A MATRIX SUBSTITUTION METHOD OF CONSTRUCTING PARTIALLY BALANCED DESIGNS[1]

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1. Introduction and summary. Vartak [6] has considered the construction of experimental designs with the help of Kronecker products of matrices. The method is equivalent to the replacement of two elements, 0 and 1, by two matrices. A generalisation of the above idea is given by the author [4], using only the incidence matrices of balanced incomplete block (BIB) designs for substitution. In the present paper the same idea is extended to the case where substitution is by the incidence matrices of partially balanced incomplete block (PBIB) designs and factorial experiments. In Sections 2 and 3 some ideas regarding canonical vectors and PBIB designs are introduced. Section 4 deals with associable designs and their properties. In Section 5 balanced matrices are defined and in Section 6 a method is given for constructing designs by substituting for the elements of a balanced matrix, the incidence matrices of associable designs. The application of this method to the construction of factorial experiments is considered in Section 7.

2. A canonical matrix. Let $\mathbf{N} = [n_{ij}]$ be the incidence matrix of a design, where n_{ij} is the number of times the ith treatment occurs in the jth block. Let the ith treatment be replicated r_i times and the jth block have k_j plots. The $\mathbf{C}$-matrix of the design is defined by

$$\mathbf{C} = \text{diag}\,(r_1, r_2, \cdots, r_v) - \mathbf{N}\,\text{diag}\,(k_1, k_2, \cdots, k_b)\mathbf{N}', \tag{2.1}$$

where diag $(a_1, a_2, \cdots, a_v)$ stands for a diagonal matrix with diagonal elements equal to $a_1, a_2, \cdots, a_v$ respectively.

If $\mathbf{l}$ is a vector such that $\mathbf{l}'\mathbf{l} = 1$ and $\mathbf{C}\mathbf{l} = a\mathbf{l}$, then the vector $\mathbf{l}$ is called a canonical vector of the design. If $\mathbf{l}_1, \mathbf{l}_2, \cdots, \mathbf{l}_v$ form a set of v mutually orthogonal canonical vectors, then the $v \times v$ matrix $\mathbf{L}$ whose ith column is $\mathbf{l}_i$ will be called a canonical matrix of the design.

The importance of a canonical matrix is quite obvious, since knowledge of it enables one to analyse even the most complicated design. For a design in which $r_1 = r_2 = \cdots = r_v$ and $k_1 = k_2 = \cdots = k_b$, the same canonical matrix $\mathbf{L}$ reduces both $\mathbf{L}'\mathbf{C}\mathbf{L}$ and $\mathbf{L}'\mathbf{N}\mathbf{N}'\mathbf{L}$ to the diagonal form. Hence the properties of a canonical matrix of such a design can be studied with reference to the matrix $\mathbf{N}\mathbf{N}'$.

Received December 10, 1958; revised April 27, 1959.

[1] This work was supported by a Research Training Scholarship of the Government of India.

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In this paper we shall consider only those designs for which $n_{ij} = 1$ or 0, $r_1 = r_2 = \cdots r_v = r$ and $k_1 = k_2 = \cdots = k_b = k$. Above conditions are satisfied in most of the designs used in practice.

The following matrix theorem (Thrall and Tornheim ([5], p. 189) will be useful in later sections.

THEOREM 2.1. *Let* $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_t$ *be a set of real symmetric matrices such that every pair commute. Then there exists an orthogonal matrix* $\mathbf{L}$ *such that* $\mathbf{L}'\mathbf{A}_i\mathbf{L} = \mathbf{D}_i$, *where each* $\mathbf{D}_i$ *is diagonal.*

3. Canonical matrix of PBIB designs. A PBIB design with m associate classes has been defined by Bose and Shimamoto [1] substantially as follows: A PBIB design with m associate classes $m \geqq 1$ is an arrangement of v treatments in b blocks of k plots each such that

(i) Each of the v treatments is replicated exactly r times and no treatment appears more than once in a block.

(ii) There exists a relationship of association between every pair of the treatments satisfying the following conditions:

(a) Any two treatments are either first, second, $\cdots$, or mth associates.

