BOUNDS FOR THE VARIANCE OF THE MANN-WHITNEY STATISTIC

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1. Summary. Let X, Y be independent random variables with continuous cumulative probability functions and let

$$p = \Pr\{Y < X\}.$$

For the variance of the Mann-Whitney statistic U, upper and lower bounds are obtained in terms of p, for the case of any X and Y as well as for the case of stochastically comparable X, Y. The results for the case of stochastic comparability are new, while the inequalities in the case of arbitrary X, Y have either been obtained by van Dantzig or are a consequence of other inequalities due to van Dantzig.

2. Introduction and statement of results. Let X and Y be independent random variables with the continuous cumulative probability distribution functions (c.d.f.'s) $F(x)$ and $G(y)$, respectively, and let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be samples of these random variables. We consider the statistic

$$U = \text{number of pairs } (X_i, Y_j) \text{ such that } Y_j < X_i, \tag{2.1}$$

introduced by Wilcoxon [1] for $m = n$ and by Mann and Whitney [2] in the general case.

To simplify arguments we shall from now on assume that $F(t)$ and $G(t)$ are both strictly increasing functions, although it can be easily seen that all conclusions remain valid without this restriction. The function

$$L(t) = F[G^{(-1)}(t)], \tag{2.2}$$

which will be called the "relative distribution function of X and Y," is a convenient means of reducing many problems involving two probability distributions to a study of a cumulative probability function on the unit interval. One verifies easily that X and Y have the same distribution if and only if $L(t) = t$ for $0 \leqq t \leqq 1$. Similarly X is stochastically smaller than Y, that is, $F(s) \geqq G(s)$ for $-\infty < s < +\infty$ if and only if $L(t) \geqq t$ for $0 < t < 1$.

Using the quantity

$$p = \Pr\{Y < X\} = \int_{-\infty}^{+\infty} G(s)\, dF(s) = \int_0^1 t\, dL(t) \tag{2.3}$$

and the relative distribution function L, one can rewrite expressions for the expectation and the variance of U obtained by van Dantzig [4] in the form

$$E(U) = mnp, \tag{2.4.1}$$

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$$\sigma^2(U) = mn[(m-1)\varphi^2 + (n-1)\gamma^2 + p(1-p)], \tag{2.4.2}$$

where

$$\begin{aligned}\varphi^2 &= \int_{-\infty}^{+\infty} F^2\, dG - \left(\int_{-\infty}^{+\infty} F\, dG\right)^2 = \int_{-\infty}^{+\infty} F^2\, dG - (1-p)^2 = \sigma^2[F(Y)]\\ &= \int_0^1 L^2(t)\, dt - (1-p)^2,\end{aligned} \tag{2.4.3}$$

$$\begin{aligned}\gamma^2 &= \int_{-\infty}^{+\infty} G^2\, dF - \left(\int_{-\infty}^{+\infty} G\, dF\right)^2 = \int_{-\infty}^{+\infty} G^2\, dF - p^2 = \sigma^2[G(X)]\\ &= \int_0^1 t^2\, dL(t) - p^2.\end{aligned} \tag{2.4.4}$$

In Sec. 3, inequalities involving φ^2 and γ^2 will be derived which will be used to obtain Theorem 3.2 on the sharp upper bound

$$\sigma^2(U) \leqq mnp(1-p) \max(m, n) \tag{2.5}$$

and Theorem 3.5 on the sharp lower bound

$$\sigma^2(U) \geqq \begin{cases} \mu\nu\left[\mu r(1-r) - \dfrac{(\mu-1)^2}{12(\nu-1)}\right] & \text{if } \dfrac{\mu-1}{\nu-1} \leqq 2r \\ \mu\nu[\tfrac{4}{3}r\sqrt{2(\mu-1)(\nu-1)r} - (\mu+\nu-2)r^2 + r(1-r)] & \\ & \text{if } \dfrac{\mu-1}{\nu-1} \geqq 2r \end{cases} \tag{2.6}$$

where $\mu = \min(m, n)$, $\nu = \max(m, n)$, $r = \min(p, 1-p)$. The upper bound (2.5) has been obtained by van Dantzig [4] and is discussed here only for the sake of completeness and convenient reference. While it is believed that (2.6) has not been stated elsewhere, the inequalities involving φ^2 and γ^2 on which it is based are essentially modifications of analogous inequalities obtained by van Dantzig [5].

