MOMENTS OF THE ABSOLUTE DIFFERENCE AND ABSOLUTE DEVIATION OF DISCRETE DISTRIBUTIONS[1]

By S. K. Katti

Iowa State University of Science and Technology

1. Introduction. Johnson [3], Crow [1] and Ramasubban [5] have discussed the evaluation of the mean difference and the mean deviation for some positive integral valued discrete distributions. These are particular cases of a more general statistic which may be defined as

$$\Delta_r = E \mid X_1 - X_2 \mid^r, \tag{1a}$$

where X_1 and X_2 are two random variables with given distributions. Statistic (1a) will be referred to as the rth moment of the absolute difference of X_1 and X_2. In this paper, Δ_r is evaluated when X_1 and X_2 are independent and both have distributions—possibly different ones—within one of the following families of distributions: (i) Poisson, (ii) Pascal, and (iii) Binomial. The case when X_1 and X_2 are distributed as two independent Logarithmic variables, and the cases when X_1 and X_2 are independent and have distributions in two different families of distributions (chosen from the Poisson, Pascal, Binomial, and Logarithmic families), can be treated along similar lines, but the results are not given here in order to conserve space. Methods are also given to evaluate Δ_r when X_2 is a fixed constant and when X_1 is distributed as (i) a Poisson (ii) a Pascal (iii) a Binomial (iv) a Hypergeometric and (v) a Logarithmic random variable. In this special case, Δ_r will be called the rth moment of the absolute deviation of X_1 about X_2 and denoted by δ_r. I am investigating two sample tests, based on the sample analogues of the Δ_r's, that may be appropriate when the two samples are from two specified but different parametric populations.

2. An expression for the rth moment of the absolute difference $\mid X_1 - X_2 \mid$. Let X_1 and X_2 be two arbitrary independent positive integral valued random variables with probabilities $P_i^{(1)}$ and $P_i^{(2)}$ of obtaining $X_1 = i$ and $X_2 = i$ respectively. Then the rth moment Δ_r is given by

$$\begin{aligned} \Delta_r &= E \mid X_1 - X_2 \mid^r \\ &= \sum_{ik} k^r P\{X_2 - X_1 = k \mid X_1 = i\} P\{X_1 = i\} \\ &\quad + \sum_{ik} k^r P\{X_1 - X_2 = k \mid X_2 = i\} P\{X_2 = i\} \\ &= \sum_{ik} k^r P_i^{(1)} P_{i+k}^{(2)} + \sum_{ik} k^r P_i^{(2)} P_{i+k}^{(1)} \end{aligned} \tag{1}$$

where the summations are over $1, 2, 3, \cdots$.

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3. Some applications of equation (1).

(a) Moments of the absolute difference for two independent Poisson random variables. Let

$$P_i^{(1)} = e^{-\lambda_1}\lambda_1^i/i!, \qquad P_i^{(2)} = e^{-\lambda_2}\lambda_2^i/i!.$$

Then

$$\begin{aligned}(2)\qquad \Delta_r &= e^{-\lambda_1-\lambda_2}\left\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{k^r(\lambda_1\lambda_2)^i\lambda_2^k}{i!(i+k)!} + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{k^r(\lambda_1\lambda_2)^i\lambda_1^k}{i!(i+k)!}\right\}\\ &= e^{-\lambda_1-\lambda_2}(A_r + B_r) \quad \text{say.}\end{aligned}$$

In order to simplify A_r, write $\lambda_1\lambda_2 = \nu$. Then A_r can be rewritten in the form

$$\begin{aligned}(3)\qquad A_r &= \left(\lambda_2\frac{\partial}{\partial\lambda_2}\right)^r\left\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\nu^i\lambda_2^k}{i!(i+k)!}\right\}\\ &= \left(\lambda_2\frac{\partial}{\partial\lambda_2}\right)^r A_0 \quad \text{say,}\end{aligned}$$

where we denote the term in curly brackets by A_0. It is apparent that A_0 can be written in the form

$$(4)\qquad A_0 = \sum_{i=0}^{\infty}\frac{\nu^i}{(i!)^2}\,{}_1F_1(1; i+1; \lambda_2),$$

where ${}_1F_1(\alpha; \gamma; x)$ is a confluent hypergeometric function [2]. Operating on (4) by $\lambda_2\partial/\partial\lambda_2$ yields

$$(5)\qquad A_1 = \left(\lambda_2 - \frac{\nu}{\lambda_2}\right)A_0 + \nu e^{2\sqrt{\nu}}\,{}_1F_1(\tfrac{3}{2}; 3; -4\sqrt{\nu}) + \frac{\nu}{\lambda_2}e^{2\sqrt{\nu}}\,{}_1F_1(\tfrac{1}{2}; 1; -4\sqrt{\nu}).$$

