one infers that the probability is greater than .95 that for a sample showing such a large deviation from the mean ($u/\sqrt{n} = 4, n = 4$) *all* the constituent elements will have deviations on the same side of the population mean. Thus if all the elements of the sample investigated are found to have deviations on the same side of the population mean, this could *not* be construed as *additional evidence* that the sample indicated an abnormal condition.

This conclusion is weaker than the facts of the example warrant, since it is based upon the *integral* of $F_n(u)$ from u' to infinity. Unfortunately the author does not have data available on the rate of convergence of these integrals.

NOTE ON A MATRIC THEOREM OF A. T. CRAIG

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An extremely elegant theorem given recently by A. T. Craig[1] and applied by him to establish a further theorem on independent χ^2 distributions may be stated as follows:

If A and B are the symmetric matrices of two homogeneous quadratic forms in n variates which are normally and independently distributed with zero means and unit variances, a necessary and sufficient condition for the independence in probability of these two forms is that $AB = 0$.

The proof given that the condition is sufficient is adequate, but Craig's treatment of its necessity consists essentially in its assertion. In view of the growing interest in such quadratic forms, for example in connection with serial correlation, the neatness of this theorem is likely to lead to a wide usefulness. It therefore seems worth while to give a complete proof of the necessity condition.

The form with matrix A is denoted by Q_1 and that with matrix B by Q_2. The characteristic functions, if defined as $Ee^{\frac{1}{2}\lambda Q_1}$ and $Ee^{\frac{1}{2}\mu Q_2}$, are respectively the reciprocals of the square roots of the determinants of the matrices $1 - \lambda A$ and $1 - \mu B$, while the characteristic function for Q_1 and Q_2 together, $Ee^{\frac{1}{2}(\lambda Q_1 + \mu Q_2)}$, is the reciprocal of the square root of the determinant of $1 - \lambda A - \mu B$. A necessary and sufficient condition for independence is therefore that

$$| 1 - \lambda A | \cdot | 1 - \mu B | \equiv | 1 - \lambda A - \mu B |$$

shall hold identically for all values of λ and μ. Since the determinant of the product of two matrices is the product of their determinants, the left member is the same as

$$| 1 - \lambda A - \mu B + \lambda\mu AB | .$$

From this it is immediately obvious that $AB = 0$ implies the independence of the two forms. The converse will now be proved.

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[1] "Note on the independence of certain quadratic forms," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 195–197.

We shall assume therefore that Q_1 and Q_2 are independent, so that the identity holds, and prove that $AB = 0$.

It must not be supposed that Q_1 and Q_2 can by the same linear transformation be reduced to forms in which product terms are absent and only terms in the squares of the variates appear. The available theorems[2] leading to this canonical form require that at least one of the quadratic forms be non-singular. But it is of the essence of the present situation that both Q_1 and Q_2 be singular, since this is implied by $AB = 0$. It does not appear possible, for example, to reduce to this canonical form the pair $x_1^2 + x_2^2$ and $x_1^2 + 2x_1x_3$.

Nevertheless a real orthogonal transformation can be found reducing Q_1 to

$$d_1x_1^2 + \cdots + d_rx_r^2,$$

where r is the rank of Q_1. Thus there exists an orthogonal P such that $A = PLP^{-1}$ and $B = PMP^{-1}$, where L and M, when partitioned so as to separate the rows and columns into successive groups of r and $n - r$, are of the forms

$$L = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right], \qquad M = \left[\begin{array}{c|c} E & C \\ \hline C' & F \end{array}\right].$$

Here D stands for an r-rowed diagonal matrix having $d_1 \cdots d_r$ in its diagonal, 0 for various matrices whose elements are all zero, and E, C and F for arbitrary matrices of appropriate dimensions. Then

$$| 1 - \lambda A | = | P(1 - \lambda L)P^{-1} | = | P | \cdot | 1 - \lambda L | \cdot | P^{-1} | = | 1 - \lambda L |,$$

and in the same way,

$$| 1 - \mu B | = | 1 - \mu M |$$

and

$$| 1 - \lambda A - \mu B | = | 1 - \lambda L - \mu M |.$$

We thus have

$$| 1 - \lambda L | \cdot | 1 - \mu M | \equiv | 1 - \lambda L - \mu M |.$$

From this identity it follows that a pair of forms Q_1^* and Q_2^*, quadratic in a set of variates normally and independently distributed with zero means and unit variances, and having matrices L and M respectively, are independent.

Since $AB = PLMP^{-1}$, the theorem will be proved if we can show that $LM = 0$. Let

$$M_1 = \left[\begin{array}{c|c} E & C \\ \hline C' & 0 \end{array}\right], \qquad M_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right],$$

so that $M = M_1 + M_2$. Since obviously $LM_2 = 0$, we need only to show that $LM_1 = 0$.

[2] E.g., Bôcher, *Introduction to Higher Algebra*, pp. 169, 305.

Let $Q_2^* = Q' + Q''$, where Q' and Q'' are quadratic forms in the particular normally distributed variates considered, and have matrices M_1 and M_2 respectively. Since Q'' obviously does not involve any of the variates occurring in Q_1^* and since all the variates are independent it follows that Q_1^* is independent of Q''. Since it has been shown above that Q_1^* is also independent of Q_2^*, it must be independent of the difference $Q_2^* - Q'' = Q'$. Therefore

$$| 1 - \lambda L | \cdot | 1 - \mu M_1 | \equiv | 1 - \lambda L - \mu M_1 | .$$

We have:

$$| 1 - \lambda L | = \prod_{i=1}^{r} (1 - \lambda d_i).$$

Also,

$$1 - \lambda L - \mu M_1 = \left[\begin{array}{c|c} 1 - \lambda D - \mu E & -\mu C \\ \hline -\mu C' & 1 \end{array}\right].$$

Consequently equating the terms of highest degree in λ on the two sides of the identity

$$\Pi(1 - \lambda d_i) \, | 1 - \mu M_1 | \equiv | 1 - \lambda L - \mu M_1 |$$

yields the identity in μ,

$$| 1 - \mu M_1 | \equiv 1,$$

or upon putting $\mu = 1/x$,

$$| M_1 - x | = (-x)^n.$$

Hence all the latent roots of the real symmetric matrix M_1 are zero. Now for a symmetric matrix the sum of the squares of the latent roots equals the sum of the squares of the elements, since both equal the trace of the square of the matrix. Therefore $M_1 = 0$. Consequently $LM_1 = 0$ and the theorem is established.

Since $M_1 = 0$, the following further result is obvious:

Two independent quadratic forms in a set of variates normally and independently distributed with zero means and a common variance can by a transformation be reduced to two forms having no variate in common.

But one of the disjunct sets of variates in the forms as thus reduced is not necessarily independent of the other set. For example, if x_1, x_2, x_3, x_4 are normally distributed with equal variances and any fixed non-vanishing correlation, the same in each of the six pairs, the sets (x_1, x_2) and (x_3, x_4) are not independent of each other, but the forms $(x_1 + x_2)^2$ and $(x_3 - x_4)^2$ are, since $x_1 + x_2$ is uncorrelated with $x_3 - x_4$.

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