In Sec. 4 similar inequalities for φ^2 and γ^2 are obtained which yield Theorem 4.2 on the sharp upper bound

$$\begin{aligned}\sigma^2(U) \leqq \mu\nu\{&\nu[\tfrac{1}{3}(1-(1-2p)^{3/2}) - p^2]\\ &+ \mu[-\tfrac{2}{3}(1-(1-2p)^{3/2}) + 2p - p^2]\\ &+ \tfrac{1}{3}[1-(1-2p)^{3/2}] - p(1-p)\},\end{aligned} \tag{2.7}$$

and Theorem 4.5 on the sharp lower bound

$$(2.8)\quad \sigma^2(U) \geqq \begin{cases} mn\{\frac{1}{3}[m + n + 1 + 2\sqrt{(m-1)(m-n)(1-2p)^3}] \\ \qquad - [m(1-p)^2 + np^2 + p(1-p)]\} & \text{if } \frac{n-1}{m-1} \leqq 2p, \\ mn\{\frac{4}{3}p\sqrt{2(m-1)(n-1)p} - (m+n-2)p^2 + p(-p)\} \\ & \text{if } 2p < \frac{n-1}{m-1} \leqq \frac{1}{2p} \\ mn\{\frac{1}{3}[m + n + 1 + 2\sqrt{(n-1)(n-m)(1-2p)^3}] \\ \qquad - [mp^2 + n(1-p)^2 + p(1-p)]\} & \text{if } \frac{1}{2p} \leqq \frac{n-1}{m-1}, \end{cases}$$

under the assumption that X is stochastically smaller than Y, that is, $F(s) \geqq G(s)$, $-\infty < s < +\infty$. These results are new. Due to the imposition of the stochastic comparability condition the bounds (2.7) and (2.8) of course are better than the bounds (2.5) and (2.6) for the general case.

Upper bounds such as (2.5) and (2.7) are useful in problems of estimating the parameter p by $\hat{p} = U / mn$ (see [3]). Lower bounds are needed in obtaining inequalities for the power of the Mann-Whitney test such as those given in [4].

3. Inequalities for φ^2, γ^2, $\sigma^2(U)$ in the general case.

3.1. Lemma. *With the notations of the preceding section we have*

$$(3.1)\qquad \int_0^1 [L(t) - t]^2\, dt \leqq \tfrac{1}{3} - p(1-p),$$

and equality holds for

$$(3.1.1)\qquad L_1(t) = \begin{cases} 0 & \text{for } 0 \leqq t < p \\ 1 & \text{for } p \leqq t \leqq 1 \end{cases}$$

and for

$$(3.1.2)\qquad L_2(t) = 1 - p \text{ for } 0 < t < 1.$$

Proof. In the identity

$$\int_0^1 \int_0^t [L(t) - L(s)]\, ds\, dt = \int_0^1 (2t-1)L(t)\, dt,$$

the integrand on the left side satisfies $0 \leqq L(t) - L(s) \leqq 1$ for $0 \leqq s \leqq t \leqq 1$, hence $L(t) - L(s) \geqq [L(t) - L(s)]^2$, and

$$\int_0^1 (2t-1)L(t)\, dt \geqq \int_0^1 \int_0^t [L(t) - L(s)]^2\, ds\, dt = \int_0^1 L^2(t)\, dt - \left[\int_0^1 L(t)\, dt\right]^2.$$

Therefore

$$\int_0^1 [L(t) - t]^2\,dt = \int_0^1 L^2(t)\,dt - 2\int_0^1 tL(t)\,dt + \tfrac{1}{3}$$

$$\leqq \left[\int_0^1 L(t)\,dt\right]^2 - \int_0^1 L(t)\,dt + \tfrac{1}{3} = \tfrac{1}{3} - p(1-p).$$

One verifies by direct computation that $L_1(t)$ and $L_2(t)$ yield equality in (3.1).

3.2. THEOREM. *The variance of U has the upper bound*

$$\sigma^2(U) \leqq mnp(1-p)\max(m, n). \tag{3.2}$$

Equality holds for L_2 if $n \geqq m$, and for L_1 if $m \geqq n$.

PROOF. We use the equality

$$\varphi^2 + \gamma^2 = \int_0^1 [L(t) - t]^2\,dt + \tfrac{2}{3} - p^2 - (1-p)^2$$

and, if $n \geqq m$, write (2.4.2) in the form

$$\begin{aligned}\sigma^2(U) &= mn\{(m-1)(\varphi^2 + \gamma^2) + (n-m)\gamma^2 + p(1-p)\}\\ &= mn\left\{(m-1)\int_0^1 [L(t)-t]^2\,dt + (n-m)\gamma^2\right.\\ &\qquad \left. + (m-1)[\tfrac{2}{3} - p^2 - (1-p)^2] + p(1-p)\right\}.\end{aligned} \tag{3.2.1}$$

Noting that

$$\gamma^2 = \int_0^1 t^2\,dL(t) - p^2 \leqq \int_0^1 t\,dL(t) - p^2 = p(1-p) \tag{3.2.2}$$

and making use of (3.1) we obtain (3.2). Since equality holds for $L_2(t)$ in (3.1) and in (3.2.2), the upper bound is attained in (3.2) for $L_2(t)$, if $n \geqq m$. The case $m \geqq n$ follows by a symmetrical argument.

