ON A GEOMETRICAL METHOD OF CONSTRUCTION OF PARTIALLY BALANCED DESIGNS WITH TWO ASSOCIATE CLASSES

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1. Introduction. The method of construction of partially balanced block designs discussed here is different than the ones known in the literature. It is based on the existence of an oval (maximum number of points no three on one line) in finite Desarguesian planes. This method can be applied to any plane of order 2^h, h a positive integer, i.e., to planes with $2^h + 1$ points on a line. No general procedure has thus far been obtained for planes with $p^n + 1$ points on a line, p an odd prime and n a positive integer. A design based on a plane with 10 points on a line will be constructed. Further generalizations of this method will be discussed later.

2. Construction of partially balanced designs based on finite Desarguesian planes with 2: + 1 **points on a line.** The total number of points in a plane with $2^h + 1$ points on a line is $2^{2h} + 2^h + 1$. Furthermore it is well known that such planes include ovals consisting of the maximum possible number of points $2^h + 2$. The lines of the plane can be classified into two categories with respect to the oval. The first category includes lines having two points of the oval, henceforth called secants. The second category of lines consists of lines not including any point of the oval. The number of lines belonging to each of the two categories is clearly $(2^{h-1} + 1)$ $(2^h + 1)$ and $2^{2h-1} - 2^{h-1}$ respectively. Consider now the points of the plane which are not on the oval. Their number is $2^{2h} - 1$. Each of them lies on $2^{h-1} + 1$ secants and 2^{h-1} lines of the second category. This leads to a construction of partially balanced block designs identifying the points with the objects and the lines with the blocks. Each of the two categories of lines taken separately gives rise to a partially balanced block design. The first design will be obtained by calling two objects first associates if the points representing them lie on one secant, second associates otherwise. The second design will be obtained by interchanging the roles of the two categories of lines.

The parameters of the first design are as follows:

$$v = 2^{2h} - 1, \qquad b = (2^{h-1} + 1)(2^h + 1),$$

$$r = 2^{h-1} + 1, \qquad \lambda_1 = 1, \qquad \lambda_2 = 0,$$

$$k = 2^h - 1, \qquad n_1 = 2^{2h-1} - 2, \qquad n_2 = 2^{2h-1}, \qquad n_2 = 2^{2h-1},$$

$$P_1 = \begin{pmatrix} 2^{2h-2} - 3, & 2^{2h-2} \\ & 2^{2h-2} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 2^{2h-2} - 1, & 2^{2h-2} - 1 \\ & 2^{2h-2} \end{pmatrix}.$$

The parameters of the second design obtained by identifying the blocks with

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the lines not including points of the oval are

$$v = 2^{2h} - 1, \quad b = 2^{2h-1} - 2^{h-1}, \quad r = 2^h - 2^{h-1}, \quad k = 2^h + 1,$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = 2^{2h-1}, \quad n_2 = 2^{2h-1} - 2,$$

$$P_1 = \begin{pmatrix} 2^{2h-2}, & 2^{2h-2} - 1 \\ & 2^{2h-2} - 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2^{2h-2}, & 2^{2h-2} \\ & 2^{2h-2} - 3 \end{pmatrix}.$$

Let us illustrate the above described method applying it to the case $h = 2$, i.e., to the finite projective plane with 5 points on a line. Take the oval represented by the unruled conic $x^2 + yz = 0$ and the point of intersection of the tangents to this conic, where x, y, z are the coordinates of a point in the plane. Let ϵ be a primitive element of the corresponding Galois field. The six points of the oval are then: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$, $(1, \epsilon, \epsilon^2)$, $(1, \epsilon^2, \epsilon)$. The remaining fifteen points of the plane are: $(0, 1, 1)$, $(0, 1, \epsilon)$, $(0, 1, \epsilon^2)$, $(1, 0, 1)$, $(1, 0, \epsilon)$, $(1, 0, \epsilon^2)$, $(1, 1, 0)$, $(1, 1, \epsilon)$, $(1, 1, \epsilon^2)$, $(1, \epsilon, 0)$, $(1, \epsilon, 1)$, $(1, \epsilon, \epsilon)$, $(1, \epsilon^2, 0)$, $(1, \epsilon^2\ 1)$, $(1, \epsilon^2, \epsilon^2)$. Thus we can exhibit a partially balanced block design with the following parameters: $v = 15, b = 15, r = 3, k = 3, \lambda_1 = 1, \lambda_2 = 0, n_1 = 6, n_2 = 8, P_1 = \begin{pmatrix} 1, & 4 \\ & 4 \end{pmatrix}, P_2 = \begin{pmatrix} 3, & 3 \\ & 4 \end{pmatrix}$. If we identify the points with the consecutive numbers 1 through 15 we obtain Table I, the plan of the design in question.

