulas define the parameters for all k.

THEOREM 1. *Let $\{\mathbf{W}_t\}$ be the walk constructed as in the previous section with the parameters $\{\mathbf{b}_k\}$, $\{a_k\}$ and $\{r_k\}$ defined as above. Then*

$$\bigcap_{n\geq 1} \overline{\{\mathbf{W}_t/t; t\geq n\}} = A(F,1) = \bigcap_{n\geq 1} \overline{\{\mathbf{c}_k,: k\geq n\}} \cup \{\pm\mathbf{e}\cdot\infty\} \quad a.s.$$

where the overbar denotes closure in $\overline{R^d}$.

Given a closed subset C of $\overline{R^d}$ containing the antipodal pair $\pm\mathbf{e}\cdot\infty$, it is always possible to find a sequence $\{\mathbf{c}_k\}$ of distinct points in R^d which satisfies (2.6) and whose limit points in the extended sense equals C. It is also clear by using an appropriate orthogonal transformation that $\pm\mathbf{e}\cdot\infty$ can be replaced by any other antipodal pair at ∞. We leave these details to the reader.

3. Outline of the Proof of Theorem 1. Put

$$T_k^m = T_k^m = \min\{t > T_k^{m-1}\colon \vartheta_t = k\} = m\text{th occurrence of } k \text{ in } \{\vartheta_t\},$$
$$\Delta'_k = [T_k, T_k^{m_k}), \qquad \Delta''_k = [T_k^{m_k}, T_{k+1}) \quad [m_k \text{ at (2.8a)}],$$
$$\Delta_k = [T_k, T_{k+1}).$$

The Δ's are, of course, random time intervals and Δ''_k and Δ_k are empty with positive probability, Δ'_k is always nonempty.

The heart of the proof consists in verifying that the *complement* of each of the following events has probability $O(1/k^\beta)$ for some $\beta > 1$:

$$\frac{1}{k^2 p_k} < T_k < \frac{k}{p_k} \quad \text{and} \quad T_k < \frac{m_k}{2p_k} < T_k^{m_k} < \frac{2m_k}{p_k} < T_{k+1}, \tag{3.1}$$

which implies $\Delta''_k \neq \varnothing$, $\Delta_k = \Delta'_k \cup \Delta''_k$;

$$S_t^k \neq 0 \quad \text{for all } t\in\Delta'_k; \tag{3.2}$$

$$S_t^k = 0 \quad \text{for some } t\in\Delta''_k; \tag{3.3}$$

$$\min\left\{\frac{|\mathbf{W}_t(1)|}{t}\colon S_t^k\neq 0,\ t\in\Delta_k\right\} \geq k^2; \tag{3.4}$$

$$\max\left\{\left|\frac{\mathbf{W}_t}{t} - \mathbf{c}_k\right|\colon t\in\Delta_k,\ S_t^k = 0\right\} \leq \frac{1}{k^{1/4}}; \tag{3.5}$$

$$\max\left\{\frac{|\mathbf{W}_t(l)|}{|\mathbf{W}_t(1)|}\colon S_t^k\neq 0,\ t\in\Delta_k,\ l=2,3,\dots,d\right\} \leq \frac{1}{k^{1/4}}. \tag{3.6}$$

If the complements of these events have probabilities $O(1/k^\beta)$ for some $\beta > 1$ as claimed, then the Borel–Cantelli lemma implies that, w.p.1. every one of the events holds for all $k\geq K$ where K is a finite random integer. Though there is more work to be done, one can see from (3.3) and (3.5) that $A(F,1)$ must include the set of extended-sense limit points of $\{\mathbf{c}_k\}$. Moreover

(3.2), (3.4) and (3.6) show that $A(F, 1)$ cannot contain any other points of $\overline{R^d}$ except possibly $\pm \mathbf{e} \cdot \infty$.

It is clear from (3.2), (3.4) (3.6) and the 0–1 law for tail events that $A(F, 1)$ does indeed contain at least one of $\pm\mathbf{e} \cdot \infty$. That both of the points $\pm\mathbf{e} \cdot \infty$ are in $A(F, 1)$ follows (with a little work) from recurrence properties of the walks S^k. (The symmetry of their distributions simplifies the proof.)

