If there does not exist a finite G_0 for which $g_n(G)$ does not become infinite, then $g_n(G)$ becomes infinite for all $G > 0$. Thus $g_n(G)$ either becomes infinite for all $G > 0$ or approaches zero for all $G > 0$. In the first case, sampling will terminate because $f_n(G) > B$ for sufficiently large n for all $G > 0$; and in the second case too, since $f_n(G) < A$ for sufficiently large n for all $G > 0$.

3. Comments. It has been possible to obtain an upper bound for the limiting value G_o but not to obtain its value uniquely. David and Kruskal [3] have provided a solution to the same problem for the sequential t-test.

4. Acknowledgement. I am most grateful to Dr. N. L. Johnson for his guidance during research on this problem, to the referee for his comments, and to the British Coal Utilisation Research Association for permission to publish this paper.

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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Washington meeting of the Institute, March 7–9, 1957)

1. Synchronization of Trajectory Images of Ballistic Missiles and the Timing Record of the Ground Telemetry Recording System, Harry P. Hartkemeier, Stanford University, (introduced by Paul R. Rider).

In order to compute the position, velocity, and acceleration of a missile, it is necessary to synchronize the image pattern from ballistic camera plate records and the timing record of the ground telemetry recording system. In the past this has been done by personal inspection. This takes too much time; consequently, a method by which the two records may be matched by high-speed electronic computers is required to speed up the work.

The missile is equipped with two strobe lights, one on each side, which are supposed to flash simultaneously when scheduled to do so by a programmer. Inside the missile there is a timing generator controlled by a tape punched according to a coding pattern. When the timing generator sends a signal for the strobe lights to flash, it also sends a signal simultaneously to the telemetry transmitter. This signal reaches the ground recording telemetry system through a radio link. A method of matching these two records by using correlation technique and an electronic computer is presented. (Received November 6, 1956.)

2. Maximum Likelihood Estimates in a Simple Queue, A. Bruce Clarke, University of Michigan, (By Title).

A simple stationary queueing process is a process having a Poisson input (with parameter λ), and a negative exponential service time (with mean $1/\mu$, $\mu > \lambda$). Let ν = the initial queue size, x_i = the time of the ith arrival, y_i = the "busy time" up to the ith departure. The sequences $\{x_i\}$ and $\{y_i\}$ then represent the transition times of *independent* Poisson processes (parameters λ and μ), and $\{\nu, \{x_i\}, \{y_i\}$ together characterize the process. By observing the process for a fixed "busy time" τ and using the above comment, maximum likelihood estimates for λ and μ may be obtained in terms of ν, m = the total number of departures, T = the time of the mth departure, and n = the total number of arrivals up to time T. Under certain conditions these estimates of λ and μ may be approximated by $(n + \nu)/T$ and $(m - \nu)\tau$. (Received November 12, 1956.)

3. A Rank Order Test for Trend in Correlated Means, Ardie Lubin, Walter Reed Army Institute of Research, Washington, D. C.

In many experiments the major interest is not in the amount of difference caused by the treatments but the rank-order which results. This is especially true when successive measurements are made on the same subject, and the "treatments" are simply varying amounts of fatigue, sleep loss, etc., i.e., some function of time. For such studies the null hypothesis is that no trend exists and generally the only alternative hypothesis is a rank-order that can be specified by the experimenter.

A. R. Jonckheere ("A distribution-free k-sample test against ordered alternatives," *Biometrika*, Vol. 41 (1954)) has used Kendall's tau to obtain a general statistic, P, for testing the agreement between a hypothesized rank-order for n objects or scores and a set of observed rankings of the n scores by m judges. From this general approach, he derives a test for trend as a special case.

As an alternative to Jonckheere's P, a statistic J based on Spearman's $S(d^2)$ is examined. It is the sum of the $S(d^2)$ values computed between the m observed rankings of the n scores and the hypothesized rank-order of the n scores. K, the average rank order correlation between the m rankings and the hypothesized rank-order, is a simple algebraic function of J.

It is shown that J is slightly more sensitive than Jonckheere's P statistic for smal values of n, but that P tends to normality faster than J. (Received November 13, 1956.)l

4. On the Stochastic Structure of Minkowski-Leontief Systems, David Rosenblatt, American University.

A linear system $x(I - A) = w$ is said to be of *Minkowski-Leontief type* if A is a finite nonnegative square matrix of order n with no row sum exceeding unity and x, w are nonnegative row vectors. A non-null solution x of such a system is called *admissible*. Theorem: *Every system of Minkowski-Leontief type $x(I - A) = w$ which exhibits at least one admissible solution is equivalent to a unique system $\tilde{x}(I - \tilde{A}) = \theta$, where $\tilde{A}$ is a stochastic matrix depending on A and w and θ is a null vector of dimension at most $n + 1$. Every admissible solution of $x(I - A) = w$ (appropriately extended or contracted) is proportional to a convex linear combination of the stationary stochastic vectors of $\tilde{A}$.* If A is nonstochastic, $w \neq \theta_n$, let $\tilde{A}$ denote the matrix $\begin{Vmatrix} A & b \\ w^* & 0 \end{Vmatrix}$ where $b = (I - A)e'$, $w^* = \lambda_w^{-1}w$, $\lambda_w = we'$, and e is the row vector with all elements unity. If $(I - A)^{-1}$ exists and $w \neq \theta_n$, there exists a single ergodic set of indices; if w is positive the stationary vector of $\tilde{A}$ is positive. Clearly,

$$\tilde{x} = (w(I - A)^{-1}, \lambda_w)$$

If $w \neq \theta_n$ and $(I - A)$ is singular, $\tilde{A}$ is taken as $\begin{Vmatrix} A_r & 0 \\ 0 & I_{n-r} \end{Vmatrix}$ where A_r is the largest stochastic principal submatrix of A. Systems of the present type occur in economic input-output analysis and generally in socio-physical models based on "balanced-margin" tables, i.e., nonnegative square matrices X such that $eX = eX'$. (Received November 20, 1956.)

