

## RICHARD VON MISES' WORK IN PROBABILITY AND STATISTICS

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Professor Richard von Mises of Harvard University died in June, 1953, shortly after his 70th birthday. A native of Austria, he took his doctor's degree at the Technical University of Vienna in 1907, and then acted as lecturer and professor at various universities in Austria and Germany until 1920, when he became professor and director of the Institute for Applied Mathematics of the University of Berlin. The Hitler regime, depriving the German universities of so many of their best men, brought von Mises to Istanbul, and finally, in 1939, to Harvard. At Harvard, he first was professor of mathematics, and in 1944 became Gordon McKay professor of aerodynamics and applied mathematics.

Richard von Mises was one of those men who have both the ability and the energy requisite for taking an active and creative interest in many widely different fields. He has made outstanding contributions to subjects as heterogeneous as literary criticism, positivistic philosophy, aerodynamics, and probability. In this short notice, we shall be concerned exclusively with those of his works that belong to the field of probability and mathematical statistics.

It is well known that Richard von Mises is one of the significant names in the history of the tremendous development that has taken place in this field during the last thirty years. As can be seen from the appended Selected Bibliography, his works on probability and mathematical statistics range from books and papers on the foundations of probability which, of course, always represented one of his main interests, to investigations dealing with special problems in various statistical applications. Only a few of these works can be explicitly mentioned here, but it will be attempted to characterize and to follow up some of the main lines of thought, along which his contributions seem to group themselves.

In the year 1919, the two basic papers (37) and (40), which were practically the first publications by von Mises on probability, appeared almost simultaneously. The first of these, "Fundamentalsätze der Wahrscheinlichkeitsrechnung," was concerned with the general theorem in mathematical probability for which, a year later, Georg Pólya was to propose the now well known name "the central limit theorem." The second paper, "Grundlagen der Wahrscheinlichkeitsrechnung," on the other hand, gave the first exposition of von Mises' views with respect to the foundations of probability theory. Each of these two papers was to become the first in a long series of works, and the two groups of works thus initiated may perhaps be looked upon as containing the most important of von Mises' contributions to the subject.

In order to judge the two basic papers correctly, it is necessary to realize the situation in mathematical probability theory about the year 1919. Since the appearance of the classical treatise of Laplace, a few mathematicians—Tcheby-

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cheff, Markov, Borel, and some others—had done important work in the field, but the conceptual foundations on which the whole subject rested were still obscure. There was no commonly accepted definition of mathematical probability, and in so far as there were any definitions at all, they were clearly inadequate for the numerous applications that were made in fields such as population statistics, molecular physics, and many others. Moreover, with few exceptions, mainly belonging to the French and Russian schools, writers on probability did not seem to feel under any obligation to conform to the standards of rigor that were regarded as obvious in other parts of mathematics. The admirable work of Lyapunov on the central limit theorem seemed to be entirely unknown among mathematicians.

In the introductions to his two above-mentioned papers, von Mises gives a review of the situation, and arrives at the conclusion, which seems entirely justified, that “today, probability theory is not a mathematical science.” He then develops his own program for building up probability theory as a mathematical science, starting from the thesis that *probability theory should be regarded as the mathematical theory of a group of observed phenomena, in the same way as, for example, geometry and theoretical mechanics*. Just as geometry gives an idealized mathematical picture of the large bulk of our observations with respect to the configuration and position of bodies in space, so probability theory should be constructed to provide a mathematical model of the statistical regularities observed in cases where a given experiment or observation may be repeated a large number of times under similar conditions.

Starting from this thesis, von Mises develops in (40) his system of foundations, which was soon to become familiar to all probabilists. We find here the concept of a collective, the definition of mathematical probability as the limit of a frequency ratio, and the two fundamental postulates, requiring the existence of the limiting values of the relevant frequencies, and their invariance under any place selection. It is shown how the main rules for operating with probabilities can be deduced from these basic principles, and a system of classification of the operations used in probability theory is worked out.

The publication of the “Grundlagen” paper aroused a great deal of interest among mathematicians, statisticians, and philosophers. Quite naturally, opinions were divided, and even if the basic view of probability theory as a mathematical theory of random phenomena was, on the whole, completely endorsed by most mathematicians and statisticians, the collective concept and the two postulates were severely criticized by many authors. An extensive literature grew up about these questions, and von Mises himself took an active part in the discussion. Besides in a number of papers dealing with special problems, particularly related to the second postulate, the foundations of the subject are discussed in his two treatises (75) and (127) and, above all, in his well known book (64). In this book, “Probability, Statistics and Truth,” which has been translated from the original German edition into English, Russian, and Spanish, he gives a detailed exposition of his system, intended for nonmathematical readers, and also his replies to various criticisms, and his comments on alternative systems proposed by others.