3.3 LEMMA. *Let F_1 and F_2 be strictly increasing continuous* c.d.f.'s *with*

$$\int_{-\infty}^{+\infty} F_1\,dF_2 = p_1, \qquad \int_{-\infty}^{+\infty} F_2\,dF_1 = p_2, \tag{3.3.1}$$

hence

$$p_1 + p_2 = 1; \tag{3.3.2}$$

and let

$$\varphi_1^2 = \int_{-\infty}^{+\infty} F_1^2\,dF_2 - p_1^2, \qquad \varphi_2^2 = \int_{-\infty}^{+\infty} F_2^2\,dF_1 - p_2^2. \tag{3.3.4}$$

Then, for any $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, $\mu_1 + \mu_2 > 0$, we have

$$\mu_1\varphi_1^2 + \mu_2\varphi_2^2 \geqq \mu_u\left[p_1 p_2 - \frac{\mu_u}{12\mu_v}\right], \tag{3.3.5}$$

for $u = 1, v = 2$, *as well as for* $u = 2, v = 1$. *Inequality* (3.3.5) *can not be improved if*

$$\frac{\mu_u}{\mu_v} \leqq \min(2p_1, 2p_2). \tag{3.3.6}$$

PROOF. Writing

$$F_u(s) = t, \qquad F_v[F_u^{(-1)}(t)] = L(t),$$

we have

$$p_u = \int_0^1 t\,dL(t), \qquad p_v = \int_0^1 L(t)\,dt,$$

$$\varphi_u^2 = \int_0^1 t^2\,dL(t) - p_u^2, \qquad \varphi_v^2 = \int_0^1 L^2(t)\,dt - p_v^2.$$

For any real α, β,

$$\begin{aligned} 0 \leqq \int_0^1 [L(t) - \alpha t - \beta]^2\,dt = \varphi_v^2 + p_v^2 + \frac{\alpha^2}{3} + \beta^2 - \alpha \\ + \alpha[\varphi_u^2 + p_u^2] - 2\beta p_v + \alpha\beta, \end{aligned} \tag{3.3.6.1}$$

and

$$\begin{aligned} \varphi_v^2 + \alpha\varphi_u^2 &\geqq \alpha - \frac{\alpha^2}{3} - \alpha p_u^2 - p_v^2 - (\beta^2 + \alpha\beta - 2\beta p_v) \\ &= 2(\alpha + \beta)p_v - (\alpha + 1)p_v^2 - \frac{\alpha^2}{12} - \left(\beta + \frac{\alpha}{2}\right)^2. \end{aligned}$$

For fixed α, the right-hand expression is maximum at $\beta = p_v - (\alpha/2)$, so that

$$\varphi_v^2 + \alpha\varphi_u^2 \geqq \alpha\left(p_1 p_2 - \frac{\alpha}{12}\right).$$

Setting $\alpha = (\mu_u/\mu_v)$, we obtain (3.3.5). Equality holds if and only if $L(t) = \alpha t + \beta$ for $0 < t < 1$, with $\alpha = (\mu_u/\mu_v)$ and $\beta = p_v - (\alpha/2)$, that is, for

$$L_3(t) = \frac{\mu_u}{\mu_v} t + p_v - \frac{\mu_u}{2\mu_v}, \qquad 0 < t < 1, \tag{3.3.7}$$

and this is a c.d.f. if and only if $L(0) \geqq 0$, $L(1) \leqq 1$, which is equivalent with (3.3.6).

3.4. LEMMA. *Under the assumptions*

$$p \leqq \tfrac{1}{2}, \tag{3.4.1}$$

$$\min\left(\frac{m-1}{n-1}, \frac{n-1}{m-1}\right) \geqq 2p, \tag{3.4.2}$$

we have

$$(3.4.3)\quad \begin{aligned} (m-1)\varphi^2 + (n-1)\gamma^2 &\geqq (m-1)(1-2p) \\ &+ \tfrac{4}{3}p\sqrt{2(m-1)(n-1)p} - [(m-1)(1-p)^2 + (n-1)p^2] \\ &= \tfrac{4}{3}p\sqrt{2(m-1)(n-1)p} - (m+n-2)p^2, \end{aligned}$$

and this inequality can not be improved.

