Upper bounds have been tabulated by these two authors and are shown to be attained for the c.d.f. $P(x)$ of which the inverse function $x(P)$ is given by

$$x(P) = \frac{(2n-1)^{\frac{1}{2}}}{\left\{2\left[1 - \frac{1}{\binom{2n-2}{n-1}}\right]\right\}^{\frac{1}{2}}}[P^{n-1} - (1-P)^{n-1}]. \tag{2}$$

However, there is no lower bound (other than zero) since $E(w_n)/\sigma$ can be made arbitrarily small for certain universes.

From the point of view of practical applications this last defect is particularly disappointing as we often have to deal with data which are clearly not normal whilst no definite alternate distribution is known. In such situations one is, however, often able to assume that the data may be graduated by a finite range distribution (not extending beyond the range $a\sigma \leqq x \leqq b\sigma$) *without making any further assumptions as to the shape of the distribution.*

It is remarkable that if the above very wide assumption is made about $P(x)$ it is, in fact, possible to derive lower bounds for $E(w_n)/\sigma$ which are well above zero and in certain cases fairly near the upper bounds.

To fix the ideas we assume that $E(x) = 0$ and $\sigma = 1$ and then proceed to consider distributions for which it is known that $-X \leqq x \leqq X$. If the range of x is $a \leqq x \leqq b$, that is, is asymmetrically placed about the mean or origin, X has to be taken as max $(-a, b)$. In this case the upper and lower bounds are not necessarily attained.

Before discussing the problem arising when x is restricted we shall derive the maximum of $E(x_n)$ in the unrestrained case, abandoning the condition of symmetry in the parental population imposed by Moriguti. An upper bound for $E(x_m)$ $(m = 1, 2, \cdots, n)$ will also be given.

3. Upper bound of the expectation of x_m. We consider first the extreme x_n.

$$E(x_n) = n \int_0^1 x(P)P^{n-1}\, dP. \tag{3}$$

This is to be maximized for functions $x(P)$ subject to

$$\int_0^1 x\, dP = 0 \quad \text{and} \quad \int_0^1 x^2\, dP = 1. \tag{4}$$

From the calculus of variations we find the stationary solution

$$P(x) = \left(\frac{ax+1}{n}\right)^{1/(n-1)}, \qquad -\frac{(2n-1)^{\frac{1}{2}}}{n-1} \leqq x \leqq (2n-1)^{\frac{1}{2}} \tag{5}$$

where $a = (n-1)/(2n-1)^{\frac{1}{2}}$, and

$$E(x_n) = \frac{n(2n-1)^{\frac{1}{2}}}{n-1}\int_0^1 P^{n-1}(nP^{n-1} - 1)\, dP = (n-1)/(2n-1)^{\frac{1}{2}}.\text{[2]} \tag{6}$$

It will now be shown that (6) gives in fact the upper bound of $E(x_n)$ and that this is attained for the distribution of (5).

[2] The details of this derivation are omitted since they are given in the preceding paper.

By Schwarz's inequality we have

$$\int_0^1 nP^{n-1}\left(x+\frac{1}{a}\right)dP \leqq \left\{\left(\int_0^1 (nP^{n-1})^2\,dP\right)\left(\int_0^1 \left(x+\frac{1}{a}\right)^2 dP\right)\right\}^{\frac{1}{2}}.$$

Hence

$$E(x_n)+\frac{1}{a} \leqq \left\{\frac{n^2}{2n-1}\left(1+\frac{1}{a^2}\right)\right\}^{\frac{1}{2}}$$

or

$$E(x_n)+\frac{(2n-1)^{\frac{1}{2}}}{n-1} \leqq \frac{n^2}{(n-1)(2n-1)^{\frac{1}{2}}}$$

that is,

(6′) $$E(x_n) \leqq (n-1)/(2n-1)^{\frac{1}{2}},$$

equality occurring for (5). This upper bound of $E(x_n)$ is tabulated in Table 1 and may be compared with Moriguti's upper bound applicable to symmetrical populations only.

Precisely as above it can be shown that

$$|\,E(x_m)\,| \leqq \left(\frac{B(2m-1,\ 2n-2m+1)}{[B(m,\ n-m+1)]^2}-1\right)^{\frac{1}{2}}.$$

TABLE 1

Sample size n	*Upper bound of* $E(x_n)$ *(symmetrical population)*	*Upper bound of* $E(x_n)$ *(any population)*
2	0.5774	0.5774
3	0.8660	0.8944
4	1.0420	1.1339
5	1.1701	1.3333
6	1.2767	1.5076
7	1.3721	1.6641
8	1.4604	1.8074
9	1.5434	1.9403
10	1.6222	2.0647
11	1.6974	2.1822
12	1.7693	2.2937
13	1.8385	2.4000
14	1.9052	2.5019
15	1.9696	2.5997
16	2.0320	2.6941
17	2.0926	2.7852
18	2.1514	2.8735
19	2.2087	2.9592
20	2.2645	3.0424

However, this upper bound can be attained by a probability distribution only if $m = n$ (or $m = 1$), as the stationary solution is

$$x \propto \frac{1}{B(m, n - m + 1)} P^{m-1}(1 - P)^{n-m} - 1$$

and this expression for x is monotonic only if $m = n$ or $m = 1$.