Successive application s times of the operator $\lambda_2\partial/\partial\lambda_2$ leads to the recursion formula

$$(6)\qquad A_{s+1} = \sum_{i=0}^{s}\binom{s}{i}\left(\lambda_2 + (-1)^{i+1}\frac{\nu}{\lambda_2}\right)A_{s-i} + (-1)^s\frac{\nu}{\lambda_2}e^{2\sqrt{\nu}}\,{}_1F_1(\tfrac{1}{2}; 1; -4\sqrt{\nu}).$$

A_0 can be calculated from formula (4), using the tables given by Nath [4] to get the values of the confluent hypergeometric functions involved therein, and then A_r can be calculated by the repeated application of (5) and (6). Calculation of B_r follows along similar lines. Equation (2) can then be employed to calculate Δ_r.

For the particular case when $\lambda_1 = \lambda_2 = \lambda$, i.e. when X_1 and X_2 are Poisson variates with the same mean, Δ_1 reduces to

$$(7)\qquad \Delta_1 = 2\lambda\{{}_1F_1(\tfrac{3}{2}; 3; -4\lambda) + {}_1F_1(\tfrac{1}{2}; 1; -4\lambda)\}.$$

On using the facts that the Bessel function of the first kind $J_\nu(x)$ [2] and the modified Bessel function of the first kind $I_n(x)$ [5] are given by

$$J_\nu(x) = \sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{x}{2}\right)^{\nu+2i}}{i!(\nu+i)!} = \frac{\left(\frac{x}{2}\right)^{\nu}e^{-ix}}{\Gamma(\nu+1)}\,{}_1F_1(\tfrac{1}{2}+\nu; 1+2\nu; 2ix)$$

and

$$I_n(x) = \sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2i}}{i!(n+i)!},$$

we obtain

$$\Delta_1 = 2\lambda e^{-2\lambda}\{I_0(2\lambda) + I_1(2\lambda)\}, \tag{8}$$

which agrees with formula (2.18) of Ramasubban [5].

(b) Moments of the absolute difference for two Pascal or two Binomial random variables. First, let X_1 and X_2 be two independent Pascal random variables. Write

$$P_i^{(1)} = q_1^{k_1}\binom{k_1+i-1}{i}\left(\frac{p_1}{q_1}\right)^i, P_i^{(2)} = q_2^{k_2}\binom{k_2+i-1}{i}\left(\frac{p_2}{q_2}\right)^i.$$

From (1) (by changing k to j), we have

$$\begin{aligned}\Delta_r = q_1^{-k_1}q_2^{-k_2}\Bigg\{&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} j^r\binom{k_1+i-1}{i}\binom{k_2+i+j-1}{i+j}\left(\frac{p_1\,p_2}{q_1\,q_2}\right)^i\left(\frac{p_2}{q_2}\right)^j \\ &+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} j^r\binom{k_1+i+j-1}{i+j}\binom{k_2+i-1}{i}\left(\frac{p_1\,p_2}{q_1\,q_2}\right)^i\left(\frac{p_1}{q_1}\right)^j\Bigg\} \\ = q_1^{-k_1}q_2^{-k_2}&(A_r+B_r), \quad \text{say.}\end{aligned} \tag{9}$$

In order to simplify A_r, we now write $p_1p_2/q_1q_2 = \nu_1$ and $p_2/q_2 = \nu_2$. Consequently, we can write

$$\begin{aligned}A_r = \left(\nu_2\frac{\partial}{\partial\nu_2}\right)^r\Bigg\{&\sum_{i=0}^{\infty}\nu_1^i\binom{k_1+i-1}{i}\binom{k_2+i-1}{i} \\ &\cdot {}_2F_1(k_2+i, 1; i+1; \nu_1)\Bigg\}\end{aligned} \tag{10}$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function [2]. Denote the term in the curly bracket by A_0. A recurrence formula for calculating A_{s+1} is

$$\begin{aligned}A_{s+1} = \sum_{i=0}^{s}\binom{s}{i}A_i\left\{\left(\nu_2\frac{\partial}{\partial\nu_2}\right)^{s-j}\left[\frac{k_2}{1-\nu_2}+\frac{k_1\,\nu_1}{\nu_1-\nu_2}\right]\right\} \\ + \nu_1\,k_1\,{}_2F_1(k_2, k_1+1; 1; \nu_1)\left(\nu_2\frac{\partial}{\partial\nu_2}\right)^s\left(\frac{1}{\nu_2-\nu_1}\right).\end{aligned} \tag{11}$$

