THE DISTRIBUTION OF RADIAL ERROR

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1. Summary. An expression (equation 2) suitable for computation is obtained for the probability distribution of $\zeta = (\xi^2 + \eta^2)^{\frac{1}{2}}$ where ξ and η are independent Gaussian variables with, in general, unequal means and unequal variances.

2. Introduction. The probability distribution for ζ has direct application in considering the radial error in situations where a point must be located by two coordinates as in navigation. It also has application in the study of turbulence in fluids [1] and in other fields. In these applications ξ and η are not necessarily uncorrelated. However since a rotation of coordinates reduces the mathematical problem in which the variables to be combined are correlated to one where they are uncorrelated, ξ and η are considered to be uncorrelated in this note.

Before deriving equation (2) it is desirable to point out how the present result fits in with related work in the literature. An expression for the characteristic function for ζ' is given by Patnaik [3] whose result in fact holds for the n dimensional case, $n \geqq 1$. If the variances of ξ and η are each σ^2, the variable ζ^2/σ^2 is governed by the noncentral χ^2 probability distribution with two degrees of freedom. The noncentral χ^2 distribution in n dimensions is given in terms of an infinite series in [3] and in Tang [6]. The series essentially represents a Bessel function $I_j(x)$ of the first kind, imaginary argument and order $j = \frac{1}{2}n - 1$ so that one can write for the probability density of $\sigma^2\chi^2$.

$$g(x) = \frac{1}{2\sigma^2} e^{-(a+x)/2\sigma^2} \left(\frac{x}{\lambda}\right)^{(n-2)/4} I_{\frac{1}{2}n-1}\left(\frac{\sqrt{\lambda x}}{\sigma^2}\right) \tag{1}$$

where λ is the sum of squares of the means of the n variables. This result in the two dimensional case is used to represent the probability density for the noise plus signal following a quadratic detector in electrical circuit theory and is derived and applied in [2].

The "method of mixtures" due to Robbins and Pitman is used by them in [4] to express the noncentral χ^2 distribution as an infinite sum of central χ^2 distributions in the n dimensional case and the method can be applied to the present problem involving unequal variances. In addition a form of the distribution for ζ in the two dimensional case where one mean is zero is given by Frenkiel in [1]. Frenkiel's result is an infinite series in which the nth term involves the $(2n)$th derivative of $I_n(x)$. Both this result and the result by the method of mixtures appear to the writer to be less adaptable to computation than equation (2).

3. Result and derivation. The method used here to obtain $p(r)$, the probability density or frequency function of ζ, is simply to write the bivariate Gaussian

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distribution in polar coordinates and integrate over the angular variable. The immediate result is a double summation of products of Bessel functions which is then reduced to a single summation by application of an addition theorem for Bessel functions. The end result is

$$p(r) = Ar \exp\left[\frac{-r^2(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right] \left[I_0(ar^2)I_0(dr) + 2\sum_{j=1}^{\infty} I_j(ar^2)I_{2j}(dr)\cos 2j\psi\right], \tag{2}$$

where

$$A = \frac{1}{\sigma_1\sigma_2}\exp\left[-\frac{m_1^2\sigma_2^2 + m_2^2\sigma_1^2}{2\sigma_1^2\sigma_2^2}\right],$$

$$a = \frac{\sigma_1^2 - \sigma_2^2}{4\sigma_1^2\sigma_2^2}, \quad b = \frac{m_1}{\sigma_1^2}, \quad c = \frac{m_2}{\sigma_2^2}, \quad d^2 = b^2 + c^2,$$

and

$$\tan\psi = \frac{m_2\sigma_1^2}{m_1\sigma_2^2}.$$

Here σ_1^2 and σ_2^2 are the variances of ξ and η and m_1 and m_2 their means. (In [8] the $I_j(x)$ are tabulated for $j = 0(1)20$, $x = 0(.1)20$ or 25. In [5] they are tabulated for $j = 0(1)22$ and $x = 0(.01$ or $.02)5$.) Note that equation (2) with $\sigma_1 = \sigma_2 = \sigma$ reduces, as it should, to $2rg(r^2)$ when $g(r)$ is given by equation (1) evaluated for $n = 2$.