(b) Each treatment has exactly n_i ith associates ($i = 1, 2, \cdots, m$).

(c) Given any two treatments which are ith associates, the number of treatments that are both jth associates of the first and kth associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also $p^i_{jk} = p^i_{kj}$.

(iii) Two treatments which are ith associates occur together in exactly λ_i blocks.

Further we shall define each treatment to be its own 0th associate and 0th associate of no other treatment. Consistently we define

$$\lambda_0 = r, n_0 = 1, \qquad p^0_{st} = \delta_{st}n_s, \qquad p^t_{0s} = p^t_{s0} = \delta_{st}, \tag{3.1}$$

where δ_{ij} is the Kronecker delta which is defined for all pairs of natural numbers i, j as $\delta_{ij} = 1$, if $i = j$; and $\delta_{ij} = 0$, if $i \neq j$.

Each of the associate classes of a PBIB design can define the corresponding association matrix $\mathbf{B}_t = [B^t_{ij}]$ ($t = 1, 2, \cdots, m$), where $B^t_{ij} = 1$, if the ith and jth treatments are the tth associates and $B^t_{ij} = 0$ otherwise. Now, from the definition, it can be shown that for a PBIB design with incidence matrix $\mathbf{N}$,

$$\mathbf{NN}' = \sum \lambda_t \mathbf{B}_t. \tag{3.2}$$

It should be noted that the results of this and the next two sections lean almost entirely on the fact that $\mathbf{NN}' = \sum \lambda \mathbf{B}$, or on its canonical equivalents, such as $\mathbf{N}_1\mathbf{N}_2' = \sum \mu \mathbf{B}$ of Theorem 4.1.

For the sake of brevity, we shall often denote the design by incidence matrix, say $\mathbf{N}_c$, and its parameters by $v(c)$, $b(c)$, $r(c)$, $k(c)$, $m(c)$, $\lambda_i(c)$, $n_i(c)$, $p^k_{ij}(c)$, and the association matrices of the design by $\mathbf{B}_t(c)$ $t = 0, 1, \cdots, m(c)$. In two PBIB designs $\mathbf{N}_1$ and $\mathbf{N}_2$, if $v(1) = v(2)$, $m(1) = m(2)$ and $\mathbf{B}_t(1) = \mathbf{B}_t(2)$,

$\{t = 0, 1, \cdots, m(1)\}$; then the ith and jth treatments of $\mathbf{N}_1$ are pth associates if and only if the ith and jth treatments of N_2 and pth associates. Hence $n_i(1) = n_i(2)$ and $p^i_{jk}(1) = p^i_{jk}(2)$. Consequently, it follows that the equality of association matrices implies the equality of the secondary parameters n_i and p^i_{jk} but the converse is not true in general. This point is important, since we shall be concerned with the equivality of association matrices in Theorems 3.1, 4.1 and Definition 5.2.

The following four designs with given parameters and a suitable and appropriate association scheme can be considered as PBIB designs. (The parameters n_i and p^i_{jk} naturally depend upon the association scheme.)

(a) A null design with the incidence matrix $\mathbf{O}(v, b)$(a $v \times b$ matrix with all the elements equal to zero). Parameters: $v, b, r = k = \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

(b) A randomized block design with the incidence matrix $\mathbf{E}(v, b)$(a $v \times b$ matrix with all the elements equal to unity). Parameters: v, b, $r = b$, $k = v$, $\lambda_1 = \lambda_2 = \cdots = \lambda_m = b$.

(c) A BIB design of v^* treatments, each replicated r^* times in b^* blocks of k^* plots each, such that each pair of treatments occurs together in exactly λ^* blocks. Parameters: $v = v^*$, $b = b^*$, $k = k^*$, $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda^*$.