5. The Joint Distribution of a Set of Sufficient Statistics for the Parameters of a Simple Telephone Exchange Model, VÁCLAV EDVARD BENEŠ, Bell Telephone Laboratories, (By Title).

This paper considers a simple telephone exchange model which has an infinite number of trunks and in which the traffic depends on two parameters, the calling-rate and the mean holding-time. It is desired to estimate these parameters by observing the model continuously during a finite interval, and noting the calling-time and hang-up time of each call, insofar as these times fall within the interval. It is shown that the resulting information may, for the purpose of this estimate, be reduced without loss to four statistics. These statistics are the number of calls found at the start of observation, the number of calls arriving during observation, the number of calls leaving during observation, and the average number of calls existing during the interval of observation. The joint distribution of these sufficient statistics is determined (in principle) by deriving a generating function for it. From this generating function, the means, variances, covariances, and correlation coefficients are obtained. Various estimators for the parameters of the model are compared, and some of their distributions, means, and variances presented. (Received November 29, 1956.)

6. On the Stochastic Structure of Minkowski-Leontief Systems, II, DAVID ROSENBLATT, American University, (By Title).

Consider a system $x(I - A) = w$ of Minkowski-Leontief type such that $(I - A)^{-1}$ exists. Clearly, $(I - A)^{-1}$ exists if and only if A contains no stochastic principal submatrix. In a static economic input-output context the element a_{ij} is designated as the input (per unit output) to industry or activity i procured from industry j; w_j, x_j are respectively final output and total output (or activity level) of the jth industry. Consider the uniquely corresponding system $\tilde{x}(I - \tilde{A}) = \theta$, where $\tilde{A}$ is stochastic. The unique stationary stochastic vector of $\tilde{A}$ is given by $(p_{n+1}w^*(I - A)^{-1}, p_{n+1})$. The "multiplier" $\mu = \sum_{j=1}^{n+1} x_j/\lambda$ is given by $1/p_{n+1}$, where $\lambda = x_{n+1} = we'$. Given a nonsingular matrix $(I - A)$, the following relation holds in components of an admissible solution for any w: $\sum_{j=1}^{n} (1 - r_j)x_j - x_{n+1} = 0$, where $x_{n+1} = we'$ and r_j is the jth row sum in A. The latter relation is the technical production-possibility function of the economy in an input-output sense; $-\Delta x_j/\Delta x_k = (1 - r_k)/(1 - r_j)$, $\Delta x_j/\Delta x_{n+1} = 1/(1 - r_j)$, $j \neq k$, $j, k = 1, \ldots, n$, are the invariant "substitution ratios" of the system, obviously independent of w. Let x be an admissible solution of $x(I - A) = w$, $(I - A)$ singular or not, and let $D(x, \lambda)$ be a diagonal matrix with components of x and λ on the diagonal. Then $D(x, \lambda)\tilde{A}$ is a "balanced-margin" table. Consistent with a noted "substitution" result, $\sum_{j=1}^{n} K_j w_j = x_{n+1} = \lambda$, where $K_j = 1$ for all j independently of w. (Received December 17, 1956.)

7. A Further Contribution to the Theory of Systematic Statistics, JUNJIRO OGAWA, University of North Carolina.

Up to 1945 the main interest of statistical estimation has been in the "efficient estimator," but from the point of view of practical use, it seems reasonable to inquire whether comparable results could have been obtained by a smaller expenditure. F. Mosteller (1946) proposed the use of systematic statistics in this connection. The author (1952) developed a systematic theory of estimation and testing hypotheses with respect to the location and scale parameter of a population whose density depends on only these two parameters.

There are many cases in which the samples are by their very nature ordered in magnitude, for example in a life test of electric lamps. In such cases the population probabilty distributions are usually supposed to be exponential. Thus, at least for the exponential distribution, estimation and testing of a hypothesis based upon systematic statistics are of great importance from the standpoint of practical application.

There will be presented in this paper the table of the optimum spacings of the selected sample quantiles, corresponding best estimators, and a discussion on the testing procedure of a statistical hypothesis on the scale parameter σ of the exponential distribution $f(x) = (1/\sigma)e^{-(x/\sigma)}$ for $x > 0$. (Received January 7, 1957.)