4. Upper bound of $E(w_n)$ for $-X \leqq x \leqq X$. When introducing the finite variate range $-X \leqq x \leqq X$ we stated that this restriction raised the lower bound of $E(w_n)$. The restraint may, however, also cause a reduction of the "free" upper bound of $W_n = E(w_n)$ found by Plackett and Moriguti. Since Moriguti confines himself to finding the upper bound for symmetrical distributions we first show that his solution applies generally and provides the maximum ratio of mean range to standard deviation in the competitor class of unrestrained $x(P)$. To show this let us start with a general $x(P)$ and write $y(p) = x(p + \frac{1}{2})$ which we split into a symmetrical and anti-symmetrical part by setting $y(p) = o(p) + e(p)$ where $o(-p) = -o(p)$ and $e(-p) = e(p)$. Now $(p + \frac{1}{2})^{n-1} - (\frac{1}{2} - p)^{n-1}$ is clearly odd in p. Hence W_n is unchanged if $y(p)$ is replaced by $o(p)$, but

$$\sigma^2\{o(p)\} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} o^2(p)\,dp$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} y^2(p)\,dp - \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^2(p)\,dp \leqq \int_{-\frac{1}{2}}^{+\frac{1}{2}} y^2(p)\,dp = 1.$$

Hence W_n/σ is increased by the removal of $e(p)$. Finally "scaling up" $o(p)$, that is, introducing $co(p)$ so as to satisfy $\sigma^2\{co(p)\} = 1$ we have

$$W_n\{co(p)\} = cW_n\{o(p)\} \geqq W_n\{y(p)\}$$

which shows that, in finding the maximum of W given $\sigma = 1$ we may confine ourselves to symmetrical distributions, that is, odd $y(p) = x(\frac{1}{2} + p)$. This maximum is attained for the finite range distribution (2) for which we have

$$|x| \leqq \frac{(2n-1)^{\frac{1}{2}}}{\left\{2\left[1 - 1\Big/\binom{2n-2}{n-1}\right]\right\}^{\frac{1}{2}}} = X_n \tag{7}$$

say. It follows that if we seek a maximum under the restraint that $|x(P)| \leqq X$, the solution is still given by (2) provided $X \geqq X_n$ while for $X < X_n$ the restrained maximum will be reduced. The critical quantity X_n is tabulated below:

n	2	3	4	5	6	7	8	9	10	
X_n	1.732	1.732	1.919	2.137	2.350	2.551	2.739	2.916	3.082	
n	11	12	13	14	15	16	17	18	19	20
X_n	3.240	3.391	3.535	3.674	3.808	3.937	4.062	4.183	4.301	4.416

We proceed to find the maximum when $| x(P) | \leqq X < X_n$. The solution $x_0(P)$ (say) suggested by the calculus of variation is now

$$\begin{aligned} a_x x_0(P) &= n(P^{n-1} - Q^{n-1}) && \text{for } P_1 \leqq P \leqq 1 - P_1 \\ x_0(P) &= -X && \text{for } P \leqq P_1 \\ x_0(P) &= X && \text{for } 1 - P_1 \leqq P. \end{aligned} \tag{8}$$

where P_1 is related to the constant $a_x > 0$ by

$$a_x X = n(-P_1^{n-1} + (1 - P_1)^{n-1}). \tag{9}$$

a_x is determined to satisfy $\sigma^2(x_0) = 1$, and $Q = 1 - P$. To prove that this solution $x_0(P)$ does in fact yield a maximum we denote by $x_1(P)$ any other competitor function satisfying the conditions $\sigma^2(x_1) = 1$ and $| x_1 | \leqq X$ and write with obvious notation:

$$\frac{1}{n}[W_n(x_1) - W_n(x_0)] = \int_0^1 (x_1 - x_0)(P^{n-1} - Q^{n-1})\,dP - \frac{1}{2n} a_X \int_0^1 (x_1^2 - x_0^2)\,dP$$

$$= \int_0^1 (x_1 - x_0)\left(P^{n-1} - Q^{n-1} - \frac{a_X}{n} x_0\right) dP - \frac{a_X}{2n}\int_0^1 (x_1 - x_0)^2\,dP$$

$$= \int_0^{P_1} (x_1 + X)\left(P^{n-1} - Q^{n-1} + \frac{a_X X}{n}\right) dP$$

$$+ \int_{1-P_1}^1 (x_1 - X)\left(P^{n-1} - Q^{n-1} - \frac{a_X X}{n}\right) dP - \frac{a_X}{2n}\int_0^1 (x_1 - x_0)^2\,dP$$

in virtue of (8).