(d) An identity design with the incidence matrix $\sigma(v)$(a $v \times v$ Identity matrix). Parameters: $v = b$, $r = k = 1$, $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

THEOREM 3.1. *If there are s* PBIB *designs* $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_s$ *such that* $v(1) = v(2) = \cdots = v(s) = v$, $m(1) = \cdots = m(s) = m$ *and* $\mathbf{B}_t(1) = \mathbf{B}_t(2) = \cdots = \mathbf{B}_t(s) = \mathbf{B}_t$ *for* $t = 1, 2, \cdots, m$, *then there exists an orthogonal matrix* $\mathbf{L}$, *which is a canonical matrix for each of the s designs.*

PROOF: From the definition of a PBIB design it can be shown that

$$\mathbf{B}_t\mathbf{B}_x = \sum_{i=0}^{m} p^i_{tx}\mathbf{B}_i . \tag{3.3}$$

Since $p^i_{jk} = p^i_{kj}$, it follows that

$$\mathbf{B}_t\mathbf{B}_x = \mathbf{B}_x\mathbf{B}_t , \tag{3.4}$$

Hence by Theorem 2.1, there exists an orthogonal matrix L such that $\mathbf{L}'\mathbf{B}_i\mathbf{L}$ is diagonal for $i = 0, 1, \cdots, m$. Since

$$\mathbf{N}_c\mathbf{N}_c' = \sum_{i=0}^{m} \lambda_i(c)\mathbf{B}_i , \tag{3.5}$$

it follows that $\mathbf{L}'\mathbf{N}_c\mathbf{N}_c'\mathbf{L}$ is diagonal for all $c = 1, 2, \cdots, s$. Hence $\mathbf{L}$ is a canonical matrix for each of the s designs.

4. Associable designs.

DEFINITION 4.1. The s designs $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_s$, each in v treatments and b blocks, will be called associable designs, if there exists an orthogonal matrix $\mathbf{L}$, such that $\mathbf{L}'\mathbf{N}_i\mathbf{N}_j'\mathbf{L}$ is diagonal for all $i, j = 1, 2, \cdots, s$. The matrix $\mathbf{L}$ will be called a canonical matrix of association.

LEMMA 4.1. *Any design* $\mathbf{N}$ *is associable with itself* ($\mathbf{N}$) *and its complementary design* $\{\mathbf{E}(v, b) - \mathbf{N}\}$.

LEMMA 4.2. *Any design is associable with a null design, or a randomised block design, provided the numbers of treatments and blocks are the same for the different designs.*

LEMMA 4.3. *The identity design is associable with any design whose incidence matrix is a symmetric* $v \times v$ *matrix.*

THEOREM 4.1. *If two* PBIB *designs* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *are such that*

(i) $v(1) = v(2) = v$; $b(1) = b(2) = b$; $m(1) = m(2) = m$; $\mathbf{B}_t(1) = \mathbf{B}_t(2) = \mathbf{B}_t$, $t = 1, 2, \cdots, m$;

(ii) *and if the* b *'double blocks' formed by amalgamating* jth *block of* $\mathbf{N}_1$ *with the* jth *block of* $\mathbf{N}_2$ *are such that a treatment* i *of* $\mathbf{N}_1$ *and a treatment* j *of* $\mathbf{N}_2$ *occur together in exactly* $\mu_p(1, 2)$ *double blocks if and only if the* ith *and* jth *treatments are the* pth *associates in either of the designs;*
then $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *are associable.*

PROOF. Let $\mathbf{N}_1\mathbf{N}_2' = [m_{ij}]$. Then $m_{ij} = \sum n_{ik}(1) \cdot n_{jk}(2) =$ the number of double blocks in which the treatment i of $\mathbf{N}_1$ and the treatment of j of $\mathbf{N}_2$ occur together $= \mu_p(1, 2)$, but $\mu_p(1, 2)$ also appears in the ith row and jth column of $\sum_p \mu_p(1, 2)\ \mathbf{B}_p$. Hence $\mathbf{N}_1\mathbf{N}_2' = \sum_p \mu_p(1, 2)\mathbf{B}_p$, and similarly for $\mathbf{N}_2\mathbf{N}_1'$, whence the result on applying an argument similar to that leading from (3.5) to the conclusion of Theorem 3.1.

