ADMISSIBLE TESTS FOR THE MEAN OF A RECTANGULAR DISTRIBUTION[1]

By Allan Birnbaum

Columbia University

1. Summary. Explicit characterizations are given of the minimal complete class and a minimal essentially complete class of tests of a simple hypothesis specifying the mean of a uniform distribution of known range. Examples are given of tests which are optimal against various alternatives.

2. Introduction. Let $p_\theta(x)$ be the density function of a uniform distribution with mean θ and known range R. Without loss of generality we may assume $R = 1$, so that

$$p_\theta(x) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} < x \leqq \theta + \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

for any real value of θ.

Consider the problem of testing a simple hypothesis specifying the value of θ on the basis of a sample of $n(n \geqq 2)$ random independent observations $x_1, x_2, \ldots x_n$. Without loss of generality we may take the hypothesis to be $H_0:\theta = 0$. We shall consider tests of H_0 against the general composite alternative hypothesis $H_1:\theta \neq 0$.

It is known that a minimal sufficient statistic [1] for $p_\theta(x)$ is (u, v), where $u = \min_i x_i$, $v = \max_i x_i$. For all statistical purposes we may restrict our attention to the range of (u, v) as sample space rather than the range of $(x_1, x_2, \cdots x_n)$, as is shown for example in [2]. Thus we take as sample space $T = \{(u, v) \mid u \leqq v < u + 1\}$. Any test procedure may then be represented by a decision function $\delta(u, v)$, where $\delta(u, v)$ is a real-valued Lebesgue-measurable function defined on T, satisfying $0 \leqq \delta(u, v) \leqq 1$, and such that the test procedure rejects H_0 with probability $\delta(u, v)$ when (u, v) is observed. Hereafter by "a test δ" we shall mean a test represented by a Lebesgue-measurable function $\delta = \delta(u, v)$ of the kind just described, and hereafter "measure" will refer to Lebesgue measure. It is to be noted that in the following sections "the class of all tests" will be understood to be $\mathfrak{D} = \{\delta(u, v)\}$, and not the class of all decision functions $\delta(x_1, x_2, \cdots x_n)$ defined on the original sample space.

The distribution of (u, v) is given by the density function

$$p_\theta(u, v) = n(n - 1)\, k_\theta(u, v)(v - u)^{n-2}$$

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where

$$k_\theta(u, v) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} < u \leqq v \leqq \theta + \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Characterization of the minimal complete classes. For any test δ, let $\beta_\delta(\theta)$ be the probability of accepting H_0 when θ is the true mean; that is,

$$\beta_\delta(\theta) = \int_T p_\theta(u, v)(1 - \delta(u, v))\, du\, dv.$$

Let $r_\delta(\theta)$, the risk function of the test δ, be defined as the probability of an incorrect decision; that is,

$$r_\delta(\theta) = \begin{cases} \beta_\delta(\theta), & \theta \neq 0, \\ 1 - \beta_\delta(\theta), & \theta = 0, \end{cases}$$

Let $\mathfrak{A}'$ be the class of nonrandomized decision functions each having as acceptance region the interior of the subset of $T_0 = \{(u, v) \mid -\frac{1}{2} < u \leqq v < \frac{1}{2}\}$ which lies above the graph of an arbitrary nondecreasing function $v(u)$.

The class $\mathfrak{A}$ of tests is defined as follows: $\delta \in \mathfrak{A}$ if and only if there exists a $\delta' \in \mathfrak{A}'$ such that the set

$$\{(u, v) \mid \delta(u, v) \neq \delta'(u, v)\}$$

has measure 0.

THEOREM 1. *$\mathfrak{A}$ is the class of essentially unique Bayes solutions.*

PROOF. For any test δ and any cumulative distribution function $\xi(\theta)$ we have

$$r_\delta(\xi) = \int_{-\infty}^{\infty} r_\delta(\theta) d\xi(\theta).$$

From the definition of $r_\delta(\theta)$ we have, letting $\gamma = \xi(0+) - \xi(0-)$,

$$\begin{aligned} r_\delta(\xi) &= \int_{-\infty}^{\infty} \beta_\delta(\theta)\, d\xi(\theta) + \int_{0-}^{0+} (1 - 2\beta_\delta(\theta))\, d\xi(\theta) \\ &= \gamma + \int_{T_0} [\xi(u + \tfrac{1}{2}) - \xi(v - \tfrac{1}{2}) - 2\gamma]\, p_0(u, v)(1 - \delta(u, v))\, du\, dv \\ &\quad + \int_{T - T_0} \left[\int_{-\infty}^{\infty} p_\theta(u, v)\, d\xi(\theta)\right] (1 - \delta(u, v))\, du\, dv. \end{aligned}$$