PROOF. For any $\alpha > 0$, $0 \leqq \beta \leqq 1$, $\alpha + \beta \geqq 1$, we have $0 \leqq \dfrac{1-\beta}{\alpha} \leqq 1$, and

$$(3.4.4)\quad \begin{aligned} \int_0^1 [L(t) - \alpha t - \beta]^2\,dt &\geqq \int_{(1-\beta)/\alpha}^1 [L(t) - \alpha t - \beta]^2\,dt \\ &\geqq \int_{(1-\beta)/\alpha}^1 (\alpha t + \beta - 1)^2\,dt = \frac{(\alpha+\beta-1)^3}{3\alpha}. \end{aligned}$$

From this and

$$(3.4.4.1)\quad \begin{aligned} \int_0^1 [L(t) - \alpha t - \beta]^2\,dt = \varphi^2 + \alpha\gamma^2 + (1-p)^2 + \alpha p^2 + \frac{\alpha^2}{3} \\ - \alpha + \beta^2 + \alpha\beta - 2\beta(1-p) \end{aligned}$$

follows

$$\varphi^2 + \alpha\gamma^2 \geqq \alpha - \frac{\alpha^2}{3} - \alpha p^2 - (1-p)^2 - \alpha\beta - \beta^2 + 2\beta(1-p) + \frac{(\alpha+\beta-1)^3}{3\alpha}.$$

For fixed p and α, the right side is maximum for $\beta = 1 - \sqrt{2\alpha p}$. This value satisfies the conditions $0 \leqq \beta \leqq 1$, $\alpha + \beta \geqq 1$, if and only if

$$(3.4.5)\quad 2p \leqq \alpha \leqq \frac{1}{2p},$$

and then we obtain

$$(3.4.6)\quad \varphi^2 + \alpha\gamma^2 \geqq 1 - 2p + \tfrac{4}{3}p\sqrt{2\alpha p} - [\alpha p^2 + (1-p)^2].$$

If $m \geqq n$, then (3.4.2) becomes $[(n-1)/(m-1)] \geqq 2p$, so that $\alpha = [(n-1)/(m-1)]$ satisfies (3.4.5), and for this value of α inequality (3.4.6) yields (3.4.3).

If $m < n$, then (3.4.2) becomes $[(m-1)/(n-1)] > 2p$, the value $\alpha = [(n-1)/(m-1)]$ again satisfies (3.4.5) and we obtain (3.4.3) from (3.4.6).

Equality holds in (3.4.4) if and only if

$$L(t) = \begin{cases} \alpha t + \beta & \text{for } 0 < t \leqq \dfrac{1-\beta}{\alpha} \\ 1 & \text{for } \dfrac{1-\beta}{\alpha} < t \leqq 1, \end{cases}$$

so that equality is attained in (3.4.3) for

$$L_4(t) = \begin{cases} \dfrac{n-1}{m-1}t + 1 - \sqrt{2\dfrac{n-1}{m-1}p}, & 0 < t \leqq \sqrt{2p\dfrac{m-1}{n-1}} \\ 1, & \sqrt{2p\dfrac{m-1}{n-1}} < t \leqq 1. \end{cases}$$

3.5. THEOREM. *Under the assumption*

$$p \leqq \tfrac{1}{2} \tag{3.5.1}$$

and with the notations

$$\mu = \min(m, n), \qquad \nu = \max(m, n), \tag{3.5.2}$$

we have

$$\sigma^2(U) \geqq \mu\nu\left[\mu p(1-p) - \frac{(\mu-1)^2}{12(\nu-1)}\right], \quad \text{if } \frac{\mu-1}{\nu-1} \leqq 2p, \tag{3.5.3}$$

$$\sigma^2(U) \geqq \mu\nu[\tfrac{4}{3}p\sqrt{2(\mu-1)(\nu-1)p} - (\mu+\nu-2)p^2 + p(1-p)], \quad \text{if } \frac{\mu-1}{\nu-1} > 2p, \tag{3.5.4}$$

and these inequalities can not be improved.

PROOF. Assumption (3.5.1) constitutes no loss of generality since, in case it is not satisfied for p defined by (2.3), it will be satisfied if F and G are interchanged. Using the notations (3.5.2) and setting in Lemma 3.3: $p_1 = p$, $p_2 = 1 - p$, $\mu_1 = m - 1$, $\mu_2 = n - 1$, $\varphi_1^2 = \gamma^2$, $\varphi_2^2 = \varphi^2$, $\mu_u = \mu - 1$, $\mu_v = \nu - 1$, we obtain (3.5.3) from (3.3.5) and (2.4.2). Inequality (3.5.4) follows immediately from Lemma 3.4 and (2.4.2).