It should be noted that the conditions of Theorem 4.1 are not necessary, but in some of the later results we shall assume that these sufficient conditions are satisfied and then the parameters $\mu_p(1, 2)$ will be called the parameters of association. Further when a PBIB is associable with itself, its complementary design, a null design, or a randomised block design, the sufficient conditions given in Theorem 4.1 are satisfied.

5. Balanced matrices.

DEFINITION 5.1. If there exist $s(u \times w)$ incidence matrices $\mathbf{N}_1^*, \mathbf{N}_2^*, \cdots, \mathbf{N}_s^*$, such that $\sum_{i=1}^s \mathbf{N}_1^* = \mathbf{E}(u, w)$, and if there exists an orthogonal matrix $\mathbf{L}^*$ such that $\mathbf{L}^{*\prime}(\mathbf{N}_i^*\mathbf{N}_j^{*\prime} + \mathbf{N}_j^*\mathbf{N}_i^{*\prime})\mathbf{L}^*$ is diagonal for all $i, j = 1, 2, \cdots, s$, then the matrix $\mathbf{A} = \sum_1^s i\mathbf{N}_i$ will be called a canonically balanced matrix in s integers $1, 2, \cdots, s$.

There is great resemblance between the conditions imposed on $\mathbf{N}_i$ in Definition 4.1 and $\mathbf{N}_i^*$ in Definition 5.1. The condition, that all $\mathbf{L}^{*\prime}(\mathbf{N}_i^*\mathbf{N}_j^{*\prime} + \mathbf{N}_j^*\mathbf{N}_i^{*\prime})\mathbf{L}^*$ are diagonal, implies that all the designs with incidence matrices $\mathbf{N}_i^*$ and $\mathbf{N}_i^* + \mathbf{N}_j^*$ have the same canonical matrix $\mathbf{L}^*$. On the other hand, the condition that all $\mathbf{L}'(\mathbf{N}_i\mathbf{N}_j')\mathbf{L}$ are diagonal is slightly stronger and implies that not only all the designs with incidence matrices $\mathbf{N}_i$ and $\mathbf{N}_i + \mathbf{N}_j$ have the same canonical matrix $\mathbf{L}$ but also each $\mathbf{N}_i\mathbf{N}_j'$ is symmetric or $\mathbf{N}_i\mathbf{N}_j' = \mathbf{N}_j\mathbf{N}_i'$.

DEFINITION 5.2. If there exist s PBIB designs with $s(u \times w)$ incidence matrices $\mathbf{N}_1^*, \mathbf{N}_2^*, \cdots, \mathbf{N}_s^*$ such that $\sum_{i=1}^s \mathbf{N}_i^* = \mathbf{E}(u, w)$ and such that the designs $\mathbf{N}_i^*$ and $\mathbf{N}_i^* + \mathbf{N}_j^*(i > j = 1, 2, \cdots, s)$ all have the same association matrices B_t, $t = 1, 2, \cdots, m$, then the matrix $\mathbf{A} = \sum_{i=1}^{s} i\mathbf{N}_i^*$ will be called a partially balanced matrix.

LEMMA 5.1. *A partially balanced matrix is also a canonically balanced matrix.*

PROOF. The condition $\sum \mathbf{N}_i^* = \mathbf{E}(u, w)$ is satisfied in both Definitions 5.1 and 5.2. Now, if $\mathbf{N}_i^*$ and $\mathbf{N}_i^* + \mathbf{N}_j^*$ are PBIB designs, then from (3.2), we have

$$\mathbf{N}_i^*\mathbf{N}_i^{*\prime} = \sum_{t=0}^{m} \lambda_t^*(i)\mathbf{B}_t \tag{5.1}$$

and

$$(\mathbf{N}_i^* + \mathbf{N}_j^*)(\mathbf{N}_i^* + \mathbf{N}_j^*)' = \sum_{t=0}^{m} \mu_t^*(ij)\mathbf{B}_t\,. \tag{5.2}$$

Hence

$$\mathbf{N}_i^{*\prime}\mathbf{N}_j^* + \mathbf{N}_j^*\mathbf{N}_i^{*\prime} = \sum_{t=0}^{m} \{\mu_t^*(ij) - \lambda_t^*(i) - \lambda_t^*(j)\}\mathbf{B}_t\,, \tag{5.3}$$

whence the result on applying an argument similar to that leading from (3.5) to the conclusion of Theorem 3.1.

