mal). The numbers would have equal probabilities insofar as this is attainable by chaining. To obtain a random three-digit decimal series it would be necessary to reject the numbers above 999 (decimal). This would amount to only 2.34% of the available data. As before, rejection could be accomplished easily in the binary series by use of a ten-stage electronic counter.

Several promising devices are being considered for tabulating random numbers in accordance with the principles discussed herein. Electronic or electrical systems actuated by cosmic rays seem to be the most desirable. Tabulating equipment may be wired to turn out random numbers, possibly as a by-product of other card runs.

If only a few random numbers are needed, they can be obtained by much simpler methods. For example, a coin may be tossed, letting heads and tails represent $+1$ and $-1$, respectively The product of $k$ successive tosses would be tabulated as the random binary variable. Products equal to $+1$ and $-1$ would be coded as 1 and 0, respectively. Blocks of binary symbols would then be converted to the decimal system as described above.

## REFERENCE

[1] Tippett, L. H. C., *Random Sampling Numbers*, Tracts for Computers, No. 15, Cambridge University Press, 1927.

# NOTE ON THE ERROR IN INTERPOLATION OF A FUNCTION OF TWO INDEPENDENT VARIABLES

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Suppose that $g$ is a functon of one real variable $x$ and $h$ is an interpolation function such that $g(x) = h(x)$ for $x = x_1, x_2, \cdots, x_n$. Let $f(x) = g(x) - h(x)$ and suppose that $\frac{d^n}{dx^n} f(x)$ exists in an interval containing the points $x_0, x_1, \cdots, x_n$. Then the error in interpolation may be estimated from the well-known relation

$$f(x_0) = \frac{f^{(n)}(\xi)}{n!} (x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n), \tag{1}$$

where $\xi$ is some point in the smallest interval containing $x_0, x_1, \cdots, x_n$.

In the most usual case, where $h(x)$ is a polynomial of degree less than $n$, we have $f^{(n)}(\xi) = g^{(n)}(\xi)$.

It is natural to consider the corresponding situation for functions of two independent real variables $x$ and $y$. Let $g$ and $h$ be two functions such that $g(x, y) = h(x, y)$ for $n$ points $x = x_i, y = y_i (i = 1, 2, \cdots, n)$. Setting $f(x, y) = g(x, y) - h(x, y)$ as before, we have $f(x_i, y_i) = 0$ for $i = 1, 2, \cdots, n$. Then if $(x_0, y_0)$

is a point at which $g$ and $h$ are defined, we may ask whether there is any formula corresponding to (1) from which the error $f(x_0, y_0)$ can be estimated.

Some restrictions must be placed upon the function $f$ if any interesting results are to be obtained. Let us suppose that $f(x, y)$ can be expanded in a Taylor series about each of the points $(x_i, y_i)(i = 0, 1, \cdots, n)$ with a region of convergence sufficient to include all the points of the set. These conditions are more stringent ones than will be required for obtaining the later results; on the other hand, they would almost always be satisfied in any practical problem of interpolation, so it scarcely seems worthwhile to look for the weakest possible conditions at this point.

The first case of real interest is $n = 3$. It follows from the general statement of Taylor's theorem with the remainder that

$$\begin{aligned} 0 = f(x_i, y_i) &= f(x_0, y_0) + (x_i - x_0) f_x(x_0, y_0) + (y_i - y_0) f_y(x_0, y_0) \\ &+ \tfrac{1}{2}[(x_i - x_0)^2 f_{xx}(\xi_i, \eta_i) + 2(x_i - x_0)(y_i - y_0) f_{xy}(\xi_i, \eta_i) \\ &+ (y_i - y_0)^2 f_{yy}(\xi_i, \eta_i)] \qquad (i = 1, 2, 3), \end{aligned} \tag{2}$$

where $(\xi_i, \eta_i)$ is a point on the line segment joining $(x_0, y_0)$ and $(x_i, y_i)$ for $i = 1, 2, 3$.