Since the value of A_0 can be computed to any degree of accuracy from its formula, the quantities we need to know before we can use (11) are

$$\left(\nu_2\frac{\partial}{\partial\nu_2}\right)^{s-i}\left[\frac{k_2}{1-\nu_2}+\frac{k_1\nu_1}{\nu_1-\nu_2}\right] \quad \text{and} \quad \left(\nu_2\frac{\partial}{\partial\nu_2}\right)^s\frac{1}{\nu_2-\nu_1} \qquad \text{for all } s.$$

Since the derivatives with respect to ν_2 of the functions involved are relatively simple, we will give here a method for expressing $(\nu_2\partial/\partial\nu_2)^s f$ in terms of $(\partial/\partial\nu_2)^s f$.

First, we observe that

$$\left(\nu_2 \frac{\partial}{\partial \nu_2}\right)^s f = \nu_2 \frac{\partial}{\partial \nu_2} f + \nu_2^2 \frac{\partial^2}{\partial \nu_2^2} f. \tag{12}$$

It follows that $(\nu_2 \partial/\partial \nu_2)^s f$ has the form

$$\sum_{i=1}^{s} a_i^s \nu_2^i \left(\frac{\partial}{\partial \nu_2}\right)^i f, \tag{13}$$

where $a_i^{(s)}$ are constants, not involving f. To evaluate $a_i^{(s)}$, we note that

$$\left(\nu_2 \frac{\partial}{\partial \nu_2}\right)^{s+1} f + \sum_{i=1}^{s+1} (i a_i^{(s)} + a_{i-1}^{(s)}) \nu_2^i \left(\frac{\partial}{\partial \nu_2}\right)^i f \equiv \sum_{i=1}^{s+1} a_i^{(s+1)} \nu_2^i \left(\frac{\partial}{\partial \nu_2}\right)^i f, \tag{14}$$

with the convention that $a_i^{(s)} = 0$ for $i < s$. This yields us the set of recurrence formulae

$$a_i^{(s+1)} = i a_i^{(s)} + a_{i-1}^{(s)}. \tag{15}$$

It is evident that $a_1^{(1)} = 1$. Hence (15) can be used successively to obtain the various values of $a_i^{(s)}$ and then (13) used to obtain $(\nu_2(\partial/\partial \nu_2))^s f$.

The calculation of B_r follows along similar lines. Δ_r can then be calculated from (9). For the particular case when X_1 and X_2 are Pascal random variables with the same parameters, i.e.

$$P(x = i) = q^{-k} \binom{k + i - 1}{i} \left(\frac{p}{q}\right)^i,$$

Δ_1 reduces to

$$\begin{aligned} \Delta_1 &= \left(\frac{2kp}{q}\right) q^{-2k} \\ &\cdot \left\{ {}_2F_1\left(k + 1, k + 1; 1; \frac{p^2}{q^2}\right) + (k + 1) \frac{p}{q} \, {}_2F_1\left(k + 2, k + 1; 2; \frac{p^2}{q^2}\right) \right\}. \end{aligned} \tag{16}$$

Upon using the relation (see [2], eqn. (36) pp. 113) that

$$\begin{aligned} &F(a, b; a - b + 1; Z) \\ &= (1 + \sqrt{Z})^{-2a} {}_2F_1(a, a - b + \tfrac{1}{2}; 2a - 2b + 1; 4\sqrt{Z}(1 + \sqrt{Z})^{-2}), \end{aligned} \tag{17}$$

we obtain

$$\Delta_1 = 2kpq \, {}_2F_1(k + 1, \tfrac{1}{2}; 2; -4pq) \tag{18}$$

which agrees with formula (2.12) of Ramasubban [5].

Suppose now that X_1 and X_2 are Binomial random variables with

$$P_i^{(1)} = \binom{n_1}{i} p_1^i q_1^{n_1 - i} \quad \text{and} \quad P_i^{(2)} = \binom{n_2}{i} p_2^i q_2^{n_2 - i}.$$

It is apparent that the formulae for this case can be obtained from those given for the Pascal case by changing the k's to $(-n)$'s, p's to $(-p)$'s and the quantities $\binom{k+i-1}{i}$ to $(-1)^i\binom{n}{i}$. Hence no separate discussion need be given here for this case.