LEMMA 5.2. *If s_2 integers are divided in s_1 groups each group containing at least one integer, and if all the integers of a group are replaced by an indentical integer, then a canonically balanced matrix in s_2 integers will be reduced to a canonically balanced matrix in s_1 integers. A similar result holds also for a partially balanced matrix.*

DEFINITION 5.3. Let $\mathbf{A} = [a_{ij}](i = 1, 2, \cdots, u; j = 1, 2, \cdots, w)$ be a matrix whose elements a_{ij} take any one of the s values $1, 2, \cdots, s$. The matrix A will be called a partially balanced matrix, if it satisfies the following conditions:

(a) The number of times the integer c occurs in a row is the same for all the rows and is equal to $\alpha(c)$, say.

(b) The number of times the integer c occurs in any column is the same for all the columns and is equal to $\beta(c)$, say.

(c) The rows have an association scheme similar to the treatments of a PBIB design with parameters n_i^* and $p_{ij}^{k*}(i, j, k = 0, 1, \cdots, h)$. The number of times the combination of integers $\begin{bmatrix} c \\ d \end{bmatrix}$ and $\begin{bmatrix} d \\ c \end{bmatrix}$ occur in any pair of rows, which are the ith associates, is the name for all the pairs of rows, which are the ith associates, and is equal to $\gamma_i(c, d)$. (The combinations $\begin{bmatrix} c \\ d \end{bmatrix}$ and $\begin{bmatrix} d \\ c \end{bmatrix}$ are considered to be identical.)

The pair of integers $\begin{bmatrix} c \\ d \end{bmatrix}$ or $\begin{bmatrix} d \\ c \end{bmatrix}$ will be used in Theorem 6.2 to "mesh" so to speak with the "double block" notion of Theorem 4.1; since the substitution of integers c and d with matrices N_c and N_d will form double blocks of the form $\begin{bmatrix}\mathbf{N}_c\\ \mathbf{N}_d\end{bmatrix}$ or $\begin{bmatrix}\mathbf{N}_d\\ \mathbf{N}_c\end{bmatrix}$.

The Definitions 5.2 and 5.3 are equivalent. This follows, since, starting with Definitions 5.3, if we replace the integer c of the matrix $\mathbf{A}$ by 1 and the remaining $s - 1$ integers by 0, and if we call the resultant matrix $\mathbf{N}_c^*(c = 1, 2, \cdots, s)$, then from Definition 5.3, it can be shown that $\mathbf{N}_c^*$ and $\mathbf{N}_c^* + \mathbf{N}_d^*$ are PBIB designs with the same association scheme.

6. Construction of designs. The operator '$\times$' will denote the Kronecker product of matrices defined by

$$(6.1) \qquad \mathbf{A} \times \mathbf{B} = [a_{ij}] \times \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1w}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2w}\mathbf{B} \\ \vdots & & & \vdots \\ a_{u1}\mathbf{B} & a_{u2}\mathbf{B} & \cdots & a_{uw}\mathbf{B} \end{bmatrix}.$$

THEOREM 6.1. *If there exists a canonically balanced matrix* $\mathbf{A}$ *in* s *integers* $1, 2, \cdots, s$ *given by* $\sum_{i=1}^{s} i\mathbf{N}_i^*$ *with the corresponding orthogonal matrix* $\mathbf{L}^*$, *if there exist* s *mutually associable designs with incidence matrices* $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_s$ *with the canonical matrix of association equal to* $\mathbf{L}$, *and if the integer* c *in* $\mathbf{A}$ *is replaced by the matrix* $\mathbf{N}_c(c = 1, 2, \cdots, s)$, *then the matrix* $\mathbf{A}$ *will be converted into an incidence matrix* $\mathbf{N}$ *of a design whose canonical matrix is* $\mathbf{L}^* \times \mathbf{L}$.