But clearly

$$P^{n-1} - Q^{n-1} + a_X X/n \leqq 0 \text{ and } x_1 \geqq -X \qquad \text{for } P \leqq P_1$$

and

$$P^{n-1} - Q^{n-1} - a_X X/n \geqq 0 \text{ and } x_1 \leqq X \qquad \text{for } P \geqq 1 - P_1$$

so that $w_n(x_1) < w_n(x_0)$. To evaluate the maximum of W_n when $X < X_n$ we note from (1), (8) and (9)

$$W_n = n\int_{P_1}^{1-P_1} \frac{n}{a_X}(P^{n-1} - Q^{n-1})^2\,dP + 2X(1 - P_1^n - (1 - P_1)^n)$$

$$= \frac{2nX\left\{(1 - P_1)^{2n-1} - P_1^{2n-1} - (2I_{1-P_1}(n, n) - 1)\Big/\binom{2n-2}{n-1}\right\}}{(2n-1)((1-P_1)^{n-1} - P^{n-1})} + 2X(1 - (1 - P_1)^n - P_1^n). \tag{10}$$

Here P_1 is given by the condition

$$\int_{\frac{1}{2}}^1 x_0^2(P)\,dP = \tfrac{1}{2} \tag{11}$$

that is, from the equation

$$(12)\quad \frac{X^2}{(2n-1)((1-P_1)^{n-1}-P_1^{n-1})^2} \left\{(1-P_1)^{2n-1}-P_1^{2n-1}-(2I_{1-P_1}(n,n)-1)\Big/\binom{2n-2}{n-1}\right\}+P_1X^2=\tfrac{1}{2}.$$

From (10) we therefore obtain as the upper bound

$$(13)\quad W_n = n(1-2P_1X^2)((1-P_1)^{n-1}-P_1^{n-1})/X + 2X(1-(1-P_1)^n-P_1^n)$$

where P_1 is the root of (12).

5. Lower bound of $E(w_n)$ for $-X \leqq x \leqq X$. Reverting to the probability integral $P(x)$ in place of its inverse $x(P)$ we turn now to the problem of minimizing

$$(14)\qquad W\{P\} = \int_{-X}^{X}(1-P^n-Q^n)\,dx$$

subject to

$$(15)\qquad \mu = X - \int_{-X}^{X}P(x)\,dx = 0$$

$$(16)\qquad \sigma^2 = X^2 - 2\int_{-X}^{X}xP(x)\,dx = 1$$

and

$$(17)\qquad -X \leqq x \leqq X.$$

Without loss of generality we may confine ourselves to step-functions of (say) m "internal" steps, namely

$$(18)\qquad \begin{cases} P(x) = 0 & x \leqq x_1 \\ P(x) = P_i & x_i < x < x_{i+1} \qquad (i = 1, 2 \cdots n) \\ P(x) = 1 & x \geqq x_{m+1} \end{cases}$$

where $0 < P_1 < \cdots < P_m < 1$; for by the Euler-Maclaurin theorem we can' with any accuracy desired, approximate to the integrals (14), (15) and (16) by step-functions, provided m is taken sufficiently large. Hence the lower bound of W given by (14) may be determined for the step-functions (18).

LEMMA A. *For any m-step-function $P(x)$ (see (18)) with $m \geqq 3$, satisfying* (15), (16) *and* (17), *a new step-function $P^*(x)$ also satisfying* (15), (16) *and* (17) *can be found for which the number of steps is reduced by at least one and* $W\{P^*\} \leqq W\{P\}$.

PROOF. Keeping the x_i unchanged we define P^* as follows:

$$(19)\qquad \begin{cases} P^* = P_i + \Delta_i^* & i = 1, 2, 3 \\ P^* = P_i & i > 3 \end{cases}$$

where the Δ_i^* will be determined in the form $\Delta_i^* = \rho\Delta_i$ with the common scale-factor, ρ, being subsequently chosen.

In order that P^* should satisfy (15) and (16) we have

$$\text{(20)} \qquad \begin{cases} \sum_{i=1}^{3} \Delta_i(x_{i+1} - x_i) = 0 \\ \sum_{i=1}^{3} \Delta_i(x_{i+1}^2 - x_i^2) = 0. \end{cases}$$

Condition (17) is automatically satisfied.

From (18) we have

$$\text{(21)} \qquad W\{P\} = \sum_{i=1}^{m} (1 - P_i^n - Q_i^n)(x_{i+1} - x_i)$$

and hence, using Taylor's expansion up to the second-order term, that

$$\text{(22)} \qquad \begin{aligned} W\{P^*\} - W\{P\} = &-n\sum_{i=1}^{3} (P_i^{n-1} - Q_i^{n-1})(x_{i+1} - x_i)\Delta_i^* \\ &-\tfrac{1}{2}n(n-1)\sum_{i=1}^{3}[(P_i + \vartheta_i\Delta_i^*)^{n-2} + (Q_i - \vartheta_i\Delta_i^*)^{n-2}](x_{i+1} - x_i)\Delta_i^{*2} \end{aligned}$$

where $0 < \vartheta_i < 1$. Consider now the 3×3 matrix

$$M = \left\| \begin{matrix} x_{i+1} - x_i \\ x_{i+1}^2 - x_i^2 \\ (P_i^{n-1} - Q_i^{n-1})(x_{i+1} - x_i) \end{matrix} \right\| \qquad \text{with columns } i = 1, 2, 3.$$