Here is what is going on: the recurrent *director* walk, $\{a_k S_t^k \mathbf{e}\}$, enters the structure of $\mathbf{W}$ for the first time at $t = T_k$ and plays its biggest role during the special interval Δ_k. After that time its influence on the *averages* $\{\mathbf{W}_t/t\}$ rapidly diminishes due to the large denominators, $t \gg a_k$. During its special time interval, the director's effect completely vanishes [at the zeros of $\{S_t^k\}$] and this is what allows the averages $\mathbf{W}_t/t$ to approach the currently appropriate centering vector $\mathbf{c}_k$; see (2.5). However, these vanishing times must not come too soon because at the beginning of the interval the influence of the preceding director, which has not completely subsided, must be prevented from affecting the averages in an unpredictable fashion. The main effect of the director during its special interval at times when it does not vanish is to drag the ratios $\mathbf{W}_t/t$ out toward $\pm\mathbf{e} \cdot \infty$. (Recall that S^k is integer valued so $a_k|S_t^k| \geq a_k$ when $S_t^k \neq 0$.)

4. Some details of the proof of Theorem 1. In what follows we will derive $O(1/k^\beta)$ estimates using inequalities derived inductively from (2.8). These inequalities are only claimed to be valid for k "sufficiently large." However, it seems that these inequalities are actually valid for $k \geq 5$, but the proof is left to the interested reader.

Let us also note that in some of the estimates, terms such as $P[\Delta_k = \varnothing]$ or $P[\Delta_k'' = \varnothing]$ logically ought to be included on the right-hand sides of some of the derivations. [See (4.4) and (4.5), for example.] No harm is done in omitting them (to help reduce the clutter) for they are all $O(1/k^2)$ as may be seen from the very next step.

4.1. *Proof that* $1 - P[$*events in* $(3.1)] = O(1/k^2)$. First let us note some useful inequalities. The random variable $T = T_k$ has a geomtric distribution with mean $1/p (p = p_k)$ and variance $(1-p)/p^2$, and $T^\nu = T_k^\nu$ is a sum of ν independent copies of T. So, for $p \leq 1/2$, $\nu \geq k^2$, and $k \geq 2$,

$$\text{(4.1a)} \qquad 1 - P\left[\frac{1}{x^2 p} < T < \frac{x}{p}\right] = P\left[T \leq \frac{1}{x^2 p}\right] + P\left[T \geq \frac{x}{p}\right] \leq 1 - (1-p)^{1/(x^2 p)} + \frac{2}{x^2} \leq \frac{4}{x^2},$$

$$\text{(4.1b)} \qquad 1 - P\left[\frac{\nu}{2p} < T^\nu < \frac{2\nu}{p}\right] \leq P\left[\left|T^\nu - \frac{\nu}{p}\right| \geq \frac{\nu}{2p}\right] \leq \frac{8}{k^2}.$$

The probability of the complement of (3.1) is dominated by the sum

$$1 - P\left[\frac{1}{k^2 p_k} < T_k < \frac{k}{p_k}\right] + P\left[T_k \geq \frac{m_k}{2p_k}\right]$$
$$+ 1 - P\left[\frac{m_k}{2p_k} < T_k^{m_k} < \frac{2m_k}{p_k}\right] + P\left[T_{k+1} \leq \frac{2m_k}{p_k}\right].$$

Taking first $x = k$ and then $x = m_k/2$ in $(4.1\mathrm{a})_k$ we see that the first term is bounded by $4/k^2$ and the second term by $16/m_k^2$. The latter is clearly $O(1/k^2)$ by (4.2). (See below). Next the third term is bounded by $8/k^2$ by $(4.1\mathrm{b})_k$ with $\nu = m_k$. Finally, the fourth term is $O(1/k^2)$ by $(4.1\mathrm{a})_{k+1}$ with $x^2 = p_k(2m_k p_{k+1})^{-1}$. [The latter is bigger than $(k+1)^2$ by (4.2).] This completes the proof that $1 - P[(3.1)]$ is $O(1/k^2)$.

In the preceding we have used (and will use again, implicitly) the important inequality

$$k^{13} \leq m_k \leq \frac{\frac{1}{2} p_k}{(k+1)^2 p_{k+1}}, \tag{4.2}$$

for all k sufficiently large.