It is particularly interesting to read in this book, as well as in the discussion with Doob published in (117), his comments on the measure-theoretic approach to the subject favored by a certain number of contemporary mathematicians and statisticians. Even though this approach, as pointed out, for example, in the well known book by Kolmogoroff, starts from a conception of the object and character of probability theory, which is very close to the one advocated by von Mises himself, he takes a strongly critical position against this method of introducing the concept of mathematical probability and formulating the axioms expressing its basic properties. In a chapter with the expressive title, "A part of the theory of sets? No!," he declares that probability theory "remains in all circumstances a theory of certain observable phenomena, which are idealized in the concept of a collective." So far, many of his opponents would be prepared to agree, at least partly. But then he goes on to say that, from this point of view, he finds it impossible to "concede the existence of a separate concept of probability based on the theory of sets, which is sometimes said to contradict the concept of probability based on the notion of relative frequency," and that "there can be also no question of reconciling these two concepts." This seems to be a final expression of his standpoint in the question. The discussion will undoubtedly be continued during many years to come, but, however the question of the best choice of axiomatic foundations of probability theory may be decided (if it will ever be decided at all), it will always stand out as the great achievement of von Mises to have been the first to draw general attention to the problem, to have indicated the way along which its possible solutions should be sought, and to have given one such solution.

Let us now pass to the second main group of von Mises' writings in our field, the one that begins with the "Fundamentalsaetze" paper (37). As already mentioned, this paper is concerned with the central limit theorem, and with certain other problems belonging to the same general range of ideas. A proof of the asymptotic normality of the distribution of a sum of independent random variables is given under certain rather restrictive conditions. There is a detailed discussion of the asymptotic behavior of the distribution of the sum in the case when the distributions of the terms belong to one of those simple classes that are usually encountered in the applications. Similar results are given in respect of the asymptotic behavior of the a posteriori distributions obtained by applying Bayes' theorem to a sample of observations.

Von Mises returned to this subject in a great number of his works, continually improving his results and extending the field of problems considered. The most important papers belonging to this group are (87), (89), (93), (102), (105) and (126). With respect to the central limit theorem itself, his main results were superseded by others, but he soon generalized the problem in a very interesting way, where he was able to find important new results, and which still seems to open possibilities for further research. We shall give a brief characterization of the problem considered in this group of his writings, which may be said to culminate in the paper (126). Let  $U(x_1, \dots, x_n)$  be a symmetric function of the independent random variables  $x_i$ , which are assumed to have a common distribution

function  $F(x)$ . (Von Mises considers the general case of unequal distributions.) Then  $U$  may be regarded as a function  $V(S_n)$  of the repartition function  $S_n = S_n(x)$ , where  $nS_n(x)$  denotes for every  $x$  the number of  $x_i \leq x$ . It is assumed that the function  $V$  can be defined on a convex domain  $D$  in the space of all distribution functions, including the given distribution function  $F$  as well as all possible repartitions  $S_n$  with  $n = 1, 2, \dots$ . It is required to study the asymptotic behavior of the distribution of the random variable  $V(S_n)$  as  $n$  tends to infinity. Following Volterra, von Mises defines the derivatives of the function  $V$  at any given point of  $D$  and shows that, subject to certain general regularity conditions, the distribution of  $V$  is asymptotically normal if the first derivative of  $V$  at the point  $F$  is different from zero. This covers, among others, the classical case of the sum of the  $x_i$ , and also most ordinary statistics based on moments. When the first derivative vanishes, certain nonnormal limiting distributions are obtained, and von Mises gives a detailed discussion of the case (including, among others, the case of the  $\chi^2$  statistic) when there is a nonvanishing second derivative. The limiting distribution in this case is shown to be intimately related to the Fredholm determinant of a certain symmetric kernel. As already mentioned, interesting problems in this direction still seem to be open for research.

Finally, we shall only briefly mention some other main groups among the works of von Mises. In the papers (96), (100), (111), and (112), the relations between various moments and other characteristics of a probability distribution, and between these characteristics and the values of the corresponding distribution function, are studied, and some important inequalities are given. The papers (109), (119) and (121) are concerned with Bayes' theorem and its various applications, a subject which also receives great attention in the treatises (75) and (127). Von Mises did not sympathize with the tendency in contemporary mathematical statistics to avoid the use of Bayes' theorem and the concept of a priori probability. In the works belonging to this group, he discusses the application of the theorem to various problems, including the problem of testing statistical hypotheses where, according to him, it leads to more reliable results than the methods now currently employed by mathematical statisticians.

As a glance at the list of publications will show, there are many works of von Mises in this field that have not been mentioned at all in the above. The majority of these papers are concerned with various applications of probability theory, in fields as diverse as physics, genetics, demography, and actuarial science.

This brief and insufficient review of a small part of the writings of Richard von Mises will certainly be enough to give the reader a strong impression of an active and powerful scientific personality. Those who knew him saw, in addition, many other sides of his personality, giving him a human charm that his friends will never forget.

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