Let r denote the rank of M. We distinguish two cases: (a) $r < 3$. In this case we can satisfy the equations (20) as well as

$$\text{(23)} \qquad n\sum_{i=1}^{3} (P_i^{n-1} - Q_i^{n-1})(x_{i+1} - x_i)\Delta_i = 0$$

by a set of Δ_i with $\sum\Delta_i^2 > 0$. Clearly the set $\rho\Delta_i$ also satisfies (20) and (23). For sufficiently small ρ we obviously have

$$\text{(24)} \qquad 0 < P_1 + \rho\Delta_1 < \cdots < P_3 + \rho\Delta_3 < P_4 < \cdots < P_m < 1.$$

Now if we increase ρ continuously, a point is reached, when one (or more) of the inequality signs of (24) is changed into an equality sign. Taking this value for ρ we have correspondingly a P^* of at most $(m - 1)$ steps. Further, from (22) and (23)

$$\text{(25)} \qquad W\{P^*\} \leqq W\{P\}.$$

(b) $r = 3$. This allows us to satisfy (20) and

$$n\sum_{i=1}^{3} (P_i^{n-1} - Q_i^{n-1})(x_{i+1} - x_i)\Delta_i = 1$$

with a set of Δ_i, not all zero, so that the set $\Delta_i^* = \rho\Delta_i(\rho > 0)$ satisfies (20) and

$$\text{(26)} \qquad n\sum_{i=1}^{3}(P_i^{n-1} - Q_i^{n-1})(x_{i+1} - x_i)\Delta_i^* = \rho.$$

We again choose ρ as in (a) to obtain P^* and in virtue of (22) and (26) reach the required result (25).

So far we have shown that the minimum of $W(P)$ can be determined by restricting $P(x)$ to step-functions of at most $m = 2$ internal steps. Writing

$$p_1 = P_1, \quad p_2 = P_2 - P_1, \quad p_3 = 1 - P_2 \tag{27}$$

we now prove a further lemma.

LEMMA B. *W cannot attain its minimum for a set of x_i, p_i $(i = 1, 2, 3)$ satisfying*

$$-X < x_1 < x_2 < x_3 < X \tag{28}$$

and

$$p_i > 0. \tag{29}$$

PROOF. In terms of p_i conditions (15) and (16) become

$$\mu = \sum_{i=1}^{3} p_i x_i = 0 \tag{30}$$

$$\sigma^2 = \sum_{i=1}^{3} p_i x_i^2 = 1. \tag{31}$$

Keeping p_i constant we alter x_i to x_i^* by

$$x_i^* = x_i + \rho \Delta x_i . \tag{32}$$

The Δx_i will be determined to satisfy, in the first instance, the conditions

$$\mu\{P^*\} = 0, \sigma^2\{P^*\} > 1 \text{ and } W\{P^*\} = W\{P\} \tag{33}$$

namely,

$$\begin{cases} \sum p_i \Delta x_i = 0 \\ \sum p_i x_i \Delta x_i = \delta \\ (1 - P_1^n - Q_1^n)(\Delta x_2 - \Delta x_1) + (1 - P_2^n - Q_2^n)(\Delta x_3 - \Delta x_2) = 0 \end{cases} \tag{33'}$$

where δ will be specified forthwith. If for the rank r of the matrix of equations (33′) we have $r < 3$ we solve for $\delta = 0$, if $r = 3$ we solve for $\delta = 1$. In either case $\sum \Delta x_i^2 > 0$. For sufficiently small ρ we have, by (28)

$$-X < x_1 + \rho\Delta x_1 < x_2 + \rho\Delta x_2 < x_3 + \rho\Delta x_3 < X$$

and as before choose the smallest $\rho > 0$ which converts at least one of the inequalities into an equality.

Since

$$\sigma^2\{P^*\} = \sigma^2\{P\} + 2\rho\delta + \rho^2 \sum p_i \Delta x_i^2$$

and all $p_i > 0$ we have $\sigma^2\{P^*\} > \sigma^2\{P\} = 1$ so that conditions (33) are satisfied. We now need merely introduce a new distribution P' which takes the discrete frequencies p_i at the $x-$ values

$$x_i' = x_i^*/\sigma\{P^*\}$$

and see at once that

$$\mu\{P'\} = 0, \qquad \sigma^2\{P'\} = 1, \quad \text{and} \quad W\{P'\} < W\{P\}$$

which proves Lemma B.

Completion of the solution. From Lemma B it follows that the minimum can occur only for the following sets of p_i and x_i:

(a) One of the p_i is zero. This corresponds to a two-point distribution and includes the cases $x_1 = x_2$ or $x_2 = x_3$.

(b) $$p_i > 0 \qquad (i = 1, 2, 3)$$

and

(i) $$-X < x_1 < x_2 < x_3 = X$$

or

(ii) $$-X = x_1 < x_2 < x_3 < X.$$

(c) $$p_i > 0 \qquad (i = 1, 2, 3)$$

and

$$-X = x_1 < x_2 < x_3 = X.$$

We proceed to rule out (b) and (c). First consider (b) where we may confine ourselves to (i).

