PROPERTIES OF SOME TWO-SAMPLE TESTS BASED ON A PARTICULAR MEASURE OF DISCREPANCY

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1. Introduction and summary. Let F and G be continuous univariate cdf's. For testing the hypothesis $F = G$ against general alternatives, E. Lehmann [4] has proposed and found certain properties of a test based on the particular measure of discrepancy $\int (F - G)^2 \, d[(F + G) / 2]$. In this note will be given some additional properties of Lehmann's test (cf. also [8]) and a closely related test proposed by Mood [2].

2. The test statistics. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independent random samples from populations with continuous cdf's F and G respectively. Let $\binom{m}{2}\binom{n}{2}Q_{mn}$ be the number of quadruples (X_i, X_j, Y_k, Y_l), $i < j$, $k < l$, for which either the maximum of the X's is less than the minimum of Y's or the maximum of the Y's is less than the minimum of X's. Then

$$Q_{mn} = \binom{m}{2}^{-1}\binom{n}{2}^{-1} \sum_{l=2}^{n} \sum_{k=1}^{l-1} \sum_{j=2}^{m} \sum_{i=1}^{j-1} X_{ijkl}, \tag{2.1}$$

where X_{ijkl} is one if $X_i, X_j \lessgtr Y_k, Y_l$ and is zero otherwise. Lehmann [4] has shown that Q_{mn} is a minimum variance unbiased estimate of the functional,

$$Q(F, G) = \tfrac{1}{3} + 2 \int (F - G)^2 \, d\left(\frac{F + G}{2}\right). \tag{2.2}$$

Replacing F and G in (2.2) by the corresponding sample cumulative distribution functions, say S and T, yields the statistic,

$$D = \int (S - T)^2 \, d\left(\frac{S + T}{2}\right), \tag{2.3}$$

which is the symmetric version of a test statistic originally proposed by Mood [2],

$$d = \int (S - T)^2 \, dT. \tag{2.4}$$

The critical region for each of the two-sample tests corresponding to the above test statistics consists of the region in the m x n dimensional sample space (or equivalently, the arrangements of the mX's and nY's) for which the test statistic takes on its largest values. In [8] the distribution of Q_{mn} when $F = G$ has been tabled for a selection of small sample sizes.

3. The two-sample statistics expressed in terms of ranks. In order to see

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more closely the similarities among the above statistics, it is enlightening to express them in terms of ranks. Let R_i be the rank of the ith ordered X and r_j the rank of the jth ordered Y in the combined ordered sample of mX's and nY's. Lehmann [4] has given the following relation between Q_{mn} and these ranks,

$$\binom{m}{2}\binom{n}{2} Q_{mn} = \sum_{j=1}^{n} \left[(n - j) \binom{r_j - j}{2} + (j - 1) \binom{m - r_j + j}{2} \right]. \tag{3.1}$$

From the definition of d, (2.4), we have

$$d = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{j}{n} - \frac{r_j - j}{m} \right)^2, \tag{3.2}$$

and, by symmetry,

$$D = \frac{1}{2n} \sum_{j=1}^{n} \left(\frac{j}{n} - \frac{r_j - j}{m} \right)^2 + \frac{1}{2m} \sum_{i=1}^{m} \left(\frac{i}{m} - \frac{R_i - i}{n} \right)^2. \tag{3.3}$$

The above relations, after expansion and reduction, become

$$\begin{aligned} 2\binom{m}{2}\binom{n}{2} Q_{mn} = {} & \sum_{j=1}^{n} \Big[(n - 1)r_j^2 - 2(m + n - 2)jr_j \\ & - (n - 2m + 1)r_j \Big] + (n + 2m - 3) \frac{n(n + 1)(2n + 1)}{6} \\ & + (n + m^2 - 3m + 1) \frac{n(n + 1)}{2} - mn(m - 1), \end{aligned} \tag{3.4}$$

$$m^2n^2d = \sum_{j=1}^{n} [nr_j^2 - 2(m + n)jr_j] + (m + n)^2 \frac{(n + 1)(2n + 1)}{6}, \tag{3.5}$$

$$\begin{aligned} 2m^2n^2D = {} & \sum_{j=1}^{n} [nr_j^2 - 2(m + n)jr_j] + \sum_{i=1}^{m} [mR_i^2 - 2(m + n)iR_i] \\ & + (m + n)^2 \left[\frac{(n + 1)(2n + 1)}{6} + \frac{(m + 1)(2m + 1)}{6} \right]. \end{aligned} \tag{3.6}$$