8. On the Stochastic Structure of Minkowski-Leontief Systems, III, David Rosenblatt, American University, (By Title).

Consider any system $x(I - A) = w$ of M-L type. The following "aggregation" problem is of interest. Let an *aggregation matrix* C be an $(n \times r)$ stochastic matrix of incidence type, $1 \leqq r < n$. Let $B = f(A)$ be a M-L matrix of order r. We consider conditions under which $\hat{x}AC = \hat{x}CB$ obtains for admissible solutions $\hat{x}$ of a system $x(I - A) = w$. The following case is of special interest. Let a *weight matrix* E be an $(n \times n)$ diagonal matrix with nonnegative entries on the principal diagonal. A *consolidation* of a matrix A of M-L type is an $(r \times r)$ matrix $B \equiv B(A; C, E) = (C'EC)^{-1}C'EAC$, $1 \leqq r < n$. "Faithful consolidation" of a stochastic system $x(I - A) = \theta$ is characterized from the standpoint of ergodic structure; the condition $AC = CB(A; C, E)$ is of particular interest. A general consolidation condition for M-L systems is related to the "combining-of-classes" condition of stochastic learning theory. The following is of economic interest: the existence of $(I - B)^{-1}$ does not in general imply the existence of $(I - A)^{-1}$, and conversely. In the static input-output model of II, the ergodic structure of $\tilde{A}$ of the equivalent system (and the role of mean recurrence time $1/(p_{n+1})$) suggest that the stationary stochastic vector $\tilde{y}$ of $\tilde{A}$ be computed iteratively using successive powers of $\tilde{A}$, yielding $\tilde{x}$, in lieu of matrix inversion with or without consolidation; in most applications, $\lim_{k\to\infty}\tilde{A}^k$ exists. (Received January 14, 1957.)

9. The Use of Incomplete Block Designs for Asymmetrical Factorial Arrangements, Marvin Zelen, National Bureau of Standards.

Let A_s $(s = 1, 2, \ldots, m)$ denote the sth factor in a m-factor factorial experiment such that A_s has m_s levels. Let $i = (i_1, i_2, \ldots, i_m)$ represent a particular experimental combination of the m-factors and let the mathematical model underlying the measurements be

$$y_{ij} = \mu + \sum_{s=1}^{m} (a_s)_{i_s} + \sum_{t=2}^{m} \sum_{s=1}^{t} (a_{st})_{i_s t} + \cdots + (a_{12\cdots m})_{i_{12\cdots m}} + b_j + \epsilon_{ij},$$

where $(a_s)_{i_s}$, $(a_{st})_{i_s t}$, $\ldots$, $(a_{12\cdots m})_{i_{12\cdots m}}$ represent the various main effects and interactions, b_j represents the block effect, and the ϵ_{ij} are NID $(0, \sigma^2)$. Algorithms are given for using the balanced incomplete and the group divisible designs for asymmetrical factorial arrangements. Let $M(s)$ be the square matrix (of dimension M_s) $M(s) = m_sI - J$ where J is a matrix having all elements unity, and define the direct product of p such matrices by $M(1, 2, \ldots, p) = [M(1) \times M(2) \times \ldots \times M(p)]$ $(p \leqq m)$. Then the variance-covariance matrix of a p-factor interaction for the G.D. case can be written as $M(1, 2, \ldots, p)\ \sigma^2/(E_trv)$ $(t = 1$ or $2)$. For the BIBD, the same expression holds with $E_1 = E_2$. The correlations between the different interactions are all zero and since $M^2(1, 2, \ldots, p) = M(1, 2, \ldots, p) \prod_1^p m_s$, $[E_tr/\prod_1^p m_s] \sum (a_{12\cdots p})^2_{i_{12\cdots p}}$ follows a $\sigma^2\chi^2$ with $\prod_1^p (m_s - 1)$ degrees of freedom under the hypothesis of no p-factor interaction effects. (Received January 16, 1957.)

10. An Extension of the Cramér-Rao Inequality, John J. Gart, Virginia Polytechnic Institute, (By Title).

Consider a frequency function $f(x \mid \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_s)$, the function being specified when $\boldsymbol{\theta}$ is specified. The parameter $\boldsymbol{\theta}$ has a density $g(\boldsymbol{\theta})$ independent of x. Let $\mathbf{X} =$

$(x_1, x_2, \ldots, x_n)$ be a random sample from a randomly chosen population having the specified frequency function. Then if $\phi = \prod_{i=1}^{n} f(x_i \mid \theta)$ and t_k (independent of θ) is an estimate of θ_k, $1 \leqq k \leqq s$, there follows a form similar to the Cramér-Rao Inequality, $EE[(t_k - \theta_k)^2 \mid \theta] \geqq \{E[E(t_k \mid \theta) - \theta_k]\}^2 + E^2\{[\partial E(t_k \mid \theta)]/\partial\theta_k\}\{EE[(\partial \ln\phi/\partial\theta_k)^2 \mid \theta]\}^{-1}$. The equality is reached if and only if t_k is an unbiased sufficient statistic having the normal distribution with constant variance. In this case the equality hoids regardless of the form of $g(\theta)$. (Received January 17, 1957.)

11. **Multivariate Analysis of Variance,** S. N. Roy, University of North Carolina.