Suppose that the minimum did occur for a set of x_i, p_i satisfying (b), then clearly this set would have to satisfy the necessary conditions resulting from Lagrange's method of undetermined multipliers for the "free" variables x_1, x_2, p_1, p_2, p_3 with side conditions $\sigma^2 = 1$, $\mu = 0$, $\sum p_i = 1$. It will be shown that a set satisfying these equations can not provide a minimum. We may write $W(P)$ as

$$W = A_1(x_2 - x_1) + A_3(x_3 - x_2) \tag{34}$$

(where $A_j = 1 - p_j^n - (1 - p_j)^n$ $j = 1, 3$) and the variables are subject to the side conditions

$$\sum p_i = 1, \qquad \sum p_i x_i = 0, \qquad \sum p_i x_i^2 = 1 \tag{35}$$

and

$$0 < p_i < 1, \qquad -X < x_1 < x_2 < x_3 = X. \tag{36}$$

It follows from partial differentiation with respect to x_1, x_2, p_1, p_2, p_3 that

$$\alpha p_1 x_1 + \beta p_1 - A_1 = 0 \tag{37}$$

$$\alpha p_2 x_2 + \beta p_2 + A_1 - A_3 = 0 \tag{38}$$

$$\tfrac{1}{2}\alpha x_1^2 + \beta x_1 + \gamma + (x_2 - x_1)A_1' = 0 \tag{39}$$

$$\tfrac{1}{2}\alpha x_2^2 + \beta x_2 + \gamma = 0 \tag{40}$$

$$\tfrac{1}{2}\alpha x_3^2 + \beta x_3 + \gamma + (x_3 - x_2)A_3' = 0, \tag{41}$$

α, β, γ being Lagrangean multipliers. From (39) minus (40) we find

$$\tfrac{1}{2}\alpha(x_1 + x_2) + \beta - A_1' = 0, \tag{42}$$

from (40) minus (41) we find

$$\tfrac{1}{2}\alpha(x_2 + x_3) + \beta + A_3' = 0, \tag{43}$$

from (42) minus (43) we find

$$\tfrac{1}{2}\alpha(x_1 - x_3) = A_1' + A_3', \tag{44}$$

from (37) and (38) we find

$$\alpha(x_1 - x_2)\, p_1 p_2 = p_2 A_1 - p_1(A_3 - A_1), \tag{45}$$

from (37) and (38) we find

$$\tfrac{1}{2}\alpha(x_1 + x_2) + \beta = \tfrac{1}{2}[A_1/p_1 - (A_1 - A_3)/p_2], \tag{46}$$

from (44 and (45) we find

$$\frac{x_2 - x_1}{x_3 - x_1} = \frac{p_1(A_1 - A_3) + p_2 A_1}{2p_1 p_2(A_1' + A_3')} = Q, \tag{47}$$

say, from (42) and (46) we find

$$(A_1 - A_3)/p_2 = (A_1 - 2p_1 A_1')/p_1. \tag{48}$$

Now, if $q_j = 1 - p_j$,

$$\begin{aligned} A_1' + A_3' &= n(q_1^{n-1} - p_1^{n-1} + q_3^{n-1} - p_3^{n-1}) \\ &= n[(p_2 + p_3)^{n-1} - p_3^{n-1} + (p_1 + p_2)^{n-1} - p_1^{n-1}] > 0. \end{aligned}$$

Hence from (44) we have $\alpha < 0$.

The matrix of second-order partial differential coefficients corresponding to equations (37) to (41) is

$$\begin{Vmatrix} (x_2 - x_1)A_1'' & 0 & 0 & \alpha x_1 + \beta - A_1' & A_1' \\ 0 & 0 & 0 & 0 & \alpha x_2 + \beta \\ 0 & 0 & (x_3 - x_2)A_3'' & 0 & -A_3' \\ \alpha x_1 + \beta - A_1' & 0 & 0 & \alpha p_1 & 0 \\ A_1' & \alpha x_2 + \beta & -A_3' & 0 & \alpha p_2 \end{Vmatrix} \tag{49}$$

It is sufficient to show that for certain x_1, x_2, p_1, p_2, p_3 satisfying the side-conditions (35) and (36) there results a value of W smaller than that computed from the above stationary solution. We keep x_2 constant and vary x_1 by an amount Δx_1. The second-order term of the Taylor expansion in the neighbourhood of stationary W is

$$\begin{aligned} I = (x_2 - x_1)A_1''(\Delta p_1)^2 &+ (x_3 - x_2)A_3''(\Delta p_3)^2 + \alpha p_1(\Delta x_1)^2 \\ &+ 2(\alpha x_1 + \beta - A_1')\,\Delta p_1 \Delta x_1. \end{aligned} \tag{50}$$

Now from (35) we have

$$(51)\qquad \begin{cases} p_1 = (1 + x_2 x_3)/[(x_2 - x_1)(x_3 - x_1)] \\ p_2 = (1 + x_3 x_1)/[(x_3 - x_2)(x_1 - x_2)] \\ p_3 = (1 + x_1 x_2)/[(x_1 - x_3)(x_2 - x_3)]. \end{cases}$$

