A NOTE ON DODGE'S CONTINUOUS INSPECTION PLAN[1]

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1. Summary. In his first continuous sampling plan [1], H. F. Dodge showed that his procedure guarantees an Average Outgoing Quality Limit (AOQL) with the assumption that the process is in a state of statistical control. It is proved in this paper that the Dodge procedure, without the assumption of control, guarantees an AOQL, although different from that specified by Dodge.

2. Introduction. In 1943, Dodge published a continuous sampling plan [1] in the *Annals of Mathematical Statistics*. The procedure, as stated by Dodge, is as follows:

"(a) At the outset, inspect 100 per cent of the units consecutively as produced and continue such inspection until i units in succession are found clear of defects.

"(b) When i units in succession are found clear of defects, discontinue 100 per cent inspection, and inspect only a fraction $1/k$ of the units, selecting individual sample units one at a time from the flow of product, in such a manner as to assure an unbiased sample.

"(c) If a sample unit is found defective, revert immediately to a 100 per cent inspection of succeeding units and continue until again i units in succession are found clear of defects, as in paragraph (a).

"(d) Correct or replace with good units all defective units found."

In his paper, Dodge studied the properties of this plan, and presented equations and charts for determining the Average Outgoing Quality Limit (AOQL) as functions of the parameters k and i, under the assumption that the process is in a state of statistical control. A production process is said to be in statistical control if there is a positive constant $p \leq 1$ such that, for every item produced, the probability that it is defective is p, and is independent of the state (defective or nondefective) of all the other items produced.

The purpose of this paper is to show that the Dodge procedure guarantees an AOQL whether or not the process is in a state of statistical control. It is proved, without the assumption of control, that for a given k and i, an AOQL is guaranteed. In fact, AOQL $= (k - 1)/k + i)$.

For a given k and i, the above value of the AOQL is always higher than that obtained with Dodge's equations. However, it is achieved when the process alternates between producing all defective items during partial inspection and producing all nondefective items during 100 per cent inspection. As Dodge points out, [4], this worst possible behavior for the process is not a realistic model. The

results, therefore, should not be interpreted as implying the existence of practical limitations in the use of the plan as Dodge recommends; especially since the AOQL is itself an upper bound, and the actual outgoing quality is usually smaller.

It is assumed throughout the paper, that observations are drawn at random, and each defective item found is replaced by a nondefective. The definition of the AOQL given in Section 3 is consistent with that given by Wald and Wolfowitz [2], and consequently many of the comments presented in their section on Fundamental Notions (pp. 30–32) are pertinent to this paper.

3. Method of proof. Define

v_j = number of defects being passed in the jth cycle ($j = 1, 2, \cdots , m$). A cycle is the period where partial inspection begins, to the time a defective is observed.

k = an integer such that when the process is on partial inspection, one out of k items is inspected.

i = number of consecutive items free from defectives before partial inspection can begin.

T_j = number of items undergoing 100 per cent inspection from the end of the $(j-1)$st cycle until i consecutive items are observed free from defects; and such that $T_j + i$ is the total number of items being inspected 100 per cent from the end of the $(j-1)$st cycle to the beginning of the jth cycle. $T_j \geqq 0$.

N_j = number of groups of k items on partial inspection in the jth cycle. $N_j \geqq 1$.

Define AOQL = smallest number L with the property that for every process the probability is zero that

$$(1) \quad \limsup_{m \to \infty} \frac{\sum_{j=1}^{m} v_j}{\sum_{j=1}^{m} (T_j + i + kN_j)} = \limsup_{m \to \infty} \frac{\sum_{j=1}^{m} v_j/m}{\sum_{j=1}^{m} (T_j + i + kN_j)/m} > L.$$

To obtain the AOQL it is evidently sufficient to consider the special class of processes where the number of defectives in every segment on partial inspection $\geqq 1$. Since $N_j \geqq 1$, $T_j \geqq 0$, we have for any such process

$$(2) \quad \limsup_{m \to \infty} \frac{\sum_{j=1}^{m} v_j/m}{\sum_{j=1}^{m} (T_j + i + kN_j)/m} \leqq \lim_{m \to \infty} \frac{\sum_{j=1}^{m} v_j/m}{i + k}.$$

(We shall show that the limit on the right-hand side exists and is equal to $k - 1$ with probability one.)