Consider a model under which we have stochastic variates $X(p \times n) = [x_1 \ldots x_n]p$ such that x_i's (for $i = 1, 2, \ldots, n$) are independent $N[E(x_i), \Sigma]$, $E(X') = A(n \times m) \times \xi(m \times p)$, A (to be called the design matrix) is a matrix of constants given by the design of the experiment, ξ is a matrix of unknown parameters, rank $(A) = r \leqq m < n$, $p \leqq n - r$ and Σ is an unknown dispersion matrix. Under this model suppose we have a testable hypothesis (the meaning and mathematical criterion for testability being discussed in the paper) $H_0: C(s \times m)\xi(m \times p)M(p \times q) = 0\ (s \times q)$, where C and M (to be called the hypothesis matrices) are given such that rank $(C) = s \leqq r$ and rank $(M) = q \leqq p$. The alternative is $H: C\xi M = \eta\ (s \times q)\ (\neq 0)$. The test is that at a level α we accept H_0 if $c_{\max}(S^*S^{-1}) \leqq c_\alpha$ and reject H_0 otherwise, where S^* and S are matrices given (in the paper) in terms of X, A, C and M, $c_{\max}(T)$ denotes the largest root of a matrix with real nonnegative roots, and c_α is a constant depending on α, min (s, q) and $n - r$, which we can pick up from tables now under construction and expected to be published shortly. (Received January 17, 1957.)

12. **Confidence Bounds Associated with Multivariate Analysis of Variance,** S. N. Roy and R. Gnanadeshikan, University of North Carolina.

We start from the same set up as in *the previous paper*. The S^* and S (to be called respectively the dispersion matrix "due to the hypothesis" and the dispersion matrix "due to the error") are the exact analogs of the variance "due to the hypothesis" and that "due to the error" in the customary univariate analysis of variance. Given any level α, we can pick up a constant c_α from the tables mentioned in *the previous paper* and make, with a probability greater than or equal to $1 - \alpha$, the confidence interval statement: $c_{\max}^{1/2}(sS^*) - [sc_\alpha]^{1/2} \times c_{\max}^{1/2}(S) \leqq c_{\max}^{1/2}[\eta' U \eta] \leqq c_{\max}^{1/2}(sS^*) + [sc_\alpha]^{1/2} c_{\max}^{1/2}(S)$, where $U(s \times s)$ is a nonsingular matrix given (in the paper) in terms of A and C, and $c_{\max}^{1/2}[\eta' U \eta]$ is zero if and only if $\eta = 0$, i.e., H_0 is true. With a joint probability greater than or equal to $1 - \alpha$ we can also make simultaneous confidence interval statements including the one given above and others exactly similar to this but in terms of $S^{(i)}$, $S^{(i)*}$, $\eta^{(i)}$ (for $i = 1, 2, \ldots, p$) and next in terms of $S^{(i,j)}$, $S^{(i,j)*}$, $\eta^{(i,j)}$ (for $i \neq j = 1, 2, \ldots, p$), and so on, where $S^{(i)}$ and $S^{(i)*}$ stand respectively for truncated matrices after cutting out the ith row and ith column from S and S^*, $\eta^{(i)}$ for η with the ith column cut out, $S^{(i,j)}$, $S^{(i,j)*}$ for S and S^* with the ith and jth rows and columns cut out, $\eta^{(i,j)}$ for η with the ith and jth columns cut out, and so on. (Received January 17, 1957.)

13. **Extension of Some Results Given by Mitra on "Statistical Analysis of Categorical Data,"** Earl Diamond, University of North Carolina.

This is a follow-up of two previous paper ([1] "Some non-parametric generalizations of analysis of variance and multivariate analysis" by S. N. Roy and S. K. Mitra, *Biometrika*, December, 1956, and [2] "Contributions to the statistical analysis of categorical data" by S. K. Mitra, North Carolina Institute of Statistics Mimeograph Series No. 142). We start from *a* product of multinomial distributions of the form $\phi = \prod_j [n_{0j}! \prod_i p_{ij}^{n_{ij}} / \prod_i n_{ij}!]$

with $\sum_i p_{ij} = 1$, $i = i_1 i_2 \ldots i_k$; $j = j_1 j_2 \ldots j_\ell$; $i_1 = 1, 2, \ldots, r_1$; $\ldots$; $i_k = 1, 2, \ldots, r_k$; $j_1 \varepsilon (s_1)_{j_2 \cdots j_\ell}$ (a subset of s_1 depending on the subscript set $j_2 \ldots j_\ell$); $j_2 \varepsilon (s_2)_{j_3 \cdots j_\ell}$; $\ldots$; $j_{\ell-1} \varepsilon (s_{\ell-1})_{j_\ell}$ and $j_\ell = 1, 2, \ldots, s_\ell$. We next consider two hypotheses $H_0^{(1)}: p_{ij} = f_{ij}^{(1)}(\theta_1, \ldots, \theta_{t_1})$ subject to $g_m^{(1)}(\theta_1, \ldots, \theta_{t_1}) = 0$ $(m = 1, 2, \ldots, u_1 < t_1)$ and $H_0^{(2)}: p_{ij} = f_{ij}^{(2)}(\theta_1', \ldots, \theta_{t_2}')$ subject to $g_m^{(2)}(\theta_1', \ldots, \theta_{t_2}') = 0$ $(m = 1, 2, \ldots, u_2 < t_2)$, where $t_1, t_2 <$ total number of cells $-$ total number of multinomial distributions. Each hypothesis is a composite one in which the θ's or θ''s are the nuisance parameters and $f_{ij}^{(1)}$, $g_m^{(1)}$, $f_{ij}^{(2)}$ and $g_m^{(2)}$ are known functions. Tests are taken over from Refs. [1] and [2], and the asymptotic powers of the tests and the conditions for asymptotic independence are derived which are extensions of similar conditions for more special cases discussed in [2]. (Received January 17, 1957.)