It follows that

$$(52)\qquad \frac{\partial p_1}{\partial x_1} = p_1 \left(\frac{1}{x_3 - x_1} + \frac{1}{x_2 - x_1} \right)$$

and

$$(53)\qquad \frac{\partial p_3}{\partial x_1} = \frac{p_1}{x_3 - x_2} \left(\frac{x_2 - x_1}{x_3 - x_1} \right) = \frac{p_1 Q}{x_3 - x_2}.$$

The last term of (50) can be rewritten from (37), (48) and (52) as

$$\begin{aligned} 2\left(\frac{A_1}{p_1} - A_1'\right) \Delta p_1 \Delta x_1 &= \left(\frac{A_1}{p_1} + \frac{A_1 - A_3}{p_2}\right) \Delta p_1 \Delta x_1 \\ &= \left(\frac{A_1}{p_1} + \frac{A_1 - A_3}{p_2}\right)\left(\frac{1}{x_3 - x_1} + \frac{1}{x_2 - x_1}\right) p_1 (\Delta x_1)^2 \end{aligned}$$

neglecting differentials of higher than the second order.

In this way it can be shown that $I < 0$ if

$$(54)\qquad | A_1'' | (1 + Q)^2/Q^2 + | A_3'' | Q/(1 - Q) > 2(A_1' + A_3')/p_1 .$$

This inequality, a sufficient condition for ruling out the stationary solution as a minimum, is easily proved for $n = 2$ or 3. For in that case we have $A = npq$ and hence from (47)

$$\begin{aligned} Q &= [A_1/p_1 + (A_1 - A_3)/p_2]/[2(A_1' + A_3')] \\ &= n(q_1 + p_1 - p_3)/[2n(1 - 2p_1 + 1 - 2p_3)] = (1 - p_3)/(4p_2). \end{aligned}$$

Also from (48)

$$n(p_1 - p_3) = n(1 - p_1) - 2n(1 - 2p_1)$$

that is,

$$2p_1 + p_3 = 1$$

so that $p_1 = p_2 = p$ (say), $p_3 = 1 - 2p$ and $Q = \frac{1}{2}$.

Thus in (54)

$$\text{L.H.S.} = 2n(q + 1) > 4n = \text{R.H.S.}$$

For $n > 3$ it became necessary to establish by *numerical evaluation* that the value of W computed from the stationary solution is always larger than that computed from the two-point distribution of (a). The procedure used was as follows.

A representative range of values of p_1, was chosen, and the corresponding p_3 calculated from (48), which with $p_2 = 1 - p_1 - p_3$ is a relation between p_1 and p_3. Next, Q was found from (47) and then W and X, which could now be determined by (35); for

$$W/(x_3 - x_1) = A_1Q + A_3(1 - Q)$$

$$\sigma/(x_3 - x_1) = 1/(x_3 - x_1) = [p_2Q^2 + p_3 - (p_2Q + p_3)^2]^{\frac{1}{2}}$$

whence

$$W = [A_1Q + A_3(1 - Q)][p_2Q^2 + p_3 - (p_2Q + p_3)^2]^{-\frac{1}{2}}.$$

Also

$$X = (1 - p_3 - p_2Q)[p_2Q^2 + p_3 - (p_2Q + p_3)^2]^{-\frac{1}{2}}.$$

In this way corresponding values of X and W were built up, and it was easily seen that the value of W obtained from this stationary solution lay *well above* that calculated from the two-point solution (a) which is further discussed below. Attention was focused on the range $n = 4$ to 20, but the tables so obtained indicated that the *two-point solution leads to the smaller W for all X and n.*

It remains to eliminate case (c). This follows readily from the above approach. Thus, setting up equations of the type (37) to (41) we reach analogously to (48)

$$A_1'(1 - 2p_1) - A_3'(1 - 2p_3) + 2(A_1 - A_3) = 0 \tag{55}$$

or $F(p_1) - F(p_3) = 0$, where $F(p) = A'(1 - 2p) + 2A$. But $F'(p) = A''(1 - 2p) < 0$ for $0 < p < \frac{1}{2}$ and clearly $0 < p_1, p_3 < \frac{1}{2}$. Hence equation (55) can be satisfied only if $p_1 = p_3 = p$, say. This makes $x_2 = 0$ and in this case the second-order term of the Taylor expansion corresponding to (50) is easily shown to be negative. Thus the only stationary solution possible in case (c) is in fact a maximum.