4. Relations among tests when $m = n$. From the definition of Q_{mn} it follows that we may replace r_j by R_j in (3.4) if we interchange m and n. Upon adding the resultant expression for $2\binom{m}{2}\binom{n}{2}Q_{mn}$ to (3.4), setting m equal to n, and employing the identities

$$\sum_{j=1}^{n} r_j + \sum_{j=1}^{n} R_j = n(2n + 1)$$

and

$$\sum_{j=1}^{n} r_j^2 + \sum_{j=1}^{n} R_j^2 = (\tfrac{1}{3})n(2n + 1)(4n + 1),$$

we obtain the following relation,

$$4\binom{n}{2}^2 Q_{nn} = (n-1)\left[\frac{4}{3}n\,(2n+1)(n+1) + n^2(3n+1) - 4\sum_{j=1}^{n} j(r_j + R_j)\right].$$

Proceeding in an analogous manner, we obtain from (3.6),

$$2n^4 D = n\left[\frac{1}{3}n(2n+1)(8n+5) - 4\sum_{j=1}^{n} j(r_j + R_j)\right].$$

Thus, when $m = n$, Q_{mn} and D are related linearly and tests using large values of these statistics as critical regions are identical.

5. Means and variances.

a. *Means.* From (2.2) the mean of Q_{mn} is

$$E(Q_{mn}) = \tfrac{1}{3} + 2\int (F - G)^2\, d\left(\frac{F+G}{2}\right). \tag{5.1}$$

From (2.4),

$$E(d) = E\left[\int (S - T)^2\, dT\right] = E\left(\int S^2\, dT\right) - 2E\left(\int ST\, dT\right) + \frac{(n+1)(2n+1)}{6n^2}. \tag{5.2}$$

Since $E(T) = G$ and $E(S^2) = [F(1-F)\,/\,m] + F^2$, the first term on the right of (5.2) becomes, with the aid of Fubini's theorem,

$$E\left(\int S^2\, dT\right) = \int \left[\frac{F(1-F)}{m} + F^2\right] dG.$$

On the application of Fubini's theorem and a special integration by parts (*vide* [7], p. 102), the second term on the right of (5.2) reduces to

$$2E\left(\int ST\, dT\right) = \frac{n-1}{n} - \frac{n-1}{n}\int G^2\, dF + \frac{2}{n}\int F\, dG.$$

Thus $E(d)$ may be written

$$E(d) = \frac{m-1}{m}\int F^2\, dG + \frac{n-1}{n}\int G^2\, dF + \frac{n-2m}{mn}\int F\, dG - \frac{n-1}{n} + \frac{(n+1)(2n+1)}{6n^2},$$

which, after substituting the identity,

$$\int F^2\, dG + \int G^2\, dF = \tfrac{2}{3} + \int (F-G)^2\, d[(F+G)/2],$$

becomes

$$E(d) = \int (F - G)^2 \, d\left(\frac{F+G}{2}\right) - \frac{1}{m}\int F^2 \, dG - \frac{1}{n}\int G^2 \, dF + \frac{n-2m}{mn}\int F \, dG + \frac{9n+1}{6n^2}. \tag{5.3}$$

From (5.3) and the symmetry of D, we obtain for $E(D)$,

$$E(D) = \int (F - G)^2 \, d\left(\frac{F+G}{2}\right) - \frac{1}{m}\int F^2 \, dG - \frac{1}{n}\int G^2 \, dF + \frac{n-2m}{2mn}\int F \, dG + \frac{m-2n}{2mn}\int G \, dF + \frac{9n+1}{12n^2} + \frac{9m+1}{12m^2}. \tag{5.4}$$