14. Testing of Hypotheses on a Mixture of Variates Some of Which are Continuous and the Rest Categorical, S. N. Roy and M. D. Moustafa, University of North Carolina.

We start from a $k + \ell$-variate distribution in which k variates are continuous and ℓ variates are categorical. The k variates are assumed to have a conditional multivariate normal distribution with respect to the ℓ categorical variates which are assumed to have a multinomial distribution. Appropriate hypotheses are framed in this situation, analogous to the customary hypotheses on a single multivariate normal distribution (or to those in Refs. [1] and [2] of the *previous abstract*), large sample tests of such hypotheses are developed and some of their properties studied. Next, instead of assuming a single multinomial distribution on the ℓ categorical variates, a product of multinomial distributions is assumed and hypotheses are framed in this situation analogous to the customary ones for several multivariate normal distributions or to those *in Refs. [1] and [2]*, and large sample tests of such hypotheses and some of their properties are studied. (Received January 17, 1957.)

15. On Statistics Independent of a Sufficient Statistic, Evan J. Williams, North Carolina State College.

It is shown that if, for a sample drawn from a population of values of x with distribution depending on a parameter θ, the statistic z is sufficient for θ, and g is any statistic whose distribution is independent of θ, then g and z are independently distributed. The method of proof is less sophisticated than that of Basu (*Sankhyā*, Vol. 15 (1955), p. 377).

The result has application to the normal distribution: the mean of a sample is distributed independently of any location-free statistic; and to the gamma distribution: the mean of a sample is distributed independently of any scale-free statistic. These well-known results follow since the sample mean is a sufficient statistic, in the former case for the location parameter, in the latter case for the scale parameter.

The limitations of the general result lie in the difficulty of deriving statistics independent of parameters other than location and scale parameters.

The connexion of the theorem with estimation theory is discussed. (Received January 17, 1957.)

16. Generalized Quantal Response in Biological Assay, John Gurland, Iowa State College.

The quantal (all-or-none) response in biological assay refers to a response in which one of two possible outcomes occurs. In a bioassay such as that of an insecticide based on mortality of the housefly, say, there are, however, three possible outcomes, namely, alive, moribund, dead. The present paper considers a generalized quantal response in which two or more outcomes are possible. Whether one uses normits (cf. probits) or logits or other transformations, a general method of analyzing the data is developed which makes explicit use of all the possible outcomes and hence is more efficient than the common procedure of pooling some outcomes (for example, moribund and dead) in order to make the response all-or-none. Further, a technique analogous to that used in discriminant functions is suggested as a method which makes more efficient use of the data than the pooling method mentioned above. (Received January 21, 1957.)

17. The Variance of Zero-Crossing Intervals, J. A. McFadden, U. S. Naval Ordnance Laboratory, (introduced by Gilbert Lieberman).

Two expressions are given for the variance of the intervals between successive zeros of a random process. It is assumed that the successive intervals form a Markoff chain. If $x(t)$ is a random process, let $y(t) = 1$ when $x(t) \geqq 0$ and $y(t) = -1$ when $x(t) < 0$. Let β be the expected number of zeros per second and let κ be the correlation coefficient between two successive zero-crossing intervals. Then the variance is $\sigma^2 = (2A/\beta)(1 + \kappa)/(1 - \kappa)$, or alternatively, $\sigma^2 = [(1 + 2B)/\beta^2] (1 - \kappa)/(1 + \kappa)$, where $A = \int_0^\infty r(\tau) \, d\tau$ and $B = \int_0^\infty [Q(\tau) - \beta] \, d\tau$. $r(\tau)$ is the autocorrelation function of the process $y(t)$ and $Q(\tau) \, d\tau$ is the conditional probability of a zero between $t + \tau$ and $t + \tau + d\tau$, given a zero at time t. (Received January 21, 1957.)

18. A Limit Theorem and Bounds for an Optional Stopping Probability, Morris Skibinsky, Michigan State University, (By Title).

Let S_j be the standardized jth partial sum of a sequence of bounded independent, identically distributed random variables, K, a positive constant, and let

$$Q(m, n, K) = \Pr\{\max_{m \leqq j \leqq n} s_j \geqq K\}.$$

It is shown by elementary methods that if $\lim_{m\to\infty}[(n - K)/m^{1/2}] = 0$, then $\lim_{m\to\infty} Q(m, n, K) = 1 - \phi(K)$, where ϕ is the standard normal c.d.f. Certain steps in the proof are then used to obtain simple bounds for $Q(m, n, K)$ when the sequence of random variables is generated from Bernouli trials. (Received January 21, 1957.)