Properties of the Solution. We may therefore evaluate the minimum under condition (a), that is we confine ourselves to two point distributions with probabilities p and q at $x_1 < 0$ and $x_2 > 0$ respectively. Without loss of generality we assume $-x_1 < x_2$. Instead of finding the minimum of W/σ under the condition $x_2 \leqq X$ we may determine the minimum for given x_2 and then consider it over the range $1 \leqq x_2 \leqq X$. But for any *fixed* x_2 the conditions $p + q = 1$, $px_1 + qx_2 = 0$, $px_1^2 + qx_2^2 = 1$ determine p, q and x_1 uniquely, in fact

$$p = x_2^2/(1 + x_2^2) \tag{56}$$

and, with this value of p, the mean range W is given by

$$W = (1 - p^n - q^n)/\sqrt{pq}. \tag{57}$$

Thus, introducing the functions $G_n(p) = (1 - p^n - q^n)/\sqrt{pq}$ and $p(x_2) = x_2^2/(1 + x_2^2)$, the mean range W is given by $W = G_n(p(x_2))$. In order to obtain the lower bound for the mean range $E(w_n)$ we must determine the minimum of

$G_n(p(x_2))$ over the range $1 \leqq x_2 \leqq X$. Now it is shown below that the minimum value of $G_n(p(x_2))$ must occur at one of the end points of the x_2- range that is, either for $x_2 = 1$ or for $x_2 = X$ so that we have the final result

$$W_{\text{lower bound}} = \min \begin{cases} G_n(p(1)) = 2(1 - (\tfrac{1}{2})^{n-1}) \\ G_n(p(X)) = (1 - p^n - q^n)/\sqrt{pq} \end{cases} \tag{58}$$

where $p = X^2/(1 + X^2)$ and $q = 1 - p$.

This is shown in Table 2 for $n = 2\ (2)\ 20$.

In the case of variate range $a \leqq x \leqq b$ we may still use the lower bound (58) with $X = \max(|a|, |b|)$. Since for the two point solution $x_1 = -1/x_2$, this involves no loss in the sharpness of the lower bound obtained unless $\min(|a|, |b|) < 1/X$. However, this situation of extreme skewness is clearly rare and it does not seem worth-while to consider it further.

It remains to show that for the range $1 \leqq x_2 \leqq X$, the function $G_n(p(x_2))$ takes its minima at either $x_2 = 1$ or at $x_2 = X$. Since $p(x_2)$ is monotonic it suffices

TABLE 2

Table of the lower bound of $E(w_n)$ given that $-X \leqq x \leqq X$

X	$n = 2$	4	6	8	10	12
1	1.000	1.750	1.938*	1.984	1.996	1.999
2	.800	1.472	1.844*	1.984†	1.996	1.999
3	.600	1.146	1.562	1.898†	1.996‡	1.999
4	.485	.915	1.296	1.633	1.932‡	1.999§
5	.392	.755	1.090	1.400	1.687	1.952§

* 1.938 is to be used for $1 \leqq X \leqq 1.20$ for larger X interpolate in Table 2.
† 1.984 is to be used for $1 \leqq X \leqq 2.64$ for larger X interpolate in Table 2.
‡ 1.996 is to be used for $1 \leqq X \leqq 3.76$ for larger X interpolate in Table 2.
§ 1.999 is to be used for $1 \leqq X \leqq 4.82$ for larger X interpolate in Table 2.

$p = \frac{X^2}{X^2+1}$	$X=$	$n=12$	14	16	18	20
.95	4.36	1.999*	2.000	2.000	2.000	2.000
.96	4.90	1.976*	2.000	2.000	2.000	2.000
.97	5.69	1.795	2.000†	2.000‡	2.000	2.000
.98	7.00	1.538	1.760†	1.973‡	2.000§	2.000**
.99	9.95	1.142	1.320	1.494	1.664§	1.831**

* 1.999 is to be used for $1 \leqq X \leqq 4.82$ for larger X interpolate in Table 2.
† 2.000 is to be used for $1 \leqq X \leqq 5.84$ for larger X interpolate in Table 2.
‡ 2.000 is to be used for $1 \leqq X \leqq 6.86$ for larger X interpolate in Table 2.
§ 2.000 is to be used for $1 \leqq X \leqq 7.89$ for larger X interpolate in Table 2.
** 2.000 is to be used for $1 \leqq X \leqq 8.90$ for larger X interpolate in Table 2.

to show that $G_n(p)$ can not have a local minimum in the range $\frac{1}{2} < p < 1$. Suppose, then, that $G_n(p)$ had a local minimum at $p = p_1$, say, with $\frac{1}{2} < p_1 < 1$. Then the necessary conditions $G'_n(p_1) = 0$ and $G''_n(p_1) \geqq 0$ would have to be satisfied where the dash denotes differentiation with regard to p. Introducing the function $A(p) = 1 - p^n - q^n$ we have with $q_1 = 1 - p_1$

$$(59) \qquad 0 = G'_n(p_1) = \{p_1q_1A' - \tfrac{1}{2}(q_1 - p_1)A\}/(p_1q_1)^{3/2}$$

where A and its derivative $A'(p) = -n(p^{n-1} - q^{n-1})$ are taken at argument p_1. From (59) we obtain

$$(60) \qquad (q_1 - p_1)A = 2p_1q_1A'$$

and it is immediately clear that for $n = 2$ (1) 5 equation (60) can not be satisfied for any p_1 between $\frac{1}{2}$ and 1. For when substituting the expressions for A and A' we find

$$(61) \quad 1 - p_1^n - q_1^n = 2np_1q_1(p_1^{n-1} - q_1^{n-1})/(p_1 - q_1) = 2np_1q_1\textstyle\sum_{i=1}^{n-1} p_1^{i-1}q_1^{n-1-i}$$

or

$$(62) \qquad 1 - p_1^n - q_1^n - \sum_{i=1}^{n-1}\binom{n}{i}p_1^i q_1^{n-i} = \sum_{i=1}^{n-1}\left(2n - \binom{n}{i}\right)p_1^i q_1^{n-i}.$$

The left-hand side of (62) is 0 while the right-hand side is positive since for $n \leqq 5$ we have $2n \geqq \binom{n}{i}$ for all i and for some i we have $2n > \binom{n}{i}$.