When $F = G$, (5.1), (5.3), and (5.4) reduce to

$$E(Q_{mn}) = \tfrac{1}{3} \tag{5.5}$$

$$E(d) = \frac{1}{6}\left[\frac{m+n}{mn} + \frac{1}{n^2}\right] \tag{5.6}$$

$$E(D) = \frac{1}{6}\left[\frac{m+n}{mn} + \frac{1}{2n^2} + \frac{1}{2m^2}\right]. \tag{5.7}$$

b. *Variances.* A method for finding the variance of Q_{mn} for general F and G has been given by Sundrum [8]. When $F = G$, he obtained

$$\sigma^2(Q_{mn}) = \frac{1}{45}\binom{m}{2}^{-1}\binom{n}{2}^{-1}\left[(m+n)(m+n-1) - 2\right]. \tag{5.8}$$

In the following there will be outlined a procedure (cf. Hoeffding [3]) for obtaining the variance of a U'_N statistic as defined in Theorem 6.5 (Q_{mn} is a particular U'_N statistic). This procedure also provides a result which will be needed in the proof of a later theorem.

Set

$$t_{ij}(x_1, \cdots, x_i, y_1, \cdots, y_j) = Et(x_1, \cdots, x_i, X_{i+1}, \cdots, X_r, y_1, \cdots, y_j, Y_{j+1}, \cdots, Y_r),$$

$$\zeta_{00} = 0,$$

$$\zeta_{ij} = E[t_{ij}^2(X_1, \cdots, X_i, Y_1, \cdots, Y_j)] - \theta^2, \qquad i, j = 0, \cdots, r.$$

Let $(s_i, \cdots, s_r)$, $(s'_i, \cdots, s'_r)$, $(t_1, \cdots, t_r)$, and $(t'_1, \cdots, t'_r)$ be four sets of r different integers, $1 \leqq s_i, s'_1 \leqq m$, $1 \leqq t_j, t'_j \leqq n$, and let a and b be the number of integers common to the sets of s's and t's respectively. Then, from the symmetry of $t(x_1, \cdots, x_r, y_1, \cdots, y_r)$, it follows that

$$E[t(X_{s_1}, \cdots, X_{s_r}, Y_{t_1}, \cdots, Y_{t_r})t(X_{s'_1}, \cdots, X_{s'_r}, Y_{t'_1}, \cdots, Y_{t'_r})] - \theta^2 = \zeta_{ab}. \tag{5.9}$$

Thus, the variance of U_N' can be written

$$\begin{aligned}\sigma^2(U_N') &= \binom{m}{r}^{-2}\binom{n}{r}^{-2} E[\sum t(X_{s_1}, \cdots, X_{s_r}, Y_{t_1}, \cdots, Y_{t_r}) - \theta]^2 \\ &= \binom{m}{r}^{-2}\binom{n}{r}^{-2} \sum_{b=0}^{r}\sum_{a=0}^{r} \sum{}^{(ab)} E[t(X_{s_1}, \cdots, X_{s_r}, Y_{t_1}, \cdots, Y_{t_r}) \\ &\qquad \cdot t(X_{s_1'}, \cdots, X_{s_r'}, Y_{t_1'}, \cdots, Y_{t_r'})] - \theta^2,\end{aligned} \tag{5.10}$$

where $\sum^{(ab)}$ stands for summation over all subscripts such that $1 \leqq s_1 < \cdots < s_r \leqq m$, $1 \leqq s_1' < \cdots < s_r' \leqq m$, $1 \leqq t_1 < \cdots < t_r \leqq n$, $1 \leqq t_1' < \cdots < t_r' \leqq n$ and exactly a equations $s_i = s_j'$ and b equations $t_k = t_l'$ are satisfied.