It is important to note that the random variables v_j are dependent. However, if it can be shown that

$$E(v_j \mid v_1, \cdots , v_{j-1}) = k - 1$$

$$\sum_{j=1}^{\infty} \frac{E(v_j^2 \mid v_1, \cdots , v_{j-1})}{j^2} < \infty,$$

the Strong Law of Large Numbers [5] can be applied to the numerator of the right-hand side of the inequality of (2) and we then obtain

$$(3) \qquad \text{AOQL} \leqq \frac{k-1}{k+i}.$$

In fact, if the process is such that the proportion of defective items is zero whenever items are on complete inspection, and the proportion of defective items is 1 whenever items are on partial inspection, then

$$\limsup_{m \to \infty} \frac{\sum_{j=1}^{m} v_j/m}{\sum_{j=1}^{m} (T_j + i + kN_j)/m} = \frac{k-1}{k+i}.$$

Hence it follows that AOQL $= (k-1)/(k+i)$.

4. A lemma on the boundedness of $E(v_j^r \mid n_{j1}, n_{j2}, \cdots)$. Let n_{ji} be the number of defectives in the ith group of k items after the start of the jth cycle, $i = 1, 2, \cdots$. By definition of the expected value of a discrete random variable

$$(4) \quad \begin{aligned} E(v_j^r \mid n_{j1}, n_{j2}, \cdots) &= \frac{n_{j1}}{k} (n_{j1} - 1)^r + \left(1 - \frac{n_{j1}}{k}\right) \frac{n_{j2}}{k} (n_{j1} + n_{j2} - 1)^r \\ &\quad + \left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)\left(\frac{n_{j3}}{k}\right)(n_{j1} + n_{j2} + n_{j3} - 1)^r + \cdots. \end{aligned}$$

The sth term of the right-hand side of equation (4) is bounded by $(1 - 1/k)^{s-1} (sk)^r$ so that

$$(5) \qquad E(v_j^r \mid n_{j1}, n_{j2}, \cdots) < \sum_{s=1}^{\infty} (1 - 1/k)^{s-1} (sk)^r.$$

For any finite r, the right-hand side of inequality (5) is finite and independent of the n_{ji}.

5. Theorems and proofs.
THEOREM 1.

$$E(v_j \mid v_1, \cdots, v_{j-1}) = k - 1.$$

PROOF. Define

$$E(v_j \mid n_{j1}, n_{j2}, \cdots) = \varphi(n_{j1}, n_{j2}, \cdots)$$

$$(6) \quad \varphi(n_{j1}, n_{j2}, \cdots) = \frac{n_{j1}}{k} (n_{j1} - 1) + \left(1 - \frac{n_{j1}}{k}\right) [n_{j1} + \varphi(n_{j2}, n_{j3}, \cdots)].$$

Using the recursion formula a finite number of times, it follows that

$$\varphi(n_{j1}, n_{j2}, \cdots) = \left(1 - \frac{1}{k}\right)\left[n_{j1} + \left(1 - \frac{n_{j1}}{k}\right)n_{j2} + \left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)n_{j3}\right.$$

$$(7) \quad + \cdots + \left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)\left(1 - \frac{n_{j3}}{k}\right)\cdots\left(1 - \frac{n_{j(p-1)}}{k}\right)n_{jp}\right]$$

$$+ \left[\left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)\cdots\left(1 - \frac{n_{jp}}{k}\right)\right]\varphi(n_{j(p+1)}, n_{j(p+2)}, \cdots).$$

Let p approach infinity in equation (7). From the results of Section 4 $\varphi(n_{ji}, n_{j(i+1)}, \cdots)$ is bounded for all i. Also,

$$\lim_{p \to \infty}\left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)\cdots\left(1 - \frac{n_{jp}}{k}\right) = 0 \text{ so that}$$

$$\varphi(n_{j1}, n_{j2}, \cdots) = \left(1 - \frac{1}{k}\right)\left[n_{j1} + \left(1 - \frac{n_{j1}}{k}\right)n_{j2}\right.$$

$$(8)$$

$$\left. + \left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)n_{j3} + \cdots\right.$$