For a proof of (4.2), note first that because $m_k \geq 1$, for k sufficiently large, (2.8b) implies $q_k > 2m_k$. Also $\sum_{j \leq k-1} \pi_j \geq 1$, k sufficiently large. Hence by (2.8d),

$$p_{k+1} \leq \frac{p_k}{(k+1)^2 q_k} \leq \frac{\frac{1}{2} p_k}{(k+1)^2 m_k},$$

and this yields the right-hand side of (4.2).

From (2.8a), (2.7), (2.8c), (2.8e) and the monotonicity of $\{p_k\}$ (more or less in that order),

$$\begin{aligned} m_{k+1} &\geq (k+1)^3 p_{k+1} \pi_k \geq k^3 p_{k+1} p_k r_k a_k^2 \\ &= k^7 m_k (p_{k+1} a_k)(p_k a_k) \geq 4k^{10} p_k a_k m_k \geq 4k^{10} p_{k+1} a_k m_k \geq 16k^{13} m_k. \end{aligned}$$

Since $m_{k-1} \geq 1$, the last inequality implies $m_k \geq 16(k-1)^{13} > k^{13}$ for all k sufficiently large. This is the left-hand side of (4.2). □

REMARK. Iterating the inequality $m_{k+1}/m_k \geq 16k^{13}$ yields the fantastic: $m_{k+1} \geq 2^{4k}(k!)^{13}$, for all large k.

4.2. *Proof that* $1 - P[(3.2)] = O(1/k^2)$. (To reduce clutter we suppress the ks when possible). Because $Y_t J_t = 0$ for $T^{j-1} < t < T^j$, $j \geq 1$, it follows that S_t is constant on these intervals. This and the independence of the T_k^js and Y^ks, implies that

$$\begin{aligned} 1 - P[(3.2)] &= P[S_t = 0 \text{ for some } t \in \Delta'_k] \\ &= P\Big[\sum_{j \leq n} Y_{T^j} = 0 \text{ for some } n \in [1, m]\Big] \\ &= P[\widehat{S}_n = 0 \text{ for some } n \in [1, m]], \end{aligned}$$

where $\widehat{S}_n = \sum_{j=1}^{n} Y_j^k$, $\widehat{S}_0 = 0$. The random variable $\widehat{S}_n$ has a symmetric bionomial distribution on the integers in $[-rn, rn]$. Hence, by refined Stirling's inequalities,

$$P\big[\widehat{S}_n = 0 \text{ for some } n \in [1, m]\big] \leq \sum_{n=1}^{m} P\big[\widehat{S}_n = 0\big] = \sum_{n=1}^{m} \binom{2rn}{rn} 2^{-2rn}$$

$$\leq \sum_{n=1}^{m} (\pi rn)^{-1/2} \leq 2(m/r)^{1/2}.$$

Taking $m = m_k$ and $r = r_k$ and noting (2.8c), one sees that the last written quantity is $O(1/k^2)$. □

4.3. *Proof that* $1 - P[(3.3)] = O(1/k^2)$. Let $G_m(x, y)$ denote the expected number of visits by $\widehat{S}$ to y during $[0, m]$ starting at x, and let $\tau = \min(n \geq 0\colon \widehat{S}_n = 0)$. Then $G_m(z, 0) = \sum_{n=0}^{m} P\{\tau = n | \widehat{S}_0 = z\} G_{m-n}(0, 0) \leq G_m(0, 0) - G_m(0, 0) P\{\tau > m \mid \widehat{S}_0 = z\}$. Hence

$$P\{\tau > m \mid \widehat{S}_0 = z\} \leq \big[G_m(0, 0) - G_m(z, 0)\big]/G_m(0, 0)$$

$$= \big[G_m(0, 0) - G_m(z, 0)\big] \left[\sum_{n=0}^{m} \binom{2rn}{rn} 2^{-2rn}\right]^{-1}$$

$$\leq 2\big[G_m(0, 0) - G_m(z, 0)\big] \sqrt{r/m},$$

by another application of Stirling's inequalities. Also

$$G_m(0, 0) - G_m(z, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - e^{iz\theta}) \sum_{j=0}^{m} \Big\{E \exp(i\theta Y_1^k)\Big\}^j \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - \cos z\theta)(1 - \cos^{2r(m+1)}(\theta/2))}{1 - \cos^{2r}(\theta/2)} \, d\theta$$

$$\leq \frac{2}{\pi} \int_0^{\pi} \left(\frac{\sin z\theta/2}{\sin \theta/2}\right)^2 d\theta \leq \pi \int_0^{\infty} \left(\frac{\sin zt}{t}\right)^2 dt = \frac{\pi^2 |z|}{2}.$$