4. Inequalities for the case of X and Y stochastically comparable.

Throughout this section X will be assumed stochastically smaller, that is $F(s) \geqq G(s)$ or, in terms of the relative c.d.f.

$$t \leqq L(t), \qquad \text{for } 0 \leqq t \leqq 1. \tag{4.0.1}$$

According to (2.3) this implies

$$p \leqq \tfrac{1}{2}. \tag{4.0.2}$$

We introduce the abbreviations

$$A(L) = \int_0^1 [L(t) - t]^2\, dt, \tag{4.0.3}$$

$$B(L) = \int_0^1 L^2(t)\, dt = \varphi^2 + (1-p)^2, \tag{4.0.4}$$

$$C(L) = \int_0^1 t^2\, dL(t) = 1 - 2\int_0^1 tL(t)\, dt = \gamma^2 + p^2. \tag{4.0.5}$$

4.1. LEMMA. *Let $p \leqq \frac{1}{2}$ be given and let $L(t) \geqq t$ be such that $\int_0^1 L(t)\,dt = 1 - p$. Consider the family of functions*

$$(4.1.1)\qquad L_\tau(t) = \begin{cases} t, & 0 \leqq t < \tau \\ \tau + \sqrt{1-2p}, & \tau \leqq t < \tau + \sqrt{1-2p} \\ t, & \tau + \sqrt{1-2p} \leqq t \leqq 1 \end{cases}$$

defined for $0 \leqq \tau \leqq 1 - \sqrt{1-2p}$. For these functions we have

$$(4.1.2)\qquad \int_0^1 L_\tau(t)\,dt = 1 - p, \qquad 0 \leqq \tau \leqq 1 - \sqrt{1-2p},$$

$$(4.1.3.)\quad A(L) \leqq A(L_\tau) = \tfrac{1}{3}(1-2p)^{3/2}, \qquad 0 \leqq \tau \leqq 1 - \sqrt{1-2p},$$

$$(4.1.4)\qquad \tfrac{2}{3}(1-2p)^{3/2} + \tfrac{1}{3} = B(L_0) \leqq B(L) \leqq B(L_{1-\sqrt{1-2p}}) = 1 - 2p + \tfrac{1}{3} - \tfrac{1}{3}(1-2p)^{3/2},$$

$$(4.1.5)\qquad 2p - \tfrac{2}{3} + \tfrac{2}{3}(1-2p)^{3/2} = C(L_{1-\sqrt{1-2p}}) \leqq C(L) \leqq C(L_0) = \tfrac{1}{3} - \tfrac{1}{3}(1-2p)^{3/2}.$$

PROOF. Since a continuous $L(t) \geqq t$ can be uniformly approximated by a "saw-tooth" function, i.e., by a relative c.d.f. whose graph consists of a finite number of line-segments, either horizontal or on the line t (see Fig. 1), it will be sufficient to carry out the proof for such functions only.

Let us first consider an "isolated" tooth, such as K in Fig. 1, and translate it by $\Delta > 0$ to position K', thereby replacing L by L^*, say. It is clear that

$$1 - p = \int_0^1 L(t)\,dt = \int_0^1 L^*(t)\,dt, \qquad A(L) = A(L^*),$$

and

$$B(L^*) > B(L), \qquad C(L^*) < C(L).$$

Translating each isolated tooth as far as possible to the right we obtain a saw-tooth function L^{**} for which all teeth are adjacent and the last to the right ends with a horizontal line-segment with ordinate 1 (such as all teeth in Fig. 1, except K), and for which

$$\int_0^1 L^{**}(t)\,dt = \int_0^1 L(t)\,dt = 1 - p, \qquad A(L^{**}) = A(L),$$

$$B(L^{**}) > B(L), \qquad C(L^{**}) < C(L).$$

Now consider a pair of adjacent teeth, such as M and N in Fig. 1. If the vertices M, N of these teeth have the coordinates (t_1, u_1), $(t_2 u_2)$, we replace them by one tooth with the vertex $P(t_1', u_2)$ where