The equation (2) may be regarded as a set of three linear equations in the two quantities $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. The condition that these shall be consistent is

$$\begin{vmatrix} f(x_0, y_0) + U_1 & x_1 - x_0 & y_1 - y_0 \\ f(x_0, y_0) + U_2 & x_2 - x_0 & y_2 - y_0 \\ f(x_0, y_0) + U_3 & x_3 - x_0 & y_3 - y_0 \end{vmatrix} = 0, \tag{3}$$

where

$$U_i = \tfrac{1}{2}[(x_i - x_0)^2 f_{xx}(\xi_i, \eta_i) + 2(x_i - x_0)(y_i - y_0) f_{xy}(\xi_i, \eta_i) + (y_i - y_0)^2 f_{yy}(\xi_i, \eta_i)] \qquad (i = 1, 2, 3).$$

If the three points $(x_i, y_i)$ $(i = 1, 2, 3)$ are not in a straight line, (3) can be written in the form

$$f(x_0, y_0) = - \frac{\begin{vmatrix} U_1 & x_1 - x_0 & y_1 - y_0 \\ U_2 & x_2 - x_0 & y_2 - y_0 \\ U_3 & x_3 - x_0 & y_3 - y_0 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}. \tag{4}$$

This expression is analogous to (1), though far less simple and elegant in form.

A similar treatment can evidently be used in all cases of the type $n = \dfrac{m(m+1)}{2}$.

For example, for $n = 6$ the equation corresponding to (4) is

$$
(5) \qquad f(x_0, y_0) = -\frac{\begin{vmatrix} V_1 & x_1 - x_0 & y_1 - y_0 & (x_1 - x_0)^2 & (x_1 - x_0)(y_1 - y_0) & (y_1 - y_0)^2 \\ V_2 & x_2 - x_0 & y_2 - y_0 & (x_2 - x_0)^2 & (x_2 - x_0)(y_2 - y_0) & (y_2 - y_0)^2 \\ V_3 & x_3 - x_0 & y_3 - y_0 & (x_3 - x_0)^2 & (x_3 - x_0)(y_3 - y_0) & (y_3 - y_0)^2 \\ V_4 & x_4 - x_0 & y_4 - y_0 & (x_4 - x_0)^2 & (x_4 - x_0)(y_4 - y_0) & (y_4 - y_0)^2 \\ V_5 & x_5 - y_0 & y_5 - y_0 & (x_5 - x_0)^2 & (x_5 - x_0)(y_5 - y_0) & (y_5 - y_0)^2 \\ V_6 & x_6 - x_0 & y_6 - y_0 & (x_6 - x_0)^2 & (x_6 - x_0)(y_6 - y_0) & (y_6 - y_0)^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1^2 & x_1 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2^2 & x_2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3^2 & x_3 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4^2 & x_4 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5^2 & x_5 & x_5y_5 & y_5^2 \\ 1 & x_6 & y_6^2 & x_6 & x_6y_6 & y_6^2 \end{vmatrix}}
$$

where

$$
\begin{aligned}
V_i = \tfrac{1}{6}[(x_i - x_0)^3 f_{xxx}(\xi_i, \eta_i) &+ 3(x_i - x_0)^2(y_i - y_0) f_{xxy}(\xi_i, \eta_i) \\
+3(x_i - y_0) f_{xyy}(\xi_i, \eta_i) &+ (y_i - y_0)^3 f_{yyy}(\xi_i, \eta_i)] \quad (i = 1, 2, \cdots, 6).
\end{aligned}
$$

(Equation (5) breaks down only if the six points $(x_1, y_1) \cdots (x_6, y_6)$ lie on a single conic.)