Consequently $P\{\tau > m | \widehat{S}_0 = z\} \leq 10|z|\sqrt{r/m}$ and

$$P\big[S_t^k \neq 0 \text{ for all } t \in [T^m, T^q]\big] = P\big[\widehat{S}_n \neq 0 \text{ for all } n \in [m, q]\big]$$

$$= \sum_{z=-mr}^{+mr} P\{\tau > q - m \mid \widehat{S} = z\} P\big[\widehat{S}_m = z\big]$$

$$\leq 10\left(\frac{r}{q - m}\right)^{1/2} E|\widehat{S}_m| \leq \left(\frac{50 r^2 m}{q - m}\right)^{1/2}.$$

(The variable $\widehat{S}_m$ has mean 0 and variance $mr/2$.) Hence,

$$
\begin{aligned}
1 - P[(3.3)] &= P\big[S_t^k \neq 0 \ \forall\ t \in \Delta_k''\big] \\
&\leq P\big[\widehat{S}_n \neq 0 \ \forall\ n \in [m_k, q_k]\big] + P\big[T_k^{q_k} > T_{k+1}\big] \qquad (4.3)\\
&\leq \left(\frac{50 r_k^2 m_k}{q_k - m_k}\right)^{1/2} + P\big[T_{k+1} < 2q_k/p_k\big] + P\big[T_k^{q_k} > 2q_k/p_k\big].
\end{aligned}
$$

The first term in the last line is $O(1/k^2)$ by (2.8). The second term is also $O(1/k^3)$ by $(4.1\mathrm{a})_{k+1}$ with $T = T_{k+1}$, $p = p_{k+1}$ and

$$x = \sqrt{p_k/2q_k p_{k+1}} \geq (k+1)\left(\frac{\pi_1 + \pi_2 + \cdots \pi_{k-1}}{2}\right)^{1/2}.$$

This implies $1/x = O(k^{3/2})$. [See (2.8d) and note that the π_j are bounded away from 0.] The last probability term in (4.3) is $O(1/k^2)$ by (4.1b) with $\nu = q_k$. That $\nu \gg k^2$ (for all large k) as required in (4.1b) is clear from (4.2) because $q_k > m_k$ by (2.8b). □

4.4. *Proof that* $1 - P[(3.4)] = O(1/k^2)$. For $t \leq (k+1)p_{k+1}^{-1}$, we have $\frac{1}{2}a_k - t|\mathbf{c}_k| \geq \frac{1}{2}a_k - 2k^2 p_{k+1}^{-1} \geq \frac{1}{4}a_k$; see (2.8e). Also $m_k/(p_k p_{k+1} a_k^2) \leq 1/k^5$, by (2.8a) and (2.8e), and

$$
\begin{aligned}
\sigma^2(\mathbf{Z}_1^k) &= E\left|\sum_{j=1}^{k-1}\big(J_1^j \mathbf{X}_1^j - p_j \mathbf{b}_j\big)\right|^2 + |\mathbf{b}_k|^2 E\big(J_1^k - p_k\big)^2 \\
&\quad + 2\sum_{j=1}^{k-1} E\big[(J_1^j \mathbf{X}_1^j - p_j \mathbf{b}_j)\mathbf{b}_k (J_1^k - p_k)\big] \\
&= O\left(k\sum_{j=1}^{k-1}\pi_j + k^2/p_k\right) = O\left(k\sum_{j=1}^{k-2}\pi_j + k\pi_{k-1} + k^2/p_k\right) \\
&= O\big([1 + m_k/k^2 + k^2]/p_k\big) = O(m_k/k^2 p_k),
\end{aligned}
$$

as the reader can verify, though perhaps not willingly. [Keep in mind (2.8a), and $(2.8\mathrm{d})_{k-1}$ and (4.2).] Therefore, by Kolomogorov's inequality, [and (4.1a) with k replaced by $k+1$],