$$t_1' = u_2 - \sqrt{(u_2 - u_1)^2 - 2(t_2 - t_1)(u_2 - u_1)}.$$

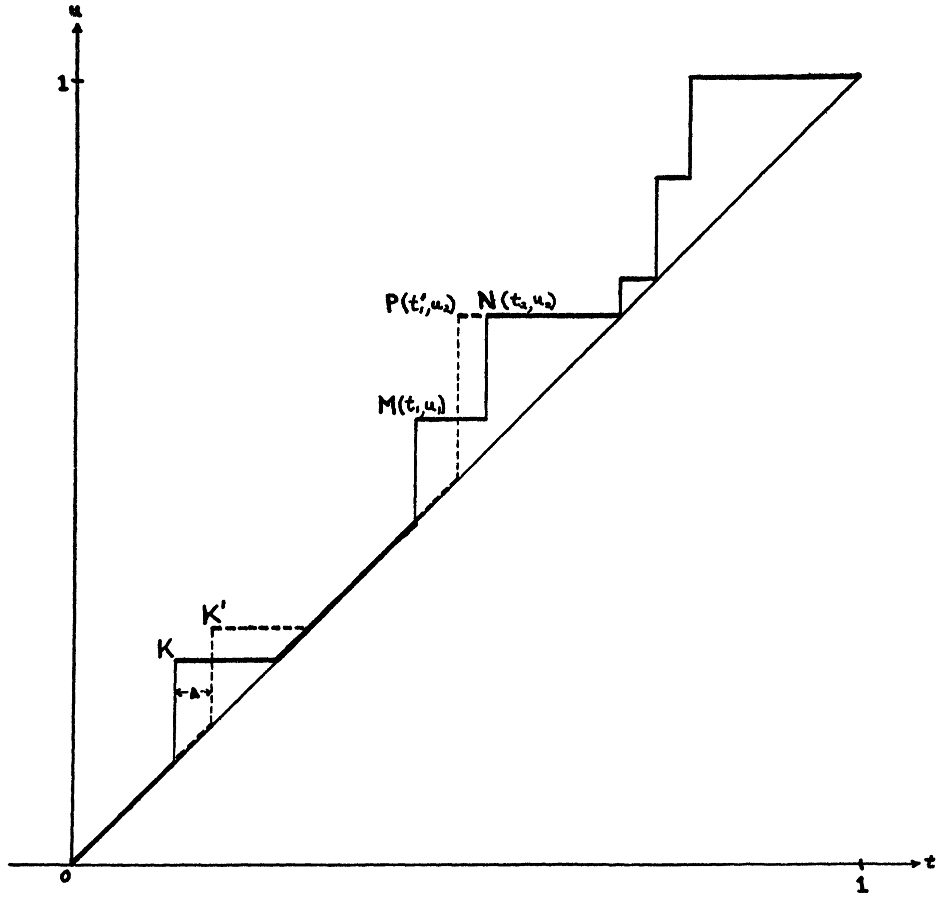

FIG. 1

Again it is clear that for the resulting L^{***} we have $\int_0^1 L^{***}(t)\,dt = \int_0^1 L^{**}(t)\,dt$, and one verifies by direct computation that the contribution of the interval (t_1, u_2) to the integrals A, and B increases as L^{**} is replaced by L^{***}, while the corresponding contribution to the integral C decreases. After a finite number of such steps, each of which merges a tooth with its neighbor to the right, we obtain $L_{1-\sqrt{1-2p}}$, which proves the inequalities involving $L_{1-\sqrt{1-2p}}$ in (4.1.4) and (4.1.5), while (4.1.3) follows from the observation that $A(L_\tau)$ takes the same value for each value of τ as for the value $\tau = 1 - \sqrt{1 - 2p}$. The inequalities involving L_0 are obtained by an analogous argument in which first all isolated saw-teeth are translated to the left as far as possible and then each tooth is, in succession, merged with its neighbor to the left.

4.2. THEOREM. *For p given and any relative* c.d.f. $L(t) \geqq t$ *with* $\int_0^1 L(t)\,dt = 1 - p$ *(implying $p \leqq \frac{1}{2}$), the variance of U has the upper bound*

$$\begin{aligned}\sigma^2(U) \leqq \mu\nu\{\nu[\tfrac{1}{3}(1 - (1 - 2p)^{3/2}) - p^2] + \mu[-\tfrac{2}{3}(1 - (1 - 2p)^{3/2}) \\ + 2p - p^2] + \tfrac{1}{3}[1 - (1 - 2p)^{3/2}] - p(1 - p)\}.\end{aligned} \tag{4.2}$$

Equality holds for $L = L_0$ *if* $n \geqq m$ *and for* $L = L_{1-\sqrt{1-2p}}$ *if* $n \leqq m$.

PROOF. If $n \geqq m$, we write (3.2.1) in the form

$$\sigma^2(U) = mn\{(m-1)A(L) + (n-m)C(L) - (n-m)p^2 + (m-1)[\tfrac{2}{3} - p^2 - (1-p)^2] + p(1-p)\}.$$

Setting $L = L_0$ in the right side we obtain the theorem from Lemma 4.1. A symmetrical argument, stressing $B(L)$ instead of $C(L)$ completes the proof for $n \leqq m$.