From (5.9) each term in $\sum^{(ab)}$ is equal to ζ_{ab}. The number of terms in $\sum^{(ab)}$ is $\binom{r}{a}\binom{m-r}{r-a}\binom{m}{r}\binom{r}{b}\binom{n-r}{r-b}\binom{n}{r}$, so that (5.10 becomes

$$\sigma^2(U_N') = \binom{m}{r}^{-1}\binom{n}{r}^{-1}\sum_{b=0}^{r}\sum_{a=0}^{r}\binom{m-r}{r-a}\binom{n-r}{r-b}\binom{r}{a}\binom{r}{b}\zeta_{ab}. \tag{5.11}$$

To find the variance of Q_{mn} by the above method, we must first obtain the ζ_{ab}'s $a, b = 0, 1, 2$, and then combine them according to (5.11).

Since $\zeta_{00} = 0$, we have the following additional result, which will be needed in the following section.

$$\begin{aligned}\sigma^2(U_N') &\leqq \binom{m}{r}^{-1}\binom{n}{r}^{-1}\left[\binom{m}{r}\binom{n}{r} - \binom{m-r}{r}\binom{n-r}{r}\right]\max(\zeta_{ab}), \\ &= o(1) \text{ as } \min(m, n) \to \infty,\end{aligned} \tag{5.12}$$

when $\max(\zeta_{ab}) < \infty$.

From (3.5) the variance of d is

$$\begin{aligned}\sigma^2(d) = m^{-4}n^{-4}\Bigg[n^2 \operatorname{var}\left(\sum_{j=1}^{n} r_j^2\right) &+ 4(m+n)^2 \operatorname{var}\left(\sum_{j=1}^{n} jr_j\right) \\ &- 4n(m+n)\operatorname{cov}\left(\sum_{i=1}^{n} r_i^2, \sum_{j=1}^{n} jr_j\right)\Bigg].\end{aligned} \tag{5.13}$$

When $F = G$, the distribution of r_j and the joint distribution of r_j and r_k $(j \leqq k)$ are easily seen to be

$$f(r_j) = \binom{m+n}{n}^{-1}\binom{r_j - 1}{j-1}\binom{m+n-r_j}{n-j}, \qquad \begin{matrix} j \leqq r_j \leqq m+j, \\ 1 \leqq j \leqq n \end{matrix} \tag{5.14}$$

and

$$\begin{aligned}f(r_j, r_k) &= \binom{m+n}{n}^{-1}\binom{r_j-1}{j-1}\binom{r_k - r_j - 1}{k-j-1} \\ &\quad \cdot \binom{m+n-r_k}{n-k}, \qquad \begin{matrix} j \leqq r_j \leqq r_k - (k-j) \leqq m+j, \\ 1 \leqq j \leqq k \leqq n. \end{matrix}\end{aligned} \tag{5.15}$$

Equations (5.14) and (5.15) may be used to find the three terms on the right of (5.13). After lengthy but straightforward calculations, we have

$$\text{(5.16)} \quad \operatorname{var}\left(\sum_{j=1}^{n} r_j^2\right) = \frac{1}{180} mn(m+n+1)(2m+2n+1)(8m+8n+11),$$

$$\text{(5.17)} \quad \operatorname{var}\left(\sum_{j=1}^{n} jr_j\right) = \frac{1}{180} mn(m+n+1)(2n+3)(2n+1),$$

$$\text{(5.18)} \quad \begin{aligned} &\operatorname{cov}\left(\sum_{i=1}^{n} r_i^2, \sum_{j=1}^{n} jr_j\right) \\ &\qquad = \frac{1}{360} mn(m+n+1)(16n^2+16mn+14m+31n+13). \end{aligned}$$

The following relation, which will be used in Theorem 6.1, can be obtained in a similar manner,

$$\text{(5.19)} \qquad \operatorname{var}\left(\sum_{j=1}^{n} r_j\right) = \frac{1}{12} mn(m+n+1).$$

Substituting (5.16), (5.17), and (5.18) in (5.13) and simplifying, we obtain for the variance of d when $F = G$,

$$\text{(5.20)} \quad \sigma^2(d) = \frac{(m+n)(m+n+1)}{45m^2n^2} + \frac{m+n+1}{180m^3n^3}(12m^2 - 3n^2 - 2mn).$$

6. Limiting distributions.

a. *Under the null hypothesis.* The following two theorems are concerned with the limiting distribution of Q_{mn}, d, and D under the null hypothesis $F = G$.

