ON THE RANGE OF THE DIFFERENCE BETWEEN HYPOTHETICAL DISTRIBUTION FUNCTION AND PYKE'S MODIFIED EMPIRICAL DISTRIBUTION FUNCTION[1]

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0. Summary. The statistic described in the title may be used to provide a test of the hypothesis that a population has a prescribed continuous distribution function. It differs from that proposed by Kuiper for use with distributions on a circle only in that the usual empirical distribution function is replaced by Pyke's modification. This change and a theorem of Sparre Andersen make possible a computation of the exact distribution for finite sample size along the lines of the computation of the distribution of Kolmogorov's statistic. A brief comparison is made in Section 1 of this statistic, Kolmogorov's, and a statistic studied by Sherman, as distances between hypothetical and empirical distribution functions. The asymptotic distribution, due to Darling and Kuiper, appears in Section 3. Tables are included of the distribution for sample sizes 1 through 20.

1. Introduction. In [9], Lischner et al discuss an "outbreak" of deaths among term infants: of 20 deaths among about 3000 term (i.e., not premature) births, 9 occurred between about the 1900-th birth and the 2100-th birth. In considering tests of the hypothesis that the number, $N(t)$, of deaths up to "time" t (measured by births) is a Poisson process, Dr. Lischner and the author were led to a statistic which had already been introduced by Kuiper [8] in connection with distributions on a circle.

Let n be a positive integer, let T_i denote the time of occurrence of the ith event in a Poisson process, $i = 1, 2, \cdots, n$, and let T denote a fixed time. Given that n events occur in the time interval $[0, T]$, the conditional joint distribution of the ratios T_i/T, $i = 1, 2, \cdots, n$, is known to be that of the order statistics of n numbers chosen independently and at random from the interval $[0, 1]$. Let $Y_1, Y_2, \cdots, Y_n$ denote the order statistics of a random sample of size n from a population with continuous distribution function F, and set $U_i = F(Y_i)$, $i = 1, 2, \cdots, n$, $U_0 = 0$, $U_{n+1} = 1$. Then the joint distribution of the random variables U_i, $i = 1, 2, \cdots, n$, is also that of n numbers chosen independently and at random from $[0, 1]$, and Pyke's suggestion [10] that the plotted points $(U_i, i/(n + 1))$ in the Cartesian plane replace the empirical distribution function may be motivated by consideration of the random variables T_i as partial

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sums of exponential random variables. Certain desirable "small sample" properties of the modified empirical distribution function are mentioned in [10].

Let $F_n(x)$ denote the empirical distribution function. Set

$$D_n^+ = \sup_{-\infty<x<\infty} [F_n(x) - F(x)],$$

$D_n^- = \sup_{-\infty<x<\infty} [F(x) - F_n(x)]$, $C_n^+ = \max_{1\leqq i\leqq n} (i/(n+1) - U_i)$, $C_n^- = \max_{1\leqq i\leqq n} (U_i - i/(n+1))$. Kolmogorov's statistic is

$$D_n = \max(D_n^+, D_n^-) = \sup_{-\infty<x<\infty} |F_n(x) - F(x)| .$$

Pyke's modification is $C_n = \max(C_n^+, C_n^-) = \max_{1\leqq i\leqq n} |i/(n+1) - U_i|$. Kuiper's statistic is $V_n = D_n^+ + D_n^-$, the range of the difference $F - F_n$, and the one studied in this paper is $K_n = C_n^+ + C_n^-$.