19. A Limit Theorem of Cramér and Its Generalization, Junjiro Ogawa, University of North Carolina, (By Title).

As a generalization of Doob's theorem, H. Cramér states the following theorem: *Suppose we have for every* $\nu = 1, 2, \ldots, y_\nu = Ax_\nu + z_\nu$, *where* x_ν, y_ν *and* z_ν *are n-dimensional random variables, while A is a matrix of order* $(n \times n)$ *with constant elements. Suppose further that as* $\nu \to \infty$, *the n-dimensional distribution of* x_ν *tends to a certain limiting distribution, while* z_ν *converges in probability to zero. Then* y_ν *has the limiting distribution defined by the linear transformation* $y = Ax$, *where x has the limiting distribution of the* x_ν (H. Cramér, *Mathematical Methods of Statistics*, Princeton, 1946, pp. 299–300). Cramér skips the proof of this theorem. In this paper, the complete proof of this theorem will be given and two theorems which are generalizations of this theorem and are useful in statistics will be proved. (Received January 22, 1957.)

20. On the Mathematical Principles Underlying the Theory of the χ^2 Test, Junjiro Ogawa, University of North Carolina, (By Title).

The rigorous proof of the theorem that the χ^2 statistic has the limiting chi-square distribution with degrees of freedom reduced by the number of the independent parameters which were estimated from the frequency data, was first given by H. Cramér in his famous book *Mathematical Methods of Statistics*, Princeton (1946), but some steps of the proof were skipped. Later on S. N. Roy and S. K. Mitra (*Biometrika*, Vol. 43 (1956)) and S. K. Mitra (Thesis, University of North Carolina, 1956) reasoned along the same lines and got theorems adjusted to various physical situations. The purposes of this paper are to present a complete and self-contained proof of Cramér's theorem on the one hand, and on the other to explain how the proof of the related theorems got by S. N. Roy and S. K. Mitra could be thrown back on that of Cramér's theorem from the mathematical point of view. (Received January 22, 1957.)

21. Minimization of Certain Integrals Subject to Linear Constraints, (Preliminary Report), C. H. Kraft and I. Olkin, Michigan State University.

Let F be the class of measures f such that $E_f q_i(x) = a_i$, $i = 1, \ldots, n$ and $E_f H(f) < \infty$. The problem of minimizing $E_f H(f)$ over F has been treated by Shannon [*Bell System Technical Journal*, Vol. 27 (1948), pp. 623–656] for $H(f) = \log f$, $q(x) = x^2$ using calculus of variations, and by Weiss [*Ann. Math. Stat.*, Vol. 27 (1956), pp. 851–853] for $H(f) = f$, arbitrary square integrable $q_i(x)$.

The following considerations apply to these cases as well as others, e.g. $H(f) = f^p$. An inequality of the form $E_f H(f) \geqq T(f, g)$ for all densities g is available, where $T(g, g) = E_g H(g)$. $T(f, g)$ is constant for $f \in F$ if and only if $g(x) = \Sigma b_i q_i(x)$. The bound is attainable if the constants b_i can be chosen so that $g \in F$. These considerations extend the proofs to not necessarily dominated families F on any measure space. (Received January 23, 1957.)

22. The Recovery of Intervariety Information, Bradley Bucher, Princeton University.

Assume, in the incomplete block model, $y_{ij} = m + b_i + v_j + e_{ij}$, that the block effects are independently distributed with mean 0 and variance β^2, the error terms e_{ij} are independently distributed with mean 0 and variance α^2, and that the variety effects $t_1, \ldots, t_k$, are fixed effects and that $t_{k+1}, \ldots, t_n$, are independently distributed with mean 0 and variance γ^2. Then in estimating any linear combination of the variety effects, say, $a_1t_1 + a_2t_2 + \cdots + a_kt_k$, we may make use of information among the varieties $t_{k+1}, \ldots, t_n$. Minimum variance linear unbiased estimates are obtained for such combinations for a large class of incomplete block designs. In general, these estimates have smaller variance than analogous estimates obtained using only inter- and intra-block recovery. For balanced incomplete blocks the estimate with intervariety recovery is shown to be the same as the combined intra- and inter-block estimate. Several techniques are developed which are useful for finding estimates using intervariety recovery. The problem of estimating γ^2 is discussed. Useful applications of the technique of intervariety recovery are considered. (Received January 24, 1957.)

23. Some Uses of Quasi-Ranges II, J. T. Chu and F. C. Leone; Case Institute of Technology and C. W. Topp, Fenn College, Cleveland, Ohio.

In "Some uses of quasi-ranges," (*Ann. Math. Stat.*, Vol. 28 (1957), No. 1), methods are given of using quasi-ranges to obtain confidence intervals for, and tests of hypotheses about, some measures of dispersion of a given distribution (such as the interquantile distance and the standard deviation). In this paper, further research is done on the selection of quasi-ranges for making inferences about the standard deviations of the normal, rectangular, and exponential distributions. The methods are also extended to the coefficient of variation, the difference and ratio of interquantile distances and standard deviations of two given distributions, etc. Tables are given to facilitate applications. Received January 24, 1957.)

24. On Selecting a Subset Which Contains All Populations Better Than a Standard, Shanti S. Gupta and Milton Sobel, Bell Telephone Laboratories.