To minimize $r_\delta(\xi)$ with respect to δ it clearly suffices to define

$$\delta(u, v) = \begin{cases} 0 & \text{if } \xi(u + \frac{1}{2}) - \xi(v - \frac{1}{2}) - 2\gamma < 0, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Now assume $\delta \in \mathfrak{A}$. Let $v(u)$ be that single-valued nondecreasing function which characterizes δ, in the manner described in the definition of $\mathfrak{A}$ above, to within a set of measure 0. Let $u(v') = \inf \{u \mid v(u) \geqq v'\}$, $-\frac{1}{2} \leqq v' \leqq \frac{1}{2}$.

Let $\xi_1(\theta)$ be any cumulative distribution function which places zero mass in the open interval $(-1, 1)$ and positive density at every other point. Let $\xi(\theta) = \frac{1}{2}(\xi_1(\theta) + \xi_2(\theta))$, where $\xi_2(\theta)$ is the cumulative distribution function defined by

$$\xi_2(\theta) = \begin{cases} 0, & \theta \leqq -1 \\ \frac{1}{3}[\frac{1}{2} + u(\theta + \frac{1}{2})], & -1 < \theta \leqq 0, \\ 2\gamma + \dfrac{\theta}{3}, & 0 < \theta \leqq 1, \quad \text{where} \quad \gamma = \xi(0), \\ 1, & 1 < \theta. \end{cases}$$

To verify that $\xi_2(\theta)$ is nondecreasing, it suffices to note that $u(v')$ is nondecreasing and never exceeds $\frac{1}{2}$, and hence $\gamma \leqq 1/3$. We obtain as a Bayes solution relative to $\xi(\theta)$, after simplification,

$$\delta_0(u, v) = \begin{cases} 0 & \text{if} \quad (u, v) \in T_0 \quad \text{and} \quad u < u(v), \\ 1 & \text{otherwise.} \end{cases}$$

It is clear δ_0 is an essentially unique Bayes solution relative to $\xi(\theta)$, and hence that δ is also.

Conversely let δ_0 be an essentially unique Bayes solution relative to some $\xi(\theta)$. Then one Bayes solution with respect to $\xi(\theta)$ is given by δ as defined in (1) above. Since δ_0 is an essentially unique Bayes solution relative to $\xi(\theta)$, $\delta_0 = \delta$ almost everywhere. Since $\xi(u + \frac{1}{2}) - \xi(v - \frac{1}{2})$ is nondecreasing in u and nonincreasing in v, it follows that $\delta \in \mathfrak{A}$ and hence that $\delta_0 \in \mathfrak{A}$.

THEOREM 2. *$\mathfrak{A}'$ is a minimal essentially complete class.*

PROOF. Let ζ be the set of $\xi(\theta)$'s which are everywhere strictly increasing, and let C_ζ be the class of Bayes solutions relative to members of ζ. Then the assumptions of Wald's Theorem 3.19 in [3] are satisfied; the conclusion of the theorem asserts that the closure $\bar{C}_\zeta$ of C_ζ is essentially complete.

To show that $\mathfrak{A} \supset \bar{C}_\zeta$, let ξ be any element of ζ, and let δ be the corresponding Bayes solution given by (1). The derivation of (1) shows that δ is essentially unique if ξ is everywhere strictly increasing. Since $\delta \in \mathfrak{A}$, $\mathfrak{A} \supset C_\zeta$. Since $\mathfrak{A}$ is closed, $\mathfrak{A} \supset \bar{C}_\zeta$. Thus $\mathfrak{A}$ is essentially complete, and it follows that $\mathfrak{A}'$ is essentially complete.

Let δ and δ' be any two different elements of $\mathfrak{A}'$. Since $\delta \neq \delta'$ on a set of positive measure, and since each is an essentially unique Bayes solution, it follows that for some θ', $r_\delta(\theta') > r_{\delta'}(\theta')$, and for some θ'', $r_\delta(\theta'') < r_{\delta'}(\theta'')$. Thus no element of $\mathfrak{A}'$ is uniformly as good as any different element of $\mathfrak{A}'$. Hence $\mathfrak{A}'$ is minimal essentially complete.