THEOREM 6.1. *If $F = G$ and $m/n \to c > 0$ as $n \to \infty$, the statistics*

$$\frac{1}{2}\frac{mn}{m+n}[Q_{mn} - E - (Q_{mn})], \qquad \frac{mn}{m+n}[D - E(D)], \qquad \frac{mn}{m+n}[d - E(d)]$$

have the same limiting distribution.

THEOREM 6.2.[1] *If $F = G$ and $m/n \to c > 0$ as $n \to \infty$, the statistic $mn/(m+n)[d - E(d)]$ has the same limiting distribution as $n\omega^2 - E(n\omega^2)$, where ω^2 is the von Mises statistic. (The limiting distribution of $n\omega^2$ is tabled in* [1]; *$E(n\omega^2) = \frac{1}{6}$.)*

It follows from the above theorems that $\frac{1}{2}[(mn / m+n)][Q_{mn} - E(Q_{mn})]$ has the same limiting distribution as $n\omega^2 - E(n\omega^2)$. In Figure 1 are compared the distribution of Q_{55} and the limiting distribution of $n\omega^2$ drawn with the appropriate scales.

PROOF OF THEOREM 6.1. From equations (3.4) and (3.5), we may write

$$\text{(6.1)} \quad \begin{aligned} &2\binom{m}{2}\binom{n}{2} Q_{mn} - m^2n^2d^2 \\ &\qquad = -\sum_{j=1}^{n} r_j^2 + 4\sum_{j=1}^{n} jr_j - (n-2m+1)\sum_{j=1}^{n} r_j + O(n^5). \end{aligned}$$

From (5.16), (5.17), and (5.19), each of the terms on the right has variance $O(n^5)$.

[1] Theorem 6.2 is due to M. Rosenblatt [6].

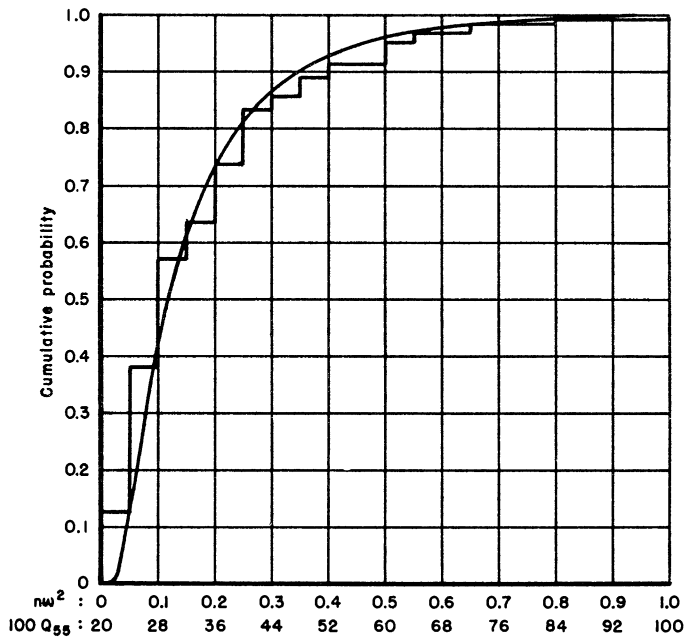

FIG. 1

Thus, aside from terms which converge to zero in probability,

$$
\begin{aligned}
\frac{1}{2}\frac{mn}{m+n}[Q_{mn} - E(Q_{mn})] &= \frac{mn}{m+n}\frac{1}{4}\binom{m}{2}^{-1}\binom{n}{2}^{-1} m^2n^2[d - E(d)] \\
&= \frac{mn}{m+n}[d - E(d)] + \frac{mn}{m+n}\frac{m+n-1}{(m-1)(n-1)}[d - E(d)].
\end{aligned} \tag{6.2}
$$

From (5.20), $\sigma^2(d) = O(1/n^2)$. Thus the second term on the right of (6.2) converges to zero in probability.