PROOF. From the method of construction, it follows that

$$(6.2) \qquad \mathbf{N} = \sum_{i=1}^{s} \mathbf{N}_i^* \times \mathbf{N}_i.$$

Since $\mathbf{N}_i\mathbf{N}_j' = \mathbf{N}_j\mathbf{N}_i'$, $\mathbf{NN}'$ can be expressed as

$$(6.3) \qquad \mathbf{NN}' = \sum_{i=1}^{s} (\mathbf{N}_i^*\mathbf{N}_i^{*\prime}) \times \mathbf{N}_i\mathbf{N}_i' + \sum_{i>j=1}^{s} (\mathbf{N}_i^*\mathbf{N}_j^{*\prime} + \mathbf{N}_j^*\mathbf{N}_i^{*\prime}) \times \mathbf{N}_i\mathbf{N}_j'.$$

Now each of the terms on the right hand side of the equation (6.3) will be reduced to the diagonal form by the orthogonal transformation $(\mathbf{L}^* \times \mathbf{L})$ by virtue of Definitions 4.1 and 5.1. Hence the matrix $(\mathbf{L}^* \times \mathbf{L})'\mathbf{NN}'(\mathbf{L}^* \times \mathbf{L})$ is diagonal. This proves the theorem.

THEOREM 6.2. *Let there be* s PBIB *designs with incidence matrices* $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_s$. *Now let the parameters of the* cth *design be* v', b', $r(c)$, $k(c)$, $\lambda_1(c)$, n_i', *and* $p_{ij}^{k\prime}$ $(i, j, k = 0, 1, \cdots, m)$. *Let the* cth *design be associable with the* dth *design, satisfying the sufficient conditions given in Theorem* 4.1, *with parameters of association equal to* $\mu_i(c, d)(i = 0, 1, \cdots, m;\ c, d = 1, 2, \cdots, s)$. *Let there be a* $u \times w$ *partially balanced matrix* $\mathbf{A}$ *in* s *integers with parameters* $\alpha(c)$. $\beta(c)$, n_i^*, p_{ij}^{k*}, $\gamma_i(c, d)(c, d = 1, 2, \cdots, s;\ i, j, k = 0, 1, \cdots, h)$ *as given in Definition* 5.3. *Now, if we replace the integer* s *in the matrix* $\mathbf{A}$ *by the matrix* $\mathbf{N}_c(c = 1, 2, \cdots, s)$, *then the matrix* $\mathbf{A}$ *will be converted into an incidence matrix, say* $\mathbf{N}$, *of a* PBIB *with* $(h + 1)(m + 1) - 1$ *non-zero associate classes and the following parameters*:

$$v = uv',$$
$$b = wb',$$
$$r = \sum_{c=1}^{s} \gamma(c) r(c), \tag{6.4}$$
$$k = \sum_{c=1}^{s} \beta(c) k(c).$$

Denoting associate classes by two subscripts $(ij)(i = 0, 1, \cdots, h; j = 0, 1, \cdots, m)$, *the other parameters are given by*

$$n_{ij} = n_i^* n_j',$$
$$p_{ik:ut}^{ij} = p_{lu}^{i*} p_{kt}^{j'},$$
$$\lambda_{0t} = \sum_{c=1}^{s} \alpha(c)\lambda_t(c), \qquad t = 0, 1, \cdots, m. \tag{6.5}$$
$$\lambda_{ij} = \sum_{c,d=1}^{s} \gamma_i(c, d)\mu_j(c, d),$$
$$i = 1, 2, \cdots, h; j = 0, 1, \cdots, m.$$

PROOF: The expressions for v, b, r, k are obvious and need no proof. The others can be proved as follows:

From the method of construction, it can be seen that the uv' rows of the new matrix $\mathbf{N}$ can be grouped in u groups corresponding to the u rows of the matrix $\mathbf{A}$. Hence, the rows of $\mathbf{N}$ can be indexed in the natural way by the double index (i, j), $i = 1, 2, \cdots, u$; $j = 1, 2, \cdots, v'$. The treatments (i, j) and (i', j') will be called (qt)th associates if the ith and i'th rows of $\mathbf{A}$ are qth associates and the jth and j'th treatments of any of the designs $\mathbf{N}_c$ are tth associates, $q = 0, 1, \cdots, h$; $t = 0, 1, \cdots, m$. The class (00) is the 0th class as in (3.1). So we have a PBIB design with $(h + 1)(m + 1) - 1$ non-zero associate classes. The expressions for n_{ij} and $p_{ik:ut}^{ij}$ follow from the above association scheme.