COROLLARY. *$\mathfrak{A}$ is the minimal complete class.*

PROOF. It was shown above that $\mathfrak{A}$ is an essentially complete class consisting of admissible tests. If δ is admissible but not in $\mathfrak{A}$, then $\mathfrak{A}$ contains a test δ' with $r_{\delta'}(\theta) \equiv r_\delta(\theta)$. Since δ' is an essentially unique Bayes solution, $\delta = \delta'$ almost everywhere, and $\delta \varepsilon \mathfrak{A}$, a contradiction. Hence there is no admissible test not in $\mathfrak{A}$, and $\mathfrak{A}$ is complete.

It is interesting that the conclusions of Wald's Theorems 5.5 and 5.7, which characterize complete classes of tests, become virtually empty in the case of the present problem because these classes contain virtually all tests with acceptance regions in T_0.

4. Examples.

Example 1. One-sided alternative. If a test of size α having high power against alternatives $\theta > 0$ is desired, the Neyman-Pearson lemma may be used to construct the (essentially unique) best test of H_0 against the simple alternative $H_1: \theta = 1 - \alpha^{1/n}$. This test is characterized by

$$v(u) = \begin{cases} u, & -\frac{1}{2} \leqq u < \frac{1}{2} - \alpha^{1/n}, \\ \frac{1}{2}, & \frac{1}{2} - \alpha^{1/n} \leqq u \leqq \frac{1}{2}. \end{cases}$$

By using the Neyman-Pearson lemma to construct best tests of H_0 of size α against any simple alternative H_1: $\theta = \theta'$, $0 < \theta' < 1 - \alpha^{1/n}$, one can verify that the above test has, at *each* θ, $\theta > 0$, the maximum power attainable by any test of size α; hence the above test is uniformly most powerful against the composite alternative H_1: $\theta > 0$.

Example 2. Two-sided alternative, locally best tests. If a test of size α, having the greatest possible power against alternatives θ close to zero, is desired, we may take the test characterized by

$$v(u) = \begin{cases} u, & -\frac{1}{2} \leqq u < -\frac{\alpha^{1/n}}{2}, \\ \frac{\alpha^{1/n}}{2}, & -\frac{\alpha^{1/n}}{2} \leqq u \leqq \frac{\alpha^{1/n}}{2}, \\ u, & \frac{\alpha^{1/n}}{2} \leqq u \leqq \frac{1}{2}. \end{cases}$$

As in Example 1 it can be verified that this test has at each θ, $|\theta| \leqq \frac{1}{2}(1 - \alpha^{1/n})$, the maximum power attainable by any test of size α; hence this test is uniformly most powerful against the composite alternative H_1: $0 < |\theta| \leqq \frac{1}{2}(1 - \alpha^{1/n})$. Again using the Neyman-Pearson lemma as above it can be shown that this test is, among all admissible tests, uniformly *least* powerful against the composite alternative H_1: $|\theta| \geqq \frac{1}{2}(1 + \alpha^{1/n})$. It is of interest to have such a simple example of a locally best test which is the worst possible of all admissible tests against certain "intermediate" alternatives (all tests A contained in T_0 being good against "distant" alternatives).

Example 3. Two-sided alternative with indifference zone. If a test of size α is desired having the greatest possible power against all alternatives except possibly values of θ close to 0, we may take the test characterized by:

CASE A. $\alpha \leqq (\frac{1}{2})^{n-1}$:

$$v(u) = \begin{cases} \left(\frac{\alpha}{2}\right)^{1/n} - \frac{1}{2}, & -\frac{1}{2} \leqq u \leqq \left(\frac{\alpha}{2}\right)^{1/n} - \frac{1}{2}, \\ u, & \left(\frac{\alpha}{2}\right)^{1/n} - \frac{1}{2} \leqq u < \frac{1}{2} - \left(\frac{\alpha}{2}\right)^{1/n}, \\ \frac{1}{2}, & \frac{1}{2} - \left(\frac{\alpha}{2}\right)^{1/n} \leqq u \leqq \frac{1}{2}; \end{cases}$$

CASE B. $\alpha > (\frac{1}{2})^{n-1}$:

$$v(u) = \begin{cases} v_0, & -\frac{1}{2} \leqq u < -v_0, \\ \frac{1}{2}, & -v_0 \leqq u \leqq \frac{1}{2}, \end{cases}$$

where v_0 is determined by

$$1 - \alpha = \int_{-\frac{1}{2}}^{-v_0} \int_{v_0}^{\frac{1}{2}} p_0(u, v)\, dv\, du,$$

Comparison with the power function in the previous example shows that there exists no uniformly most powerful unbiased test of H_0.

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REFERENCES

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