The proof that $mn / (m + n)[D - E(D)]$ has the same limiting distribution as $\frac{1}{2}[mn/(m + n)][Q_{mn} - E(Q_{mn})]$ is analogous and will be omitted.

b. *Under the alternative hypothesis.* An important subclass of the class of continuous cdf's is the class of strictly increasing continuous cdf's. The following two theorems are concerned with the problem of finding the limiting distribution of Q_{mn}, d, and D when F and G are in this subclass and $F \neq G$.

THEOREM 6.3. *If $m/n \to c > 0$ as $n \to \infty$, then the statistics*

$$\tfrac{1}{2}[mn / (m + n)]^{1/2}[Q_{mn} - E(Q_{mn})], \quad [mn / (m + n)]^{1/2}[d - E(d)], \quad and$$
$$[mn / (m + n)]^{1/2}[D - E(D)]$$

have the same limiting distribution.

PROOF. It follows from (6.1) and the inequalities,

$$\sum_{j=1}^{n} jr_j \leqq \sum_{j=1}^{n} r_j^2 < n(m + n)^2 \quad \text{and} \quad \sum_{j=1}^{n} r_j < n(m + n),$$

that we may write, aside from terms which converge to zero in probability,

$$
\begin{aligned}
\sqrt{\frac{mn}{m+n}}\,[d - E(d)] &= \sqrt{\frac{mn}{m+n}}\, m^{-2}n^{-2}2\binom{m}{2}\binom{n}{2}[Q_{mn} - E(Q_{mn})] \\
&= \frac{1}{2}\sqrt{\frac{mn}{m+n}}\,[Q_{mn} - E(Q_{mn})] \\
&\quad + \sqrt{\frac{mn}{m+n}}\,\frac{1-m-n}{2mn}\,[Q_{mn} - E(Q_{mn})].
\end{aligned} \tag{6.3}
$$

From (5.12) it follows that $\sigma^2(Q_{mn}) = o(1)$ as min $(m, n) \to \infty$, so that the second term on the right of (6.5) converges to zero in probability.

The proof that $[mn\,/\,(m+n)]^{1/2}[D - E(D)]$ has the same limiting distribution as $\frac{1}{2}[mn\,/\,(m+n)]^{1/2}[Q_{mn} - E(Q_{mn})]$ is analogous and will be omitted.

THEOREM 6.4.[2] *If $m/n = c > 0$ as $n \to \infty$, then the statistic $[mn\,/\,(m+n)]^{1/2}[Q_{mn} - E(Q_{mn})]$ has a normal limiting distribution. Excluding F and G for which either $F = G$ or $\int F\,dG = 0$ or 1, the class for which nondegeneracy occurs includes all continuous F and G which are strictly increasing throughout their range of variation.*

In the proof of Theorem (6.4) we shall need the following theorem of Lehmann's [4].

THEOREM 6.5. *Let $X_1, \cdots, X_m$, and $Y_1, \cdots, Y_n$ be independently distributed random samples from the distributions F and G respectively. Let $t(x_1, \cdots, x_r, y_1, \cdots, y_r)$ be symmetric in the x's alone and in the y's alone. Suppose that*

$$
\begin{aligned}
E[t(X_1, \cdots, X_r, Y_1, \cdots, Y_r)] &= \theta(F, G) = \theta, \\
E[t(X_1, \cdots, X_r, Y_1, \cdots, Y_r)^2] &= M < \infty.
\end{aligned}
$$

Let $m/n = c$ and let n be sufficiently large so that $r \leqq$ min (m, n). Define

$$
U_n' = \binom{m}{r}^{-1}\binom{n}{r}^{-1}\sum t(X_{\alpha_1}, \cdots, X_{\alpha_r}, Y_{\beta_1}, \cdots, Y_{\beta_r}),
$$

where the summation is extended over all subscripts

$$
1 \leqq \alpha_1 < \cdots < \alpha_r \leqq m, \qquad 1 \leqq \beta_1 < \cdots < \beta_r \leqq n.
$$