4. A method to evaluate the moments of an absolute deviation. Let X be an arbitrary positive integral valued random variable and let P_i denote the probability of obtaining $X = i$. Then the moment generating function (m.g.f.) of $|X - m|$ where m is a fixed constant is given by

$$m(t) = Ee^{t|x-m|} = \sum_{i=[m]+1}^{\infty} e^{t(i-m)}P_i + \sum_{i=0}^{[m]} e^{-t(i-m)}P_i. \tag{19}$$

Here, $[m]$ is the largest integer, less than or equal to m and the second term is considered zero when $m < 0$. Since, for $m < 0$, moments of $|X - m|$ are the same as those of $(X - m)$, we will consider only the case when $m > 0$. Equation (19) can now be simplified to

$$\begin{aligned} m(t) &= e^{-mt}\sum_{i=[m]+1}^{\infty} e^{it}P_i - e^{mt}\sum_{i=[m]+1}^{\infty} e^{-it}P_i + e^{mt}\sum_{i=0}^{\infty} e^{-it}P_i \\ &= \psi(t) - \psi(-t) + e^{mt}M(-t) \quad \text{say,} \end{aligned} \tag{20}$$

where $M(t)$ is the m.g.f. of X and wherein $\psi(t)$, the first term in (20) may be referred to as the incomplete m.g.f. of $X - m$. Since the even moments of $|X - m|$ are the same as those of $X - m$ and hence obtainable by using the regular statistical techniques, we will consider only the moments of odd order, say δ_{2r+1}. On differentiating (20) $(2r + 1)$ times and setting $t = 0$, we have

$$\delta_{2r+1} = 2\psi^{(2r+1)}(0) - E(X - m)^{2r+1}. \tag{21}$$

As remarked above, calculating of the second term poses no new problem. Our task therefore reduces to that of obtaining $\psi^{(2r+1)}(0)$.

5. To obtain the value of $\psi^{(2r+1)}(0)$ for some particular cases.

(a) The Poisson distribution: Let $P_i = e^{-\lambda}\lambda^i/i!$. Then, from the definition of $\psi(t)$

$$\psi(t) = \sum_{i=[m]+1}^{\infty} e^{(i-m)t}\frac{e^{-\lambda}\lambda^i}{i!}, \tag{22}$$

which can be written in the form $KG(0, t)$, where

$$K = \frac{e^{-\lambda}\lambda^{[m]+1}}{([m] + 1)!}, \tag{23}$$

and

$$G(0, t) = \exp\{t([m] - m + 1)\}\,{}_1F_1(1; [m] + 2; \lambda e^t). \tag{24}$$

In order to obtain a convenient method to evaluate $\psi^{(2r+1)}(0)$, let us define

(25) $G(\alpha, t) = \exp\{t([m] - m + \alpha + 1)\}{}_1F_1(\alpha + 1; \alpha + [m] + 2; \lambda e^t)$.

Differentiation of (25) yields

$$G^{(1)}(\alpha, t) = ([m] - m + \alpha + 1)G(\alpha, t) + \frac{\lambda(\alpha + 1)}{\alpha + [m] + 2} G(\alpha + 1, t). \tag{26}$$

Successive differentiation of (34) s times at $t = 0$ gives the recurrence formula

$$\begin{aligned} G^{(s+1)}(\alpha, 0) &= ([m] - m + \alpha + 1)G^{(s)}(\alpha, 0) \\ &\quad + \frac{\lambda(\alpha + 1)}{\alpha + [m] + 2} G^{(s)}(\alpha + 1, 0). \end{aligned} \tag{27}$$

Since $G(\alpha, 0) = {}_1F_1(\alpha + 1; \alpha + [m] + 2; \lambda)$ is the confluent hypergeometric function, we can obtain $G(\alpha, 0)$ for $\alpha = 0, 1, \cdots, 2r + 1$ by referring to the tables of Nath [4]. $G^{(2r+1)}(0, 0)$ can then be calculated by the repeated application of (27). Calculation of $\psi^{(2r+1)}(0, 0)$ immediately follows, since as can be easily seen, $\psi^{(2r+1)}(0, 0) = KG^{(2r+1)}(0, 0)$.