Each of the statistics D_n, V_n, and K_n, as well as a related one, L_n, studied by Sherman [11], can be interpreted as a distance between distribution functions. Let $\mathcal{E}$ denote a class of Borel subsets of the real line. Let μ_F denote the measure on the class of Borel subsets of the real line determined by a distribution function F. Then $d_{\mathcal{E}}(G, H) = \sup_{A\varepsilon\mathcal{E}} |\mu_G(A) - \mu_H(A)|$ defines a distance on the class of distribution functions, provided the class $\mathcal{E}$ is large enough that the specification of the measures of all sets in $\mathcal{E}$ determines uniquely the associated distribution function. If $\mathcal{A}$ is the class of semi-infinite intervals $(-\infty, x]$, then $D_n = d_{\mathcal{A}}(F, F_n)$. For $Y_{i-1} \leqq x \leqq Y_i$, $i = 1, 2, \cdots, n+1$, set

$$P_n(x) = \{[F(x) - F(Y_{i-1})]/[F(Y_i) - F(Y_{i-1})] + i - 1\}/(n+1),$$

if $F(Y_i) > F(Y_{i-1})$, $P_n(x) = \{[x - Y_{i-1}]/[Y_i - Y_{i-1}] + i - 1\}/(n+1)$, if $F(Y_i) = F(Y_{i-1})$ (note $F(Y_i) = F(Y_{i-1})$ with probability 0, in sampling from a population with distribution function F). Then $C_n = d_{\mathcal{A}}(F, P_n)$. If $\mathcal{B}$ denotes the class of all intervals, then $V_n = d_{\mathcal{B}}(F, F_n)$, and $K_n = d_{\mathcal{B}}(F, P_n)$. Let $\mathcal{C}$ denote the class of *all* Borel subsets of the line. One verifies that $d_{\mathcal{C}}(F, P_n) = (\frac{1}{2}) \sum_{i=1}^n |F(Y_i) - F(Y_{i-1}) - 1/(n+1)|$, the statistic studied by Sherman [11], denoted here by L_n.

It is convenient to compare K_n and L_n with C_n rather than with D_n; but C_n will be referred to as Kolmogorov's statistic, and K_n as Kuiper's.

Let H_0 denote the hypothesis that F is the population distribution function. For a random variable Z and for fixed n, define $\alpha_Z(z) = \Pr\{Z > z\}$. From the inequalities $(\frac{1}{2})K_n \leqq C_n \leqq K_n \leqq L_n$ it follows that

$$\alpha_K(2z) \leqq \alpha_C(z) \leqq \alpha_K(z) \leqq \alpha_L(z).$$

If the observed points $(Y_i, i/(n+1))$ were found all to lie on the same side of (above or below) the graph of F, we should have $C = K = L$ so that the observed result would appear of greatest significance in testing H_0 using C, of next greatest significance using K; but if $K = 2C$ the observed result would appear of greater significance in testing H_0 using K than using C. This leads to the general suggestion that one may perhaps expect Kolmogorov's statistic to be more powerful than either Kuiper's or Sherman's against alternatives specifying distribution functions lying wholly above or wholly below F, such as, for example, a different location parameter; and Kuiper's to be the most powerful of the three against an alternative specifying a distribution function whose graph rises about as high above as it descends below the graph of F. However, the author is unable to prove theorems stating even asymptotic properties of this kind.

2. Exact distribution of K_n. The statistic K_n is a function of the statistics $U_i = F(Y_i)$, $i = 1, 2, \cdots, n$, which are jointly distributed as the order statistics of a sample of n numbers chosen independently and at random from the unit interval. We shall therefore in this section assume, for convenience and without loss of generality, that $F(x) = 0$ for $x < 0$, $F(x) = x$ for $0 \leqq x \leqq 1$, $F(x) = 1$ for $x > 1$, and $U_i = Y_i$, $i = 1, 2, \cdots, n$. It will be seen that the application of a theorem of E. Sparre Andersen [1] reduces the problem of finding $\Pr\{K_n < z\}$ to that of finding the probability that all of the points $(U_i, i/(n+1))$ will lie between the lines $y = x$ and $y = x + z$. (A similar use of Andersen's theorem is made by Pyke [10], p. 575; cf., [5], p. 189.) After this reduction, any of a number of approaches that have been used to determine the distribution of D_n may be used successfully here also.