4.3 LEMMA. *Under the assumptions* (4.0.1) *and*

$$\frac{n-1}{m-1} \leqq 2p, \tag{4.3.1}$$

we have

$$\begin{aligned}(m-1)\varphi^2 + (n-1)\gamma^2 &\geqq \tfrac{1}{3}\{m + n - 2 + 2[(m-1)(m-n)(1-2p)^3]^{\frac{1}{2}}\} \\ &\quad - [(m-1)(1-p)^2 + (n-1)p^2]\end{aligned} \tag{4.3.2}$$

and this inequality can not be improved.

PROOF. If $0 \leqq \alpha < 1$ and $0 \leqq \beta \leqq 1 - \alpha$ then $0 \leqq [\beta/(1-\alpha)] \leqq 1$ and in view of (4.1) we have

$$\begin{aligned}\int_0^1 [L(t) - \alpha t - \beta]^2\, dt &\geqq \int_{\beta/(1-\alpha)}^1 [L(t) - \alpha t - \beta]^2\, dt \\ &\geqq \int_{\beta/(1-\alpha)}^1 (t - \alpha t - \beta)^2\, dt = \frac{(1-\alpha-\beta)^3}{3(1-\alpha)}.\end{aligned} \tag{4.3.3}$$

From this and (3.4.4.1) follows

$$\varphi^2 + \alpha\gamma^2 \geqq \alpha - \frac{\alpha^2}{3} - \alpha p^2 - (1-p)^2 + 2\beta(1-p) - \alpha\beta - \beta^2 + \frac{(1-\alpha-\beta)^3}{3(1-\alpha)}.$$

For fixed p and α, the right side is maximum for $\beta = \sqrt{(1-\alpha)(1-2p)}$, and this value satisfies the condition $0 \leqq \beta \leqq 1 - \alpha$ if and only if $\alpha \leqq 2p$. Consequently, for $\alpha \leqq 2p$, we have

$$\varphi^2 + \alpha\gamma^2 \geqq \tfrac{1}{3}\{1 + \alpha + 2(1-\alpha)^{\frac{1}{2}}(1-2p)^{\frac{3}{2}}\} - [\alpha p^2 + (1-p)^2].$$

Setting $\alpha = [(n-1)/(m-1)]$ which is $\leqq 2p$ by (4.3.1) we obtain (4.3.2). Equality in (4.3.3) is attained if and only if

$$L(t) = \begin{cases} \alpha t + \beta & \text{for } \ 0 < t \leqq \dfrac{\beta}{1-\alpha} \\[2ex] t & \text{for } \ \dfrac{\beta}{1-\alpha} < t \leqq 1, \end{cases}$$

so that, with $\beta = \sqrt{(1-\alpha)(1-2p)}$, $\alpha = [(n-1)/(m-1)]$, we obtain the function

$$L_5(t) = \begin{cases} \dfrac{n-1}{m-1}t + \sqrt{\left(1 - \dfrac{n-1}{m-1}\right)(1-2p)} & \\ \qquad\qquad \text{for} \quad 0 < t \leqq \sqrt{(1-2p) \Big/ \left(1 - \dfrac{n-1}{m-1}\right)} & \\ t \qquad\qquad \text{for} \quad \sqrt{(1-2p) \Big/ \left(1 - \dfrac{n-1}{m-1}\right)} < t \leqq 1. & \end{cases}$$

4.4. Lemma. *Under the assumptions* (4.0.1) *and*

$$\frac{m-1}{n-1} \leqq 2p, \tag{4.4.1}$$

we have

$$\begin{aligned} &(m-1)\varphi^2 + (n-1)\gamma^2 \\ &\qquad \geqq \tfrac{1}{3}\{m + n - 2 + 2[(n-1)(n-m)(1-2p)^3]^{\frac{1}{2}}\} \\ &\qquad\qquad - [(m-1)p^2 + (n-1)(1-p)^2], \end{aligned} \tag{4.4.2}$$

and this inequality can not be improved.

Proof. If $\alpha \geqq 1$ and $0 \leqq -\beta \leqq \alpha - 1$, then

$$0 \leqq \beta/(1-\alpha) \leqq (1-\beta)/\alpha \leqq 1$$

and in view of (4.0.1) we have

$$\begin{aligned} \int_0^1 [L(t) - \alpha t - \beta]^2\,dt &\geqq \int_0^{\frac{\beta}{1-\alpha}} + \int_{\frac{1-\beta}{\alpha}}^1 \geqq \int_0^{\frac{\beta}{1-\alpha}} (t - \alpha t - \beta)^2\,dt \\ &+ \int_{\frac{1-\beta}{\alpha}}^1 (\alpha t + \beta - 1)^2\,dt = \frac{1}{3}\left[\frac{\beta^3}{1-\alpha} + \frac{(\alpha+\beta-1)^3}{\alpha}\right]. \end{aligned} \tag{4.4.3}$$