$$
\begin{aligned}
&P\left[|\mathbf{Z}_t^k| > \frac{1}{2}a_k \text{ for some } t \in \Delta_k\right] \\
&\quad \leq P\left[|\mathbf{Z}_t^k - t\mathbf{c}_k| > \frac{1}{4}a_k \text{ for some } t \leq \frac{k+1}{p_{k+1}}\right] + P\left[T_{k+1} > \frac{k+1}{p_{k+1}}\right] \\
&\quad = O\left(\frac{k\sigma^2(\mathbf{Z}_1^k)}{p_{k+1}a_k^2} + \frac{1}{k^2}\right) = O\left(\frac{m_k}{k p_k p_{k+1} a_k^2} + \frac{1}{k^2}\right) = O\left(\frac{1}{k^2}\right).
\end{aligned}
$$

However, $|\mathbf{W}_t(1)|/t \geq (a_k|S_t^k| - |\mathbf{Z}_t^k|)/T_{k+1} \geq \frac{1}{2}a_k p_{k+1}/(k+1) \geq k^2$, whenever $S_t^k \neq 0$ and both $|\mathbf{Z}_t^k| \leq \frac{1}{2}a_k$ and $T_{k+1} \leq (k+1)/p_{k+1}$ hold. Thus

$$1 - P\big[(3.4)\big] \leq P\bigg[|\mathbf{Z}_t^k| > \frac{1}{2}a_k \text{ for some } t \in \Delta_k \neq \varnothing\bigg] + O\Big(\frac{1}{k^2}\Big) = O\Big(\frac{1}{k^2}\Big). \quad \square$$

4.5. *Proof that* $1 - P[(3.5)] = O(1/k^{3/2})$.

$$\begin{aligned}
1 - P\big[(3.5)\big] &\leq P\bigg[\Big|\frac{\mathbf{W}_t}{t} - \mathbf{c}_k\Big| > k^{-1/4} \text{ and } S_t^k = 0 \text{ for some } t \in \Delta_k''\bigg] \\
&\qquad + P\big[S_t^k = 0 \text{ for some } t \in \Delta_k'\big] \\
&\leq P\bigg[T_k^{m_k} \leq \frac{m_k}{2p_k}\bigg] + P\big[T_{k+1} \leq T_k^{m_k}\big] \qquad (4.4)\\
&\qquad + P\bigg[\Big|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\Big| > k^{-1/4} \text{ for some } t \geq \frac{m_k}{2p_k}\bigg] + O\bigg(\sqrt{\frac{m_k}{r_k}}\bigg) \\
&= P\bigg[\Big|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\Big| > k^{-1/4} \text{ for some } t \geq \frac{m_k}{2p_k}\bigg] + O\Big(\frac{1}{k^2}\Big)
\end{aligned}$$

See (2.5) and Sections 4.1 and 4.2 and (2.8c). To complete the proof we can use the Hájek–Rényi inequality. Set β = greatest integer in $m_k/(2p_k)$, then,

$$P\bigg[\Big|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\Big| > k^{-1/4} \text{ for some } t \geq \beta\bigg] \leq \frac{\sigma^2(\mathbf{Z}_1^k)}{k^{-1/2}}\bigg(\frac{1}{\beta} + \sum_{t \geq 1+\beta} \frac{1}{t^2}\bigg) = O\Big(\frac{1}{k^{3/2}}\Big).$$

4.6. *Proof that* $1 - P[(3.6)] = O(1/k^{3/2})$. Note that when (2.5) holds and $T_k \leq t < T_{k+1}$, and $S_t^k \neq 0$, then $|S_t^k| \geq 1$ (S_t^k is an integer), and $|\mathbf{W}_t(l)| = |\mathbf{Z}_t^k(l)| \leq |\mathbf{Z}_t^k|$ for $l = 2, 3, \ldots, d$ and $|\mathbf{W}_t| = |\mathbf{Z}_t^k + a_k S_t^k \mathbf{e}| \geq \big|a_k - |\mathbf{Z}_t^k|\big|$. So