In any row the integer c occurs $\alpha(c)$ times; therefore the matrix $\mathbf{N}_c$ also occurs $\alpha(c)$ times. In the design with the incidence matrix $\mathbf{N}_c$ the tth associate treatments occur together exactly $\lambda_t(c)$ times; hence the $(0t)$th associates occur together in exactly $\sum \alpha(c)\lambda_t(c)$ blocks. Thus

$$\lambda_{0t} = \sum_{c=0}^{s} \alpha(c)\lambda_t(c), \qquad t = 0, 1, \cdots, m. \tag{6.6}$$

Similarly in any pair of rows which are ith associates the combinations $\begin{bmatrix} c \\ d \end{bmatrix}$ and $\begin{bmatrix} d \\ c \end{bmatrix}$ occur exactly $\gamma_i(c, d)$ times. The lth treatment of $\mathbf{N}_c$ and the kth treatment of $\mathbf{N}_i$ occur together $\mu_j(c, d)$ times, if the kth and lth treatments are jth associates. Hence the (ij)th associate treatments occur together in exactly $\sum \gamma_i(c, d)\mu_j(c, d)$ blocks. Thus

$$(6.7) \qquad \lambda_{ij} = \sum_{c \geqq d=1}^{s} \gamma_i(c, d)\mu_j(c, d), \qquad i = 1, 2, \cdots, h; j = 0, 1, \cdots, m.$$

This proves Theorem 6.2.

7. Application to factorial experiments.

DEFINITION 7.1. If $t_1, t_2, \cdots, t_v$ are the treatment effects, then the contrast $\sum_1^v a_i t_i$ is called a normalised contrast, if $\sum_1^v a_i = 0$ and $\sum_1^v a_i^2 = 1$.

DEFINITION 7.2. A factorial experiment will be called a balanced factorial experiment (BFE), if the following conditions are satisfied.

(i) Each of the treatments is replicated exactly r times.

(ii) Each of the blocks has the same size k.

(iii) Estimates of the contrasts belonging to the different interactions are uncorrelated with each other.

(iv) For each of the interactions, all the normalised contrasts belonging to the same interaction are estimated with the same variance.

DEFINITION 7.3. In a factorial experiment, if the conditions (i), (ii) and (iii) given in Definition 7.2 are satisfied and the condition (iv) is not satisfied for some of the interactions, then the experiment will be called a partially balanced factorial experiment (PBFE).

THEOREM 7.1. *A* BFE *in m factors, $F_1, F_2, \cdots, F_m$ at $s_1, s_2, \cdots, s_m$ levels respectively, is a* PBIB *design with an association scheme given as follows: the two treatments $(x_1x_2\cdots x_m)$ and $(y_1y_2\cdots y_m)$ (where y_1, y_i represent levels of the factor F_i) are $(p_1p_2\cdots p_m)$th associates, where $p_i = 1$, if $x_i = y_i$; and $p_i = 0$, if $x_i \neq y_i$. Conversely a* PBIB *design with the above association scheme is a* BFE.

The proof of Theorem 7.1 follows from Theorem 6.1 of [3] on substituting $m_1 = m_2 = \cdots = m_h = 1$ and $h = m$.

THEOREM 7.2. *Let there be s* BFE's *with incidence matrices $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_s$ each in m factors $F_1, F_2, \cdots, F_m$ at $s_1, s_2, \cdots, s_m$ levels respectively. Let these s* BFE's *be associable* PBIB *designs satisfying the sufficient conditions given in Theorem 4.1. Now, if there exists a partially balanced matrix A in s integers* 1, 2, *$\cdots$, s with an association scheme for rows equivalent to that for the treatments of a* PBIB *design which is a* BFE *in n factors $F_{m+1}, F_{m+2}, \cdots, F_{m+n}$ at $s_{m+1}, s_{m+2}, \cdots, s_{m+n}$ levels respectively, then by substituting the matrix $\mathbf{N}_i$ for the integer i in the matrix* **A**, *the matrix* **A** *will be converted into an incidence matrix of a* BFE *in $(m + n)$ factors.*

The proof of the above theorem is obvious from Theorems 6.2 and 7.1.