From this and (3.4.4.1) follows

$$\begin{aligned} \varphi^2 + \alpha\gamma^2 \geqq \alpha - \frac{\alpha^2}{3} - \alpha p^2 - (1-p)^2 - \alpha\beta - \beta^2 + 2\beta(1-p) \\ + \frac{1}{3}\left[\frac{\beta^3}{1-\alpha} + \frac{(\alpha+\beta-1)^3}{\alpha}\right]. \end{aligned}$$

For fixed α, p, the right side is maximum for $\beta = 1 - \alpha + \sqrt{\alpha(\alpha-1)(1-2p)}$ and this satisfies the condition $0 \leqq -\beta \leqq \alpha - 1$ if and only if $(1/\alpha) \leqq 2p$. It follows that for $(1/\alpha) \leqq 2p$,

$$\varphi^2 + \alpha\gamma^2 \geqq \tfrac{1}{3}\{1 + \alpha + 2[\alpha(\alpha-1)(1-2p)^3]^{\frac{1}{2}}\} - [\alpha(1-p)^2 + p^2],$$

and for $\alpha = [(n - 1)/(m - 1)]$, this inequality yields (4.4.2). Equality in (4.4.3) holds if and only if

$$L(t) = \begin{cases} t & \text{for} \quad 0 < t \leqq \dfrac{\beta}{1-\alpha}, \\ \alpha t + \beta & \text{for} \quad \dfrac{\beta}{1-\alpha} < t \leqq \dfrac{1-\beta}{\alpha}, \\ 1 & \text{for} \quad \dfrac{1-\beta}{\alpha} < t \leqq 1, \end{cases}$$

so that for $\alpha = [(n - 1)/(m - 1)]$, $\beta = 1 - \alpha + \sqrt{\alpha(\alpha - 1)(1 - 2p)}$, we obtain the relative distribution function

$$L_6(t) = \begin{cases} t & \text{for} \quad 0 < t \leqq t_1, \\ \dfrac{n-1}{m-1}\, t + \dfrac{m-n}{m-1} + \dfrac{1}{m-1} \sqrt{(n-1)(n-m)(1-2p)} & \text{for} \quad t_1 < t \leqq t_2, \\ 1 & \text{for} \quad t_2 \leqq t \leqq 1, \end{cases}$$

where

$$t_1 = 1 - \sqrt{\frac{n-1}{n-m}(1-2p)}, \qquad t_2 = 1 - \sqrt{\frac{n-m}{n-1}(1-2p)}.$$

4.5. THEOREM. *Under the assumptions of Theorem 4.2, the variance of U has the lower bounds*

$$\begin{aligned} \sigma^2(U) \geqq mn\{\tfrac{1}{3}[m + n + 1 + 2\sqrt{(m-1)(m-n)(1-2p)^3}] \\ - [m(1-p)^2 + np^2 + p(1-p)]\} \quad \text{if} \quad \frac{n-1}{m-1} \leqq 2p, \end{aligned} \tag{4.5.1}$$

$$\begin{aligned} \sigma^2(U) \geqq mn\{\tfrac{4}{3}p\sqrt{2p(m-1)(n-1)} - (m+n-2)p^2 + p(1-p)\} \\ \text{if} \quad 2p < \frac{n-1}{m-1} \leqq \frac{1}{2p}, \end{aligned} \tag{4.5.2}$$

$$\begin{aligned} \sigma^2(U) \geqq mn\{\tfrac{1}{3}[m + n + 1 + 2\sqrt{(n-1)(n-m)(1-2p)^3}] \\ - [mp^2 + n(1-p)^2 + p(1-p)]\} \quad \text{if} \quad \frac{1}{2p} \leqq \frac{n-1}{m-1}. \end{aligned} \tag{4.5.3}$$

These lower bounds can not be improved.

PROOF. Inequality (4.5.1) follows from (2.4.2) and Lemma 4.3 with equality attained for $L_5(t)$, and (4.5.3) follows from (2.4.2) and Lemma 4.4 with equality holding for $L_6(t)$. Inequality (4.5.2) is the same as (3.5.4) which was proven for general relative c.d.f. $L(t)$, without assuming (4.0.1) and which holds whether $m \leqq n$ or $m > n$ since the right-hand side is symmetric in m, n. The lower bound

(4.5.2) cannot be improved even under assumption (4.0.1) of stochastic comparability, for $L_4(t)$ yields equality and satisfies (4.0.1).

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