As an example of the general case we may consider $n = 4$. We write

$$
\begin{aligned}
f(x_i, y_i) = f(x_0, y_0) &+ (x_i - x_0) f_x(x_0, y_0) + (y_i - y_0) f_y(x_0, y_0) \\
&+ \tfrac{1}{2}[(x_i - x_0)^2 f_{xx}(\xi_i, \eta_i) + 2(x_i - x_0)(y_i - y_0) f_{xy}(\xi_i, \eta_i) \\
&+ (y_i - y_0)^2 f_{yy}(\xi_i, \eta_i)] \qquad (i = 1, 2, 3, 4).
\end{aligned}
$$

Now,

$$
f_{xx}(\xi_i, \eta_i) = f_{xx}(x_0, y_0) + (\xi_i - x_0) f_{xxx}(\xi_i', \eta_i') + (\eta_i - y_0) f_{xxy}(\xi_i', \eta_i'),
$$

where $(\xi_i', \eta_i')$ is a point on the line segment between $(x_0, y_0)$ and $(\xi_i, \eta_i)$. Proceeding as before yields

$$
(6) \qquad f(x_0, y_0) = -\frac{\begin{vmatrix} W_1 & x_1 - x_0 & y_1 - y_0 & (x_1 - x_0)^2 \\ W_2 & x_2 - x_0 & y_2 - y_0 & (x_2 - x_0)^2 \\ W_3 & x_3 - x_0 & y_3 - y_0 & (x_3 - x_0)^2 \\ W_4 & x_4 - x_0 & y_4 - y_0 & (x_4 - x_0)^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 \\ 1 & x_2 & y_2 & x_2^2 \\ 1 & x_3 & y_3 & x_3^2 \\ 1 & x_4 & y_4 & x_4^2 \end{vmatrix}}
$$

with

$$W_i = \tfrac{1}{2}[(x_i - x_0)^2(\xi_i - x_0)f_{xxx}(\xi_i', \eta_i') + (x_i - x_0)^2(\eta_i - y_0)f_{xxy}(\xi_i', \eta_i')$$
$$+ 2(x_i - x_0)(y_i - y_0)f_{xy}(\xi_i, \eta_i) + (y_i - y_0)^2 f_{yy}(\xi_i, \eta_i)].$$

Corresponding formulas can be derived in this way for any value of $n$; in fact, several alternatives may be obtained in each case. In all cases the error $f(x_0, y_0)$ is given in terms of the derivatives of $g$ alone if a polynomial of a certain type is used for the interpolating function. For equation (4), the suitable polynomial would be $h(x, y) = a + bx + cy$; for (5), $h(x, y) = a + bx + cy + dx^2 + exy + fy^2$; for (6), $h(x, y) = a + bx + cy + dx^2$. If the interpolating function $h(x, y)$ is not so chosen, the formulas remain valid, but derivatives of $h$ will appear.

The same procedure is applicable to functions of any number of independent variables.

## ON A LEMMA BY KOLMOGOROFF

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The following lemma was proved by Kolmogoroff [1]:

*If $e_1, e_2, \cdots, e_n$ are independent events and $U$ an arbitrary event such that ($W(X)$ denoting the probability of $X$ and $W_e(X)$ the conditional probability of $X$ under the hypothesis of $e$)*

$$W_{e_k}(U) \geqq u, \qquad W(e_1 + \cdots + e_n) \geqq u.$$

Then

$$W(U) \geqq \tfrac{1}{9}u^2.$$

This result seems of some interest in itself and may also have practical applications, for it is easily seen that [2] in general if $e_1, e_2, \cdots, e_n$ are arbitrary no information about $W_{e_1+\cdots+e_n}(U)$ can be obtained from that about $W_{e_k}(U)$, $k = 1, \cdots, n$. From this point of view the constant 1/9 is interesting, though it is unimportant in Kolmogoroff's proof of the law of large numbers. Using his original method this constant can easily be improved to 1/8. However, the following method will give a better result. At the same time we shall put it into a more general form.

Let

$$W_{e_k}(U) \geqq \alpha, \qquad \sum_{k=1}^{n} W(e_k) \geqq \beta.$$