Then, as $n \to \infty$, $[mn\,/\,(m+n)]^{1/2}(U_n' - \theta)$ is asymptotically normally distributed; furthermore, if we set

$$
\begin{aligned}
\psi_1(x_1) &= E[t(x_1X_2, \cdots, X_r, Y_1, \cdots, Y_r)] - \theta, \\
\psi_2(y_1) &= E[t(X_1, \cdots, X_r, y_1, Y_2, \cdots, Y_r)] - \theta,
\end{aligned}
$$

then the limiting distribution of $[mn\,/\,(m+n)]^{1/2}(U_n' - \theta)$ is nondegenerate provided

$$
E[\psi_1^2(X_1)] + E[\psi_2^2(Y_1)] > 0.
$$

[2] Theorem 6.4 is an amended version of a statement by E. Lehmann [4, p. 173], which did not sufficiently restrict F and G for nondegeneracy.

Proof of Theorem (6.4). Set $t(X_i, X_j, Y_k, Y_l)$ equal to X_{ijkl} [which is defined following (2.1)]. Then Q_{mn} is seen to be equivalent to U'_n in Theorem 6.5, where $\theta = Q(F, G)$ and $r = 2$. The first statement of Theorem 6.4 follows immediately. To prove the second statement, we apply the second part of Lehmann's theorem. We have

$$\begin{aligned}\psi_1(x_1) &= E[t(x_1, X_2, Y_1, Y_2)] - \theta \\ &= \int_{-\infty}^{x_1}\int_{y}^{\infty} 2G(y)\, dF(x)\, dG(y) + \int_{x_1}^{\infty}\int_{-\infty}^{y} 2[1 - G(y)]\, dF(x)\, dG(y) - \theta \\ &= 2\int_{-\infty}^{x_1} (1 - F)G\, dG + 2\int_{x_1}^{\infty} F(1 - G)\, dG - \theta \\ &= 2\int_{-\infty}^{x_1} (G - F)\, dG + 2\int_{-\infty}^{\infty} F(1 - G)\, dG - \theta.\end{aligned}$$

Set $I(x_1) = \int_{-\infty}^{x_1} (G - F)\, dG$. Then $E[\psi_1^2(X_1)] = 0$ implies that

$$I(X_1) = E[I(X_1)] \tag{6.4}$$

with probability one (with respect to F).

Suppose now that the restrictions of the second statement of Theorem 6.4 hold. This implies that there exist two points x_0 and x_0', $x_0 < x_0'$, which are points of increase of both F and G. With no loss in generality it may be assumed that $G(x) - F(x) \geqq \delta > 0$ for x in the interval (x_0, x_0'). It follows that $I(x_0') > I(x_0)$ so that (6.4) can not hold with probability one.

7. Consistency and unbiasedness.

a. *Consistency*. For the class of continuous cumulative distribution functions, the test based on Q_{mn} of the hypothesis $F = G$ against the alternatives $F \neq G$ has been shown by Lehmann [4] to be consistent at each level of significance when $\min(m, n) \to \infty$.

With the aid of the theorems on limiting distributions and the fact that the means of d and D are linear functions of $\int (F - G)^2\, d(F + G)$ plus a term which is $o(1)$ as $\min(m, n) \to \infty$, it readily follows that the tests based on d and D are consistent under the conditions of the above paragraph provided the additional restriction $m/n \to c > 0$ as $n \to \infty$ is imposed.

b. *Unbiasedness*. That the tests based on Q_{mn}, d, and D are not unbiased tests of the hypothesis $F = G$ against all continuous alternatives $F \neq G$ and all m *and* n is shown by the following example.