(b) The Pascal, the Binomial and the Hypergeometric distributions: For the Pascal distribution,

$$P_i = q^{-k} \binom{k + i - 1}{i} \left(\frac{p}{q}\right)^i.$$

By proceeding along lines similar to those in (a), we can write $\psi(t)$ in the form $YH(0, t)$ where

$$Y = q^{-k} \binom{k + [m]}{[m] + 1} \left(\frac{p}{q}\right)^{[m]+1} \tag{28}$$

and

$$\begin{aligned} H(\alpha, t) &= \exp\{t[m] + 1 - m + \alpha)\}{}_2F_1 \\ &\quad \cdot \left(k + [m] + \alpha + 1, 1 + \alpha; [m] + \alpha + 2; \frac{p}{q} e^t\right). \end{aligned} \tag{29}$$

For the Hypergeometric distribution

$$P_i = \binom{Np}{i} \binom{Nq}{n - i} \Big/ \binom{N}{n} \tag{30}$$

and

$$\psi(t) = ZF(0, t) \tag{31}$$

where

$$Z = \binom{Np}{[m]+1} \binom{Nq}{n-[m]-1} \Big/ \binom{N}{n}, \tag{32}$$

$$\begin{aligned} F(\alpha, t) &= \exp\{t([m] - m + 1 + \alpha)\}{}_3F_2(-n + [m] + 1 + \alpha; \\ &- Np + [m] + 1 + \alpha, 1 + \alpha; [m] + 2 + \alpha, Nq - n + [m] + 2 + \alpha; e^t), \end{aligned} \tag{33}$$

and ${}_3F_2(\alpha, \beta, \gamma; \delta, \epsilon; Z)$ is a generalized hypergeometric function (cf. [2]). The recurrence formulae for obtaining $H^{(2r+1)}(0, 0)$ and $F^{(2r+1)}(0, 0)$ are

$$
\begin{aligned}
(34)\qquad H^{(s+1)}(\alpha, 0) &= (\alpha - m + [m] + 1)H^{(s)}(\alpha, 0) \\
&\qquad + \frac{p}{q}\frac{\alpha(k + 1 + [m] + \alpha)}{\alpha + [m] + 2} H^{(s)}(\alpha + 1, 0)
\end{aligned}
$$

and

$$
\begin{aligned}
(35)\qquad F^{(s+1)}(\alpha, 0) &= \frac{(-n + [m] + 1 + \alpha)}{([m] + 2 + \alpha)} \frac{(-Np + [m] + 1 + \alpha)(1 + \alpha)}{(Nq + n + [m] + 2 + \alpha)} \\
&\qquad \cdot F^{(s)}(\alpha + 1, 0) + ([m] + \alpha - m)F^{(s)}(\alpha, 0)
\end{aligned}
$$

respectively. Calculation of $\psi^{(2r+1)}(0, 0)$ for the two cases follows from its relationship with $H^{(2r+1)}(0, 0)$ and $F^{(2r+1)}(0, 0)$ which can be computed by the successive application of (34) and (35).

The formulae for evaluating $\psi^{(2r+1)}(0)$ in the case of the Binomial distribution can be obtained from those in the case of the Pascal distribution by making the changes suggested in 3(b).

(c) The Logarithmic distribution: For the Logarithmic distribution, $P_i = \alpha\tau^i/i$ and $-\alpha \log (1 - \tau) = 1$. Hence

$$
(36)\qquad \phi(t) = e^{mt}\psi(t) = \sum_{i=[m]+1}^{\infty} \frac{\alpha e^{it}\tau^i}{i}.
$$

In order to obtain a convenient method to compute $\psi^{(2r+1)}(0)$, we first observe that

$$
(37)\qquad \phi^{(1)}(t)(1 - \tau e^t) = \alpha\tau^{([m]+1)}e^{t([m]+1)}.
$$

This leads to the recurrence relation

$$
(38)\qquad \phi^{(s+1)}(0) = \frac{1}{1 - \tau}\,\alpha\tau^{([m]+1)}([m] + 1)^s + \tau \sum_{i=0}^{s-1} \binom{s}{i} \phi^{(i)}(0).
$$

After having calculated $\phi^{(s)}(0)$ for $s = 1, 2, \cdots, 2r + 1$ by using (38), $\psi^{(2r+1)}(0)$ can be calculated by using the recurrence formula

$$
\psi^{(2r+1)}(0) = \sum_{i=0}^{2r+1} \binom{2r + 1}{i} \phi^{(i)}(0)(-m)^{2r+1-i},
$$

which can be easily derived from (36).

6. Conclusion. The general expressions given in sections (2) and (4) can be employed to obtain methods for finding the moments of the absolute difference and absolute deviation for some well known distributions. It was shown in section (3) that the formulae for Δ_r involve as particular cases, the results obtained by Ramasubban [5]. It can be easily shown that the formulae in section (5) also lead to his results when we set $r = 0$. This has been left out of the discussion for brevity.

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