$$\begin{aligned}
1 - P\big[(3.6)\big] &\leq P\bigg[\frac{|\mathbf{Z}_t^k|}{\big|a_k - |\mathbf{Z}_t^k|\big|} \geq k^{-1/4} \text{ for some } t \in \Delta_k'\bigg] \\
&\quad + P\bigg[\frac{|\mathbf{Z}_t^k|}{\big|a_k - |\mathbf{Z}_t^k|\big|} \geq k^{-1/4},\ \Big|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\Big| \leq k^{-1/4} \\
&\qquad\qquad \text{for some } t \in \Delta_k'' \neq \varnothing\bigg] \qquad (4.5)\\
&\quad + P\bigg[\Big|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\Big| > k^{-1/4} \text{ for some } t \geq \frac{m_k}{2p_k}\bigg] \\
&\quad + P\bigg[T_k^{m_k} < \frac{m_k}{2p_k}\bigg].
\end{aligned}$$

The sum of the last three terms is $O(1/k^{3/2})$ as in (4.4).

To estimate the first term of the right of (4.5), we assume that $k \geq 4$. Then $\frac{1}{2}k^{-1/4}a_k - 2m_k|\mathbf{c}_k|/p_k \geq m_k(\frac{1}{2}k^{-1/4}k^2 - 2k)/p_k \geq k^{7/4}m_k/4p_k$ by (2.8e) and

$|\mathbf{c}_k| \le k$. Hence

$$P\left[\frac{|\mathbf{Z}_t^k|}{\left|a_k - |\mathbf{Z}_t^k|\right|} > k^{-1/4} \text{ for some } t \in \Delta_k'\right]$$

$$\le P\left[\max_{t \le 2m_k/p_k} |\mathbf{Z}_t^k| \ge \frac{k^{-1/4}a_k}{1+k^{-1/4}}\right] + O\left(\frac{1}{k^2}\right)$$

$$\le P\left[\max_{t \le 2m_k/p_k} |\mathbf{Z}_t^k - t\mathbf{c}_k| \ge \frac{a_k}{2k^{1/4}} - \frac{2m_k|\mathbf{c}_k|}{p_k}\right] + O\left(\frac{1}{k^2}\right)$$

$$\le P\left[\max_{t \le 2m_k/p_k} |\mathbf{Z}_t^k - t\mathbf{c}_k| \ge \frac{k^{7/4}m_k}{4p_k}\right] + O\left(\frac{1}{k^2}\right)$$

$$= O\left(\frac{\sigma^2(\mathbf{Z}_1^k)}{k^{7/2}(m_k/p_k)}\right) + O\left(\frac{1}{k^2}\right) = O\left(\frac{1}{k^2}\right),$$

because the variance term is $O(1/k^{11/4})$.

It remains to estimate the second term on the right-hand side of (4.5):

$$P\left[\frac{|\mathbf{Z}_t^k|}{\left|a_k - |\mathbf{Z}_t^k|\right|} \ge k^{-1/4},\ \left|\frac{\mathbf{Z}_t^k}{t} - \mathbf{c}_k\right| \le k^{-1/4} \text{ for some } t \in \Delta_k''\right]$$

$$\le P\left[\frac{a_k}{2k^{1/4}} \le |\mathbf{Z}_t^k| < t(k+1) \text{ for some } t \in \Delta_k''\right]$$

$$\le P\left[\frac{a_k}{2k^{1/4}} \le |\mathbf{Z}_t^k| \le \frac{(k+1)^2}{p_{k+1}} \text{ for some } t \le \frac{k+1}{p_{k+1}}\right]$$

$$+P\left[T_{k+1} > \frac{k+1}{p_{k+1}}\right]$$

$$\le P\left[\frac{a_k}{2k^{1/4}} \le |\mathbf{Z}_t^k| \le \frac{a_k}{k} \text{ for any } t \ge 1\right] + O\left(\frac{1}{k^2}\right) = O\left(\frac{1}{k^2}\right),$$

as soon as $k \ge 2$ because the last written probability term is then 0.

The preceding estimates yield $1 - P[(3.6)] = O(1/k^{3/2} + 1/k^2) = O(1/k^{3/2})$ as promised.