Populations $\pi_i (i = 0, 1, \dots, p)$ are given with a common Koopman-Darmois distribution of known form differing only in the value of the unknown parameter $\tau_i (i = 1, 2, \dots, p)$; cases of known and unknown (associated with the standard π_0) are treated separately. Location and scale parameter problems are both treated. In some problems π_i is defined as better than π_0 if $\tau_i > \tau_0$; in others if $\tau_i < \tau_0$. A procedure is given in each case for selecting a small subset so that, for any true configuration, the probability of including all π_i equal to or better than π_0 is at least P^*, $P^* < 1$ being preassigned. For the location parameter, with τ_0 unknown, the procedure is to retain all π_i with $\bar{w}_i = \sum_{j=1}^{n_i} w(x_{ij}) \geqq \bar{w}_0 - d/(n_i)^{1/2}$; here $\bar{w}_i$ is sufficient for $\tau_i (i = 0, 1, \dots, p)$. For scale parameter problems, with smaller τ_i more preferable, the procedure retains all π_i with $\sum_{j=1}^{n_i} w(x_{ij}) \leqq (1 + d) \sum_{j=1}^{n_0} w(x_{0j})$. In several problems the value of d is computed and tables are given for different P^* and p-values; in others transformations are used to "normalize" the problem. The normal and chi-square distributions are used as applications. Problems involving binomial and Poisson distributions are treated separately with and without normalizing transformations. (Received January 24, 1957.)

25. On the Relation Between Loss Functions and Significance Levels, (Preliminary Report), H. Robert van der Vaart, North Carolina State College and Leiden University.

Consider a one-parameter family $\{P_\theta\}$ of probability distributions. Be it asked to test $H_0: \theta = \theta_0$, against $H_1: \theta > \theta_0$. Define a loss function $L = l_0$ if H_0 is rejected when true, $L = l_1$ if H_1 is rejected when true, $L = 0$ otherwise. Suppose a family $\{w_i\}$ of subsets of the sample space is given, $P_\theta(w_i)$ being a monotonous increasing function of θ for each w_i. Then selecting a critical region w_s such that $P_\theta(w_s)$ has some fixed value α is a classical procedure, known to be minimax relatively to L provided $\alpha = l_1/(l_0 + l_1)$ (Sverdrup, 1953; Ruist, 1954). However most statisticians, while fixing $P_\theta(w_s) = \alpha$ for $\theta = \theta_0$, really want(ed) to reject H_0 only if θ differs materially from θ_0, say if $\theta > \theta_1 > \theta_0$ (cf. also Hodges and Lehmann, 1954), i.e. they test(ed) $H_0': \theta \leqq \theta_1$, against $H_1': \theta > \theta_1$ (θ_0 acting as an idealization of H_0'). Now the critical region $w_{s'}$ which is minimax in the situation described by adding a prime to each H in the definition of L has two properties: (i) $P_{\theta_0}(w_{s'}) < \alpha = l_1/(l_0 + l_1)$, depending on θ_1, (ii) $P_{\theta_0}(w_{s'})$ is smaller with more powerful test families $\{w_i\}$. Both effects (subsisting with loss functions allowing for indifference zones) indicate that fixing $P_\theta(w_s)$ upon a constant level for such "idealized null hypotheses" as $\theta = \theta_0$ may be a questionable procedure. (Received January 28, 1957.)

26. A Note on Fluctuations of Telephone Traffic, Václav Edvard Beneš, Bell Telephone Laboratories, (By Title).

Let $N(t)$ be defined as the number of calls in progress in a simple telephone exchange model characterized by unlimited call capacity, a general probability density of holding-time, and randomly arriving calls. A formula, due to Riordan, for the generating function of the transition probabilities of $N(t)$ is proved. From the generating function, expressions for the covariance function of $N(t)$ and for the spectral density of $N(t)$ are determined.

It is noted that the distributions of $N(t)$ are completely specified by the covariance function, if $N(t)$ is defined as above. (Received February 4, 1957.)

27. Randomization Procedures for the Estimation of Cross-Spectral Density Functions, A. E. GARRATT, Virginia Polytechnic Institute, (By Title).

The cross-spectral density function may be estimated by

$$\Phi_{xy}^{*}(\omega_h) = \sum_{u=1}^{n}\{x(t_u)y(t_u + k_u\Delta t)G_1(k_u) + iX(t_u)y(t_u + m_u\Delta t)G_2(m_u)\}$$

where the k_u are independently distributed according to $p_1(k)$, $k = -r, \ldots, r$; where the m_u are similarly distributed according to $p_2(m)$ and are independent of the k_u; and where $G_1(k_u)$ and $G_2(m_u)$ are arbitrary weight functions.

It is shown that the expectation of the estimator depends on the products $p_1(k)G_1(k)$ and $p_2(m)G_2(m)$, whereas the variance of the estimator depends specifically on $p_1(k)$ and $p_2(m)$. Various specifications of the products $p_1(k)G_1(k)$ and $p_2(m)G_2(m)$ and of the probability distributions $p_1(k)$ and $p_2(m)$ are considered which provide estimators with certain optimum properties. (Received February 8, 1957.)

28. Fréchet Differentiable Functional Estimates, GOPINATH KALLIANPUR, Michigan State University, (introduced by Morris Skibinsky).