Let F_1 and G_1 be cdf's with the probability density functions

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}, \qquad 0 \leqq x \leqq 1, 2 \leqq x \leqq 3 \\ &= 0, \qquad \text{otherwise,}\end{aligned}$$

and

$$\begin{aligned} g_1(y) &= 1, \qquad 1 \leqq y \leqq 2 \\ &= 0, \qquad \text{otherwise.}\end{aligned}$$

In the m x n dimensional sample space let $W_{mn}^{(1)}$ be the region for which $\max(x_1, \cdots, x_m) < \min(y_1, \cdots, y_n)$ and let $W_{mn}^{(2)}$ be the region for which $\min(x_1, \cdots, x_m) > \max(y_1, \cdots, y_n)$. Then

$$P(W_{mn}^{(1)} \mid F = G) = P(W_{mn}^{(2)} \mid F = G) = \binom{m+n}{n}^{-1},$$

$$P(W_{mn}^{(1)} \mid F = F_1, G = G_1) = P(W_{mn}^{(2)} \mid F = F_1, G = G_1) = (\tfrac{1}{2})^m.$$

Since, for fixed n and sufficiently large m, $\binom{m+n}{n} < 2^m$, there exist m_1 and n_1 such that both

$$P(W_{m_1n_1}^{(1)} \mid F = F_1, G = G_1) < P(W_{m_1n_1}^{(1)} \mid F = G),$$

$$P(W_{m_1n_1}^{(2)} \mid F = F_1, G = G_1) < P(W_{m_1n_1}^{(2)} \mid F = G),$$

so that any test of the hypothesis $F = G$ having $W_{m_1n_1}^{(1)}$, $W_{m_1n_1}^{(2)}$, or $W_{m_1n_1}^{(1)} U W_{m_1n_1}^{(2)}$ as a critical region will be biased against the alternative $F = F_1$, $G = G_1$.

Since critical regions for the tests based on Q_{mn}, d, and D are regions yielding large values of these statistics, it can be seen by examining the maxima of these statistics over the possible arrangements of X's and Y's that for each test and every m and n, one of $W_{mn}^{(1)}$, $W_{mn}^{(2)}$, or $W_{mn}^{(1)} U W_{mn}^{(2)}$ is a possible critical region. Thus, each of these tests is biased against the alternative $F = F_1$, $G = G_1$, when $m = m_1$, $n = n_1$.

8. The power of the test based on Q_{mn} for a particular class of alternatives. In [5], Lehmann has discussed the power of several two-sample distribution-free tests for the particular class of alternatives $G = F^k (k = 2, 3, \cdots,)$. One of the tests considered by Lehmann was the two-sided version of Wilcoxon's rank sum test, which we shall use here as a basis of comparison. With the aid of Lehmann's technique, the exact power of the Q_{mn} test was found for $m = n = 4$ to be 0.19 against the alternative $G = F^2$ and 0.32 against $G = F^3$, which results are identical with the corresponding results for Wilcoxon's test. For larger m and n, the approximate power of the Q_{mn} test was obtained by use of the approximate distributions indicated in Section 6. Against the alternative $G = F^2$, the approximate power was slightly larger than that of Wilcoxon's test for $5 \leqq m = n \leqq 40$ and slightly smaller for $m = n > 40$. Against the alternative $G = F^3$, the approximate power was essentially the same as that of Wilcoxon's test for $5 \leqq m = n < 15$ and slightly smaller for $m = n \geqq 15$.

REFERENCES

[1] T. W. Anderson and D. A. Darling, "Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 193–213.

[2] W. J. Dixon, "A criterion for testing the hypothesis that two samples are from the same population," *Ann. Math. Stat.*, Vol. 11 (1940), pp. 199–204.

[3] W. Hoeffding, "A class of statistics with asymptotic normal distributions," *Ann. Math. Stat.*, Vol. 19 (1948), pp. 293–325.

[4] E. L. LEHMANN, "Consistency and unbiasedness of certain nonparametric tests." *Ann. Math. Stat*, Vol. 22 (1951), pp. 165–179.

[5] E. L. LEHMANN, "The power of rank tests," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 23–44.

[6] M. ROSENBLATT, "Limit theorems associated with variants of the von Mises statistics," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 617–624.

[7] S. SAKS, *Theory of the Integral*, New York, Hafner Publishing Co., 1937.

[8] R. M. SUNDRUM, "On Lehmann's two-sample test," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 139–146.