As an application of Theorem 7.2, consider the following example:

EXAMPLE 7.1. Let us take $m = 2$, $n = 1$, $s_i = s_2 = 2$, $s_3 = 3$. Let the treatments of 2^2 design be denoted by 00, 01, 10, 11 in order. Then confounding the interaction between two factors F_1 and F_2, we get a BFE with the incidence matrix

$$(7.1) \qquad \mathbf{N}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let the balanced matrix $\mathbf{A}$ in 3 integers and 3 rows be given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}. \tag{7.2}$$

Now, putting $\mathbf{N}_2 = \mathbf{N}_3 = \mathbf{E}(4, 2) - \mathbf{N}_1$ and substituting for i in $\mathbf{A}$, the matrix $\mathbf{N}_i$, $i = 1, 2, 3$, we obtain a BFE in 3×2^2 in 6 blocks of 6 plots each. Alternatively, if we take $\mathbf{N}_2 = \mathbf{N}(4, 2) - \mathbf{N}_1$ and $\mathbf{N}_3 = \mathbf{O}(4, 2)$, we obtain a BFE in 6 blocks of 4 plots each. The first design is identical with the plan number 6.9 of Cochran and Cox [2].

In Theorem 7.2, a BFE was constructed by exact analogy with Theorem 6.2; similarly, a PBFE can be constructed by exact analogy with Theorem 6.1. The necessary and sufficient condition is that the column vectors of the matrices $\mathbf{L}$ and $\mathbf{L}^*$ given in Theorem 6.1 must form normalised treatment contrasts belonging to various interactions. The following example will illustrate the method.

EXAMPLE 7.2. Let $\mathbf{N}_1^*$ be the incidence matrix of a 3^2 factorial experiment in 3 blocks of 3 plots each, obtained by confounding only two degrees of freedom of the interaction F_1F_2. Let $\mathbf{N}_2^*$ and $\mathbf{N}_3^*$ be the matrices formed by cyclically permuting the columns of $\mathbf{N}_1^*$. Then a canonically balanced matrix B in three integers is given by

$$\mathbf{B} = \mathbf{N}_1^* + 2\mathbf{N}_2^* + 3\mathbf{N}_3^*. \tag{7.3}$$

Take $\mathbf{N}_1$ as a 2^2 BFE as given in (7.1). $\mathbf{N}_2 = \mathbf{E}(4, 2) - \mathbf{N}_1$ and $\mathbf{N}_3 = 0(4, 2)$. Then on substituting $\mathbf{N}_i$ for i in $\mathbf{B}$ we obrain a PBFE in $3^2 2^2$ in 6 blocks of 12 plots each.

The methods given may be of considerable importance for constructing confounded assymmetrical factorial designs in a large number of factors. Some of the confounded factorial designs already known and many more can be constructed by these methods.

8. Acknowledgement. The author is grateful to Prof. M. C. Chakrabarti for his help and guidance.

REFERENCES

[1] R. C. BOSE AND T. SHIMAMOTO, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.

[2] WILLIAM G. COCHRAN AND GERTRUDE M. COX, *Experimental Designs*, 2nd ed., John Wiley and Sons, New York, 1957.

[3] B. V. SHAH, "On balancing in factorial experiments," *Ann. Math. Stat.*, Vol. 29 (1948), pp. 766–779.

[4] B. V. SHAH, "On a generalisation of the Kronecker product designs," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 48–54.

[5] ROBERT M. THRALL AND LEONARD TORNHEIM, *Vector Spaces and Matrices*, John Wiley and Sons, New York, 1957.

[6] MANOHAR NARHAR VARTAK, "On application of Kronecker product of matrices to statistical designs," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 420–438.