THE DISTRIBUTION LAWS OF THE DIFFERENCE AND QUOTIENT OF VARIABLES INDEPENDENTLY DISTRIBUTED IN PEARSON TYPE III LAWS[1]

By Solomon Kullback

Although the results herein described are not entirely new, it is felt that the method of solution is of interest as presenting further illustrations of the application of characteristic functions to the distribution problem of statistics (*1*).

1. Distribution law of the difference. Let $u = x - y$, where the distribution laws of x and y are independent and given respectively by

$$f_1(x) = \frac{e^{-x} x^{p-1}}{\Gamma(p)}; \quad f_2(y) = \frac{e^{-y} y^{q-1}}{\Gamma(q)} \quad 0 \leqq x \leqq \infty; 0 \leqq y \leqq \infty. \tag{1}$$

The characteristic function of the distribution law of u is given by (*1*),

$$\varphi(t) = \int_0^\infty \frac{e^{itx-x} x^{p-1} dx}{\Gamma(p)} \int_0^\infty \frac{e^{-ity-y} y^{q-1} dy}{\Gamma(q)} \tag{2}$$

$$= \frac{1}{(1 - it)^p (1 + it)^q}. \tag{3}$$

The distribution law of u is given by (*1*),

$$D(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itu} dt}{(1 - it)^p (1 + it)^q}. \tag{4}$$

Let $1 - it = -\frac{z}{u}$,

$$D(u) = \frac{e^{-u} u^{p-1}}{2^q 2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{e^{-z} dz}{(-z)^p \left(1 + \frac{z}{2u}\right)^q}. \tag{5}$$

Now it may be shown that (*1*)

$$\frac{1}{2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{e^{-z} dz}{(-z)^p \left(1 + \frac{z}{2u}\right)^q} = \frac{e^u (2u)^{\frac{q-p}{2}}}{\Gamma(p)} W_{\frac{p-q}{2}, \frac{1-p-q}{2}} (2u) \tag{6}$$

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where $W_{k,m}(z)$ is the confluent hypergeometric function (*2*). Since $W_{k,m}(z) = W_{k,-m}(z)$ we have finally

$$(7) \qquad D(u) = \frac{u^{\frac{p+q}{2}-1}}{2^{\frac{p+q}{2}}\Gamma(p)} W_{\frac{p-q}{2}, \frac{p+q-1}{2}}(2u).$$

For $p = q$, since $W_{0,m}(2x) = \frac{x^{\frac{1}{2}} 2^{\frac{1}{2}}}{\sqrt{\pi}} K_m(x)$ where $K_m(x)$ is the Bessel Function of the second kind and imaginary argument (*1*), we obtain

$$(8) \qquad D(u) = \frac{u^{\frac{2p-1}{2}}}{2^{\frac{2p-1}{2}}\Gamma(p)\sqrt{\pi}} K_{\frac{2p-1}{2}}(u).$$

This result has been otherwise obtained by Pearson, Stouffer, and David (*3*).

2. Distribution law of the quotient. Let $u = \log x - \log y$ where x and y are defined as above.

The characteristic function of the distribution law of u is given by (*1*)

$$(9) \qquad \varphi(t) = \int_0^\infty \frac{e^{-x} x^{p-1+it}\, dx}{\Gamma(p)} \int_0^\infty \frac{e^{-y} y^{q-1-it}\, dy}{\Gamma(q)}$$

$$(10) \qquad = \frac{\Gamma(p + it)\,\Gamma(q - it)}{\Gamma(p)\,\Gamma(q)}.$$

The distribution law of u is given by (*1*)

$$(11) \qquad D(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itu}\,\Gamma(p + it)\,\Gamma(q - it)}{\Gamma(p)\,\Gamma(q)}\, dt.$$

Let $q - it = -z$, so that

$$(12) \qquad D(u) = \frac{e^{-qu}}{\Gamma(p)\,\Gamma(q)\,2\pi i} \int_{-q-i\infty}^{-q+i\infty} e^{-zu}\,\Gamma(p + q + z)\,\Gamma(-z)\, dz.$$

Now it may be shown that (*2*)

$$\frac{1}{2\pi i} \int_{-q-i\infty}^{-q+i\infty} e^{-zu}\,\Gamma(p + q + z)\,\Gamma(-z)\, dz = \Gamma(p + q)\,(1 + e^{-u})^{-(p+q)},$$

so that

$$(13) \qquad D(u) = \frac{\Gamma(p + q)}{\Gamma(p)\,\Gamma(q)} \frac{e^{pu}}{(1 + e^{u})^{p+q}}.$$

Since $e^u = \frac{x}{y} = w$, we obtain as the distribution law of the quotient

$$p(w) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} \frac{w^{p-1}}{(1+w)^{p+q}}. \tag{14}$$

If in (13) we set

$$p = \frac{n_1}{2}; \qquad q = \frac{n_2}{2}; \qquad e^u = \frac{n_1}{n_2} e^{2z},$$

we obtain

$$D(z) = \frac{2\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} e^{n_1 z}}{(n_2 + n_1 e^{2z})^{\frac{n_1+n_2}{2}}} \tag{15}$$

which result has been otherwise obtained by R. A. Fisher (*4*).

GEORGE WASHINGTON UNIVERSITY.

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