Let E^N denote Euclidean N-space. For $z = (z_1, z_2, \cdots, z_N)$ in E^N, define the *cyclic shift transformation* C by $Cz = (z_2, z_3, \cdots, z_N, z_1)$. Random variables $Z_1, Z_2, \cdots, Z_N$ will be termed *cyclically permutable* if for every Borel set $A \subset E^N$ one has

$$\Pr\{Z \varepsilon A\} = \Pr\{CZ \varepsilon A\}, \tag{2.1}$$

where Z denotes the random vector $(Z_1, Z_2, \cdots, Z_N)$. The only events to be considered will be of the form $\{Z \varepsilon A\}$ for A a Borel subset of E^N, and the same symbol A will be used for the event $\{Z \varepsilon A\}$. Thus (2.1) can be rewritten as $P(A) = P(C^{-1}A)$. An event B will be termed *cyclically symmetric* if $B = CB$ (or equivalently $B = C^{-1}B$). Set $W_r(Z) = \sum_{i=1}^{r} Z_i/r$. The following lemma, for interchangeable random variables, is due to E. Sparre Andersen [1]; it is especially clear from Spitzer's short proof [12] that it holds also for cyclically permutable random variables.

LEMMA (Andersen). *Let the random variables $Z_1, Z_2, \cdots, Z_N$ be cyclically permutable, and suppose* $\Pr\{W_i(Z) = W_j(Z)\} = 0$, *for $i \neq j$, $i, j, = 1, 2, \cdots, N$. Let B be a cyclically symmetric event. Then*

$$P(B) = NP[B \cap \{W_N(Z) < W_r(Z) \text{ for } r = 1, 2, \cdots, N-1\}].$$

Set $p_n(t) = \Pr\{U_i < i/(n+1) \leqq U_i + t/(n+1) \text{ for } i = 1, 2, \cdots, n\}$; this is the probability that all the points $(U_i, i/(n+1))$ will lie between the lines $y = x$ and $y = x + t/(n+1)$. Also

$$K_n = \max_{0 \leqq i < j \leqq n+1} |[i/(n+1) - U_i] - [j/(n+1) - U_j]|.$$

Set $Z_i = [i/(n+1) - U_i] - [(i-1)/(n+1) - U_{i-1}]$, $i = 1, 2, \cdots, n+1$, and $W_r = (1/r)\sum_{i=1}^{r} Z_i = (1/r)[r/(n+1) - U_r]$, $r = 1, 2, \cdots, n+1$; then $W_{n+1} = 0$, and $K_n = \max_{0 \leqq i < j \leqq n+1} |jW_j - iW_i|$.

THEOREM. *For* $t > 0$, $\Pr\{K_n \leqq t/(n+1)\} = (n+1)p_n(t)$.

PROOF. The random variables $Z_1, Z_2, \cdots, Z_{n+1}$ are known to be cyclically permutable (indeed, interchangeable). Since also $W_{n+1} = 0$, a cyclic permutation does not affect the value of K_n. Thus the event $B = \{K_n \leqq t/(n+1)\}$ is cyclically symmetric. The event $\{W_{n+1} < W_r \text{ for } r = 1, 2, \cdots, n\}$ may be written $\{r/(n+1) - U_r > 0 \text{ for } r = 1, 2, \cdots, n\}$, so that $B \cap \{W_{n+1} < W_r \text{ for } r = 1, 2, \cdots, n\} = \bigcap_{r=1}^n \{U_r < r/(n+1) \leqq U_r + t/(n+1)\}$. From the lemma it follows that

$$P(B) = (n+1)P[\bigcap_{r=1}^n \{U_r < r/(n+1) \leqq U_r + t/(n+1)\}],$$

or $P(B) = (n+1)p_n(t)$. This completes the proof of the theorem.