Confining ourselves then to $n \geqq 6$ we obtain from the conditions (60) and $G''_n(p_1) \geqq 0$ the inequality

$$(63) \qquad 2(p_1 - q_1)p_1q_1A'' - A' \geqq 0,$$

or substituting $A' = -n(p^{n-1} - q^{n-1})$ and $A'' = -n(n-1)(p^{n-2} + q^{n-2})$ we obtain

$$(64) \quad -n(n-1)2(p_1 - q_1)p_1q_1(p_1^{n-2} + q_1^{n-2}) + n(p_1^{n-1} - q_1^{n-1}) \geqq 0.$$

Condition (64) can clearly *not* be satisfied if $2(n-1)(p_1 - q_1)q_1 > 1$ or if $q_1 - 2q_1^2 > \frac{1}{2}(n-1)$. But this inequality holds for the range $q' \leqq q_1 \leqq q''$ where $q' = \frac{1}{4} - \sqrt{\frac{1}{16} - \frac{1}{4(n-1)}}$ and $q'' = \frac{1}{4} + \sqrt{\frac{1}{16} - \frac{1}{4(n-1)}}$ are tabled below:

n	6	7	8	10	12	14	16	18	20
q'	.138	.106	.086	.064	.051	.042	.036	.031	.028
q''	.362	.394	.414	.436	.449	.458	.464	.469	.472

We may therefore confine ourselves to the ranges $0 < q_1 \leqq q'$ and $q'' \leqq q_1 < \frac{1}{2}$, and will show that in these ranges equation (61) cannot be satisfied for $n \geqq 7$.

Dividing the left- and right-hand sides of (61) by nq and introducing the functions

$$L(q) = (1 - p^n - q^n)/nq, \qquad R(q) = 2p\sum_{i=1}^{n-1} p^{i-1}q^{n-1-i}$$

we have

$$L(q) \leqq (1 - (1 - q)^n)/nq = 1 - \tfrac{1}{2}(n - 1)(1 - \bar{q})^{n-2}q$$

where $\bar{q}$ is a mean value from the quadratic remainder term in the Taylor expansion of $(1 - q)^n$ at $q = 0$ and see that $L(q) < 1$.

On the other hand $R(q) \geqq 2p(p^{n-2} + p^{n-3}q) = R^*(q)$, say. Since for $n \geqq 7$ and $q \leqq q'$, $(dR^*(q))/(dq) < 0$, $R^*(q)$ attains its minimum for $q = q'$ and substituting the above values of q' it will be found that $R^*(q') > 1$ for $n \geqq 7$ so that for $0 < q \leqq q'$ we have that $L(q) < 1 < R(q)$ and (61) cannot be satisfied.

Turning to the range $q'' \leqq q < \frac{1}{2}$ we write $nq(L(q) - R(q))$ in the form (62), that is,

$$nq(L(q) - R(q)) = \sum_{i=1}^{n-1} \left(\binom{n}{i} - 2n\right) p^i q^{n-i}. \tag{65}$$

The only terms which are negative in the right-hand side of (65) are those for $i = 1$ and $i = n - 1$. Taking the latter first and comparing it with those for $i = n - 2$ and $i = n - 3$ we write for the sum of these three terms

$$np^{n-1}q\left\{-1 + \left(\frac{n-1}{2} - 2\right)\frac{q}{p} + \left(\frac{(n-1)\,(n-2)}{6} - 2\right)\left(\frac{q}{p}\right)^2\right\},$$

and it is clear that the quantity inside { } is positive for $n \geqq 7$ and $q'' \leqq q < \frac{1}{2}$, $p = 1 - q$. Likewise combining the terms for $i = 1$ and $i = 2$ we find

$$npq^{n-1}\left\{-1 + \left(\frac{n-1}{2} - 2\right)\frac{p}{q}\right\},$$

and it is clear that the quantity inside the { } is $\geqq 0$ for any $p \geqq q$ and $n \geqq 7$. It follows that equation (61) cannot be satisfied for $n \geqq 7$ and the above ranges of q.

For the remaining case $n = 6$ the only root p_1 of $G_6'(p_1) = 0$ over the range $\frac{1}{2} < p_1 \leqq 1 - q''; 1 - q' \leqq p_1 < 1$ was determined numerically as $p_1 = 0.5754$ and $G_6''(.5754) < 0$ verified by substitution in the left-hand side of (63).

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