Suppose $f_\theta(x)$ is a probability density over the finite range (a, b) which is independent of the unknown θ to be estimated. Let $\phi_n(x)$ denote an empirical density function (defined in the paper) of a sample of size n from the given population. Let G be a class of functionals over the Banach space L_1 satisfying the following conditions: (i) G possesses Fréchet differentials of the first two orders at the "true point" f_θ. If $g_1[f_\theta ; x]$ and $g_2[f_\theta ; x, y]$ are the Fréchet derivatives of the first and second order at f_θ, (ii) $g_1[f_\theta ; x]$ is a continuous function of x which is not zero over a set of positive measure, (iii) $| g_2[f_\theta ; x, y] | \leqq A < \infty$, A being independent of x and y. (iv) $G[f_\theta(x)] = \theta$ ("Fisher consistency"). Then assuming regularity conditions which validate differentiation with respect to θ, etc., and assuming $E_\theta(g_1[f_\theta ; x]) = 0$ without loss of generality, it is shown that $\sqrt{n}\{G[\phi_n] - \theta\}$ is asymptotically normally distributed with zero mean and asymptotic variance $E_\theta(g_1^2[f_\theta ; x])$ which satisfies Fisher's inequality $E_\theta(g_1^2[f_\theta ; x]) \geqq \{E_\theta[(\partial \log f_\theta)/\partial\theta]^2\}^{-1}$. An earlier paper by C. R. Rao and the author (Sankhyā, 1955) discusses similar problems for functionals of the empirical c.d.f. (Work done under ONR project at Columbia University.) (Received February 14, 1957.)

29. The Efficiency of Nonparametric Tests, GOTTFRIED E. NOETHER, Boston University.

Given two tests of the same hypothesis and the same significance level. If for the same power with respect to the same alternative one requires a sample of size n_1 and the other a sample of size n_2, the relative efficiency of the second test with respect to the first test is given by the ratio n_1/n_2. The paper surveys existing results on the relative efficiency of important nonparametric tests with respect to corresponding parametric as well as other nonparametric test procedures. In particular, the following problems are considered: one-sample and paired comparison tests, two-sample tests, analysis of variance tests, tests of independence and regression, goodness of fit tests. As a general conclusion, it can be said that the employment of the more efficient nonparametric methods instead of the customary parametric methods rarely involves an appreciable loss of information, but may lead to a considerable gain. (Received March 1, 1957.)

30. **On a Problem in Abelian Groups and the Construction of Fractionally Replicated Designs,** R. C. BOSE, University of North Carolina AND R. C. BURTON, National Bureau of Standards.

Consider an Abelian group of order s^n, generated by n letters $A_1, A_2, \ldots, A_n$, with the relations $A_1^s = A_2^s = \ldots = A_n^s = I$, where I is the identity and s is a prime. If $G = A_1^{x_1}A_2^{x_2} \ldots A_n^{x_n}$ is any element of the group, then the number of non-zero exponents x_i may be called the length of G. Given an integer $r < n$, the problem is to find a subgroup of order s^r, generated by r independent elements $G_i = A_1^{x_{i1}}A_2^{x_{i2}} \ldots A_n^{x_{in}}$ such that the minimum length of the elements in the subgroup (except the length of the unit element) is greater than or equal to k. Consider the finite projective space $PG(r - 1, s)$. To any point $x = (x_1, x_2, \ldots, x_r)$ of this space, assign a non-negative integer m, which may be considered the measure of x, in such a way that the total measure for the space is n. To a point of measure m associate m different letters chosen out of $A_1, A_2, \ldots, A_n$, each of these letters being assigned to one and only one point. Let $G_i = A_1^{x_{i1}}A_2^{x_{i2}} \ldots A_n^{x_{in}}$ where x_{ij} is the ith coordinate of the point to which A_j is associated. It is proved that the length of the element $G_1^{\lambda_1}G_2^{\lambda_2} \ldots G_r^{\lambda_r}$ is the measure of the set of points not lying on the linear space $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r = 0$. For example let $n = 10$, $r = 4$, $s = 3$. We can find exactly 10 points on an unruled quadric in $PG(3, 3)$. If we take the corresponding subgroup as the fundamental identity for generating a $\frac{1}{3}^4$ fraction in a factorial design with 10 factors, then all the aliases of a main effect will have five or more factors, and all the aliases of two factor interaction will have four or more factors. (Received January 21, 1957.)

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Professor Felix Bernstein, the founder and director Emeritus of the Institute of Mathematical Statistics, University of Goettingen, Germany, died December 3, 1956 in Zuerich, Switzerland. Professor Bernstein was also a member of the International Statistics Institute, a fellow of the Royal Statistics Society, a fellow of the AAAS, and was professor of biometrics, New York University from 1936–1945. In 1950 he was American Fulbright professor at the Institute of Statistics, Rome, Italy.

Dr. Robert M. Blumenthal has been appointed to an instructorship at the University of Washington.

Glenn L. Burrows has been appointed Staff Statistician at the Knolls Atomic Power Laboratory, Schenectady, New York.

Victor Chew resigned on February 1, 1957 from the position of Assistant Professor of Statistics, University of Florida, to become Asst. Statistician at Institute of Statistics at Raleigh, North Carolina, and do work towards a Ph.D. in experimental statistics.

Professor Kai Lai Chung, on leave from Syracuse University, is a Visiting Professor at the University of Chicago during 1956–57.

George E. Ferris is now with the Statistics Department of General Foods' Corporation in Hoboken, New Jersey.