Dividing $\varphi(n_{j1}, n_{j2}, \cdots)$ by $k - 1$ in equation (8)

$$\frac{\varphi(n_{j1}, n_{j2}, \cdots)}{k - 1} = \frac{n_{j1}}{k} + \left(1 - \frac{n_{j1}}{k}\right)\frac{n_{j2}}{k}$$

$$(9)$$

$$+ \left(1 - \frac{n_{j1}}{k}\right)\left(1 - \frac{n_{j2}}{k}\right)\left(\frac{n_{j3}}{k}\right) + \cdots.$$

From probabilistic considerations, the right-hand side of (9) sums to 1, since it is just the probability of obtaining one defective item in the jth cycle, which is 1. Consequently,

$$(10) \qquad \varphi(n_{j1}, n_{j2}, \cdots) = E(v_j \mid n_{j1}, n_{j2}, \cdots) = k - 1.$$

The proof of the theorem follows from the identity

$$(11) \quad E(v_j \mid v_1, \cdots, v_{j-1}) = E[E\{v_j \mid v_1, \cdots, v_{j-1}; n_{j1}, n_{j2}\} \mid v_1, \cdots, v_{j-1}].$$

But the probability distribution of $v_j \mid v_1, \cdots, v_{j-1}, n_{j1}, n_{j2}, \cdots$ is a function only of $n_{j1}, n_{j2}, \cdots$. Hence

$$(12) \quad E\{v_j \mid v_1, v_2, \cdots, v_{j-1}, n_{j1}, n_{j2}, \cdots\} = E\{v_j \mid n_{j1}, n_{j2}, \cdots\} = k - 1$$

so that $E(v_j \mid v_1, \cdots, v_{j-1}) = k - 1$, and the theorem is proved.

THEOREM 2.

$$\sum_{j=1}^{\infty} \frac{E(v_j^2 \mid v_1, \cdots, v_{j-1})}{j^2} < \infty.$$

PROOF. Letting $r = 2$ in equation (5), it follows that $E(v_j^r \mid n_{j1}, n_{j2}, \cdots) < 2k^5 - k^4$. Once again, using the identity

$$E(v_j^2 \mid v_1, \cdots, v_{j-1}) = E[E\{v_j^2 \mid v_1, v_2, \cdots, v_{j-1}, n_{j1}, \cdots\} \mid v_1, \cdots, v_{j-1}]$$

and the fact that the probability distribution of $v_j^2 \mid v_1, \cdots, v_{j-1}, n_{j1}, n_{j2}, \cdots$ is a function only of $n_{j1}, n_{j2}, \cdots$, it follows that

$$(13) \qquad\qquad E(v_j^2 \mid v_1, \cdots, v_{j-1}) < 2k^5 - k^4.$$

Consequently

$$\sum_{j=1}^{\infty} \frac{E v_j^2 \mid v_1, \cdots, v_{j-1}}{j^2} < \infty,$$

and the theorem is proved.

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ON THE POWER OF A ONE-SIDED TEST OF FIT FOR CONTINUOUS PROBABILITY FUNCTIONS[1]

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Summary. If $F(x)$ is a continuous distribution function of a random variable X, and $F_n(x)$ the empirical distribution function determined by a sample X_1, $X_2, \cdots, X_n$, then the probability $\Pr\{F(x) \leqq F_n(x) + \epsilon \text{ for all } x\}$ is known [1] to be a function $P_n(\epsilon)$, independent of $F(x)$. A closed expression for $P_n(\epsilon)$ and a table of some of its values were presented in [2]. In the present paper $P_n(\epsilon)$ is used to test a hypothesis $F(x) = H(x)$ against an alternative $F(x) = G(x)$. The power of this test is studied and sharp upper and lower bounds for it are obtained for alternatives such that $\sup_{-\infty < x < +\infty}\{H(x) - G(x)\} = \delta$, with preassigned δ. The results of [2] are assumed known.