To derive the result (2) let $p(x, y)$ be the bivariate Gaussian probability density which governs ξ and η. Then in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, the corresponding probability density is

$$P(r, \theta) = \frac{Ar}{2\pi}\exp\left[\frac{-r^2(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right]\exp\left[ar^2\cos 2\theta + br\cos\theta + cr\sin\theta\right]. \tag{3}$$

The second exponential is to be integrated over θ from 0 to 2π. Call the resulting integral L. By substituting into L the expansion for the usual generating function for the Bessel coefficients $J_n(z)$;

$$e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n \tag{4}$$

expressed in the forms

$$e^{z\sin\theta} = \sum_{n=-\infty}^{\infty} I_n(z)e^{ni(\theta+\frac{1}{2}\pi)}, \quad e^{z\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(z)e^{ni\theta} \tag{5}$$

one obtains

$$L = \int_0^{2\pi} d\theta \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \left\{I_j(ar^2)I_k(br)I_l(cr)\exp\left[i\theta(2j + k + l) + \frac{3i\pi l}{2}\right]\right\}. \tag{6}$$

The order of integration and summation may be interchanged. From the definition of L it is clear that L is real so that only the real part of this expression need be integrated. The only nonvanishing contributions to the integral occur when

$$2j + k + l = 0, \qquad l = 2n \tag{7}$$

where n is an integer. The result is that

$$L = 2\pi \sum_{j=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^n I_j(ar^2) I_{2n+2j}(br) I_{2n}(cr). \tag{8}$$

To reduce equation (8) to a single summation the following special case of an addition theorem given in Watson [7] is applied to the summation over n;

$$I_n(\sqrt{Z^2 + z^2}) \cos n\psi = \sum_{r=-\infty}^{\infty} (-1)^r I_{n+2r}(Z) I_{2r}(z), \quad \tan \psi = z/Z. \tag{9}$$

The result leads directly to equation (2).

REFERENCES

[1] F. N. Frenkiel, "Frequency distributions of velocities in turbulent flow," *J. Meteorol.*, Vol. 8, (1951), pp. 316–320.

[2] J. L. Lawson and G. E. Uhlenbeck, *Threshold Signals*, McGraw-Hill Book Co., (1950), p. 195.

[3] P. B. Patnaik, "The non-central χ^2 and F-distributions and their applications," *Biometrika*, Vol. 36 (1949), pp. 202–232.

[4] Herbert Robbins and E. J. G. Pitman, "Application of the method of mixtures to quadratic forms in normal variates," *Ann. Math. Stat.*, Vol. 20 (1949), pp. 552–560.

[5] Wasao Shibagaki, "Tables of the Modified Bessel Functions," Kyushi University Physics Department, 1946.

[6] P. C. Tang, "The power function of the analysis of variance tests with tables and illustrations of their use," *Statistical Research Memoirs, II*, University of London, (1938), pp. 126–149.

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Macmillan, 1948, Equation 7, p. 361.

[8] *Tables of the Bessel-Functions*, Part II, Vol. 10, "Functions of positive integer order," British Association for the Advancement of Science, 1952.

CORRECTION TO "ON CERTAIN CLASSES OF STATISTICAL DECISION PROCEDURES"*

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I am indebted to Dr. L. Le Cam for pointing out an error in the above-named paper (*Annals of Math. Stat.*, Vol. 24 (1953), pp. 440–448). Let

$$\mathfrak{D}^{(a)} = \{\delta \,\varepsilon\, \mathfrak{D}: r(F, \delta) \text{ is bounded by a function of } F\}.$$

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