TABLE I

The Plan of the Design

1	2	3	3	5	13	6	9	15
1	4	7	3	9	11	7	8	9
1	12	15	4	5	6	7	10	13
2	8	14	4	11	14	10	11	12
2	6	10	5	8	12	13	14	15

Remark 1. The described method of construction of partially balanced block designs also gives a method of construction of Hadamard matrices of order 2^{2h}, a method that differs from the one given by Paley [1]. The elements of these matrices can be rearranged so that they have a constant element on the diagonal and are symmetric about the main diagonal. They have further properties which will not be described here. The association matrices of the design [2] yield such matrices. Here is an example of such a matrix for $h = 2$ based on the partially balanced block design with blocks represented by the second category of lines. The diagonal elements are zeros and the below diagonal part of the matrix is as follows in Table II.

Remark 2. It seems worthwhile to point out that the method of construction of partially balanced block designs was applied to Desarguesian planes only because of the fact that they do include an oval consisting of $2^h + 2$ points. If the same would hold with respect to non-Desarguesian planes, then conceivably one could obtain, using the same method, designs that are in general non-isomorphic with the ones already constructed.

TABLE II

1
11
111
1111
10100
100101
1000111
11000111
010100101
0010110100
01001001111
001101100110
0001101101111
01100100111110

3. Construction of a partially balanced block design based on planes of order p^n. No general theory is yet available for construction of designs based on projective planes with $p^n + 1$ points on a line, p an odd prime, n a positive integer. An indication of the approach will be given by examining the case $p = 3$, $n = 2$. The total number of points in this plane is 91; the oval consists of 10 points. The 81 remaining points can be classified into five categories depending on the number of secants passing through the points. It will be convenient to name the lines including just one point of the oval tangents. Let us denote the number of points in each of the five categories by x, y, z, u, w respectively. The classification is summarized in Table III.

TABLE III

Category	Lines passing through the point			
	Number of points	Sceants	Tangents	Lines not including points of the oval
I	x	5	—	5
II	y	4	2	4
III	z	3	4	3
IV	u	2	6	2
V	w	1	8	1

Clearly $x + y + z + u + w = 81$. Further equations are obtained counting the number of intersections of tangents, secants, separately and the number of those intersections of the secants with the tangents that are not points of the oval. This yields the following equations:

$$10x + 6y + 3z + u = 630,$$

$$y + 6z + 15u + 28w = 45,$$

$$2y + 3z + 3u + 2w = 90.$$

It is easy to show that the only possible positive integer solutions of the above equations are $x = 36$, $y = 45$. This leads to a partially balanced block design if we identify, e.g., the objects with the y's and the blocks with the tangents excluding the points of the oval. The parameters of the design are: $v = 45$, $b = 10$, $r = 2$, $n_1 = 16$, $n_2 = 28$, $\lambda_1 = 1$, $\lambda_2 = 0$, $k = 9$, $P_1 = \begin{pmatrix} 8 & 7 \\ & 21 \end{pmatrix}$, $P_2 = \begin{pmatrix} 4 & 12 \\ & 15 \end{pmatrix}$. The plan of the design is shown in Table IV.

TABLE IV

1	2	3	4	5	6	7	8	9
1	10	11	12	13	14	15	16	17
2	10	18	19	20	21	22	23	24
3	11	18	25	26	27	28	29	30
4	12	19	25	31	32	33	34	35
5	13	20	26	31	36	37	38	39
6	14	21	27	32	36	40	41	42
7	15	22	28	33	37	40	43	44
8	16	23	29	34	38	41	43	45
9	17	24	30	35	39	42	44	45

It may be noticed that this design could be obtained by not using any theory but the method will apply to more complicated cases.

In conclusion I wish to thank R. C. Bose for a stimulating discussion which led to this paper.

REFERENCES

[1] R. E. A. C. Paley, "On orthogonal matrices," *J. Math. Phys.*, Vol. 12 (1933), pp. 311–320.
[2] R. C. Bose and Dale Mesner, "On linear associative algebras corresponding to association schemes of partially balanced block designs," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 21–38.