5. An example for infinite limit points. As mentioned in the introduction, any closed subset of $S^{d-1}\infty$ can be the set of infinite limit points of some sequence of averages $\{\mathbf{W}_t/t\}$. However, if the given set is contained in an open half-space (at infinity), then $\{\mathbf{W}_t/t\}$ cannot have finite limit points. In an earlier version of this paper, a distribution was constructed for which $A_\infty(F,1)$ was any given closed *convex* subset of the unit sphere at infinity. An anonymous referee suggested a modification that would produce a distribution that could be made to yield any closed subset of $S^{d-1}\cdot\infty$, convex or not. Here are the details.

Let $D\cdot\infty$ be the given closed set where D is a closed subset of the unit sphere S^{d-1} in R^d. Let $\{R_i\}$ and $\{\mathbf{\Theta}_i\}$ be two sequences of independent identically distributed random variables and independent of each other such that the R_i are real valued and positive and the $\mathbf{\Theta}_i$ take values in S^{d-1}. Put

$$\mathbf{X}_i = R_i\mathbf{\Theta}_i, \qquad \mathbf{W}_t = \sum_{i\le t}\mathbf{X}_i, \qquad t \ge 1.$$

THEOREM 2. *If*

$$\lim_t \frac{\max_{i\le t} R_i}{R_1 + R_2 + \cdots + R_t} = 1 \quad a.s. \tag{5.1}$$

and the support of the common distribution of the $\mathbf{\Theta}_i$ *equals D, then*

$$A(\{\mathbf{W}_t/t\}) = D\cdot\infty.$$

Here is the proof of the theorem. Let $M_t = \max_{i\le t} R_i$. Then (5.1) implies

$$\lim_t \frac{\sum_{i\le t} R_i - M_t}{M_t} = 0 \quad \text{a.s.}$$

Moreover (5.1) also implies $E(R_1) = \infty$ so that

$$\lim_t \frac{R_1 + \cdots + R_t}{t} = \lim_t \frac{M_t}{t} = \infty \quad \text{a.s.}$$

Thus any limit points of $\{\mathbf{W}_t/t\}$ must be infinite limit points. See (1.4). If $\sigma(t)$ denotes the smallest index of the maximal term among $R_1,\dots R_t$, so $M_t = R_{\sigma(t)}$, then as $t\uparrow\infty$, $\mathbf{W}_t = R_{\sigma(t)}\mathbf{\Theta}_{\sigma(t)} + \mathbf{o}(R_{\sigma(t)})$ and

$$\frac{\mathbf{W}_t}{|\mathbf{W}_t|} = \frac{R_{\sigma(t)}\mathbf{\Theta}_{\sigma(t)} + \mathbf{o}(R_{\sigma(t)})}{R_{\sigma(t)} + \mathbf{o}(R_{\sigma(t)})} = \mathbf{\Theta}_{\sigma(t)} + \mathbf{o}(\mathbf{1}). \tag{5.2}$$

The sequence of random integers $\{\sigma(t)\}_{t\ge1}$ is totally independent of the sequence $\{\mathbf{\Theta}_t\}_{t\ge1}$ and $\sigma(t)\to\infty$. Therefore, by standard 0–1 laws, for every $\mathbf{z}\in S^{d-1}1$ we have

$$P\Big[\liminf_t |\mathbf{\Theta}_{\sigma(t)} - \mathbf{z}| = 0\Big] = 1 \text{ or } 0$$

according as $\mathbf{z}$ is in D or not. This and (5.2) imply the conclusion of the theorem.

EXAMPLE. Consider an i.i.d. sequence $\{R_i\}$ with a common distribution which satisfies

$$P[R_1 > r] = \frac{1}{\ln r} \quad \text{for } r \ge e,$$

and $P[R_1 \leq r] = 0$ for $r \leq e$. That $\{R_i\}$ satisfies (5.1) follows from a result of Pruitt (1987) and the easily verified consequence of the above form for the tail of the distribution of the R_i,

$$\sum_k \big(P[2^k < R_1 \leq 2^{k+1} | R_1 > 2^k]\big)^2 < \infty.$$

(The convergence of this series is the main hypothesis of the Pruitt result.)

We leave it to the reader to solve the problem of constructing an i.i.d. sequence $\mathbf{\Theta}_i$ with a common distribution having support exactly equal to a given closed subset of the unit sphere in R^d.

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