Methods which have been used to discuss the distributions of D_n and D_n^+ (cf. in particular [7], [6], and [2]) apply also to the problem of determining $p_n(t)$, and are applied here to the more general problem: if n points are chosen independently and at random on [0, 1], and if $U_1, U_2, \cdots, U_n$ are the order statistics, determine the probability that all the points $(U_i, i/(n+1))$, $i = 1, 2, \cdots, n$, will lie between a prescribed pair of parallel lines of which the lower passes through or below $(0, 0)$ and through or above $(1, 1)$. Let the lines be

$$L_{1,n}: y = [1 + v/(n+1)]x - a/(n+1)$$

and

$$L_{2,n}: y = [1 + v/(n+1)]x + b/(n+1),$$

where $0 \leqq a \leqq v$, $b \geqq 0$. For $\lambda > 0$ we introduce events:

$$A_n(h; \lambda) = \bigcap_{i=1}^n \{i \leqq \lambda U_i + h\},\ B_n(a, b; v)$$
$$= \bigcap_{i=1}^n \{(n+1+v)U_i - a < i \leqq (n+1+v)U_i + b\}$$

and $C_n(a, b; v) = A_n(b; n+1+v) - B_n(a, b; v)$. The event $B_n(a, b; v)$ is the event of primary interest here, that all points $(U_i, i/(n+1))$ will lie between the lines $L_{1,n}$ and $L_{2,n}$. Set $P_n(h; \lambda) = P[A_n(h; \lambda)]$, $P_0(h; \lambda) = 1$, $p_n(a, b; v) = P[B_n(a, b; v)]$, $p_0(a, b; v) = 1$, $q_n(a, b; v) = P[C_n(a, b; v)]$, $q_0(a, b; v) = 0$; then $p_n(a, b; v) + q_n(a, b; v) = P_n(b; n+1+v)$. For $i = 0, 1, 2, \cdots, n-1$, set $f_n^i = P[D_n^i]$, where $D_n^i(a, b; v) = \bigcap_{r=1}^n \{r \leqq (n+1+v)U_r + b$ $\bigcap_{j=1}^i \{j > (n+1+v)U_j - a\} \cap \{i+1 \leqq (n+1+v)U_{i+1} - a\}$. One finds

$$f_n^i = \binom{n}{i}[(a+i+1)/(n+1+v)]^i[1 - (a+i+1)/(n+1+v)]^{n-i}\cdot$$

$$p_i(a, b; a)P_{n-i}(a+b+1, n-i+v-a),\ q_n(a, b; v) = \sum_{i=0}^{n-1} f_n^i.$$

The changes of variable $\rho_n(a, b; v) = (n+1+v)^n p_n(a, b; v)/n!$, $Q_n(h; k) = (n+k)^n P_n(h+1; n+k)/n!$ then yield $\rho_n(a, b; v) = Q_n(b-1; v+1) - \sum_{i=0}^{n-1} \rho_i(a, b; a)Q_{n-i}(a+b; v-a)$. The Q_j are known, so that *these formulas* *give the probabilities* $p_n(a, b; v)$ *in terms of the probabilities* $p_n(a, b; a)$. Henceforth we set $p_n(a, b) = p_n(a, b; a)$, $\rho_n(a, b) = \rho_n(a, b; a)$, and $Q_n(h) = Q_n(h; 0)$. Then

$$\rho_n(a, b) = Q_n(b - 1; a + 1) - \sum_{i=0}^{n-1} \rho_i(a, b) Q_{n-i}(a + b). \tag{2.2}$$

Set $\alpha(x; a, h) = \sum_{r=0}^{\infty} \rho_r(a, h)x^r$, $\beta(x; h) = \sum_{r=0}^{\infty} Q_r(h)x^r$, and $\gamma(x; a, b) = \sum_{r=0}^{\infty} Q_r(b - 1; a + 1)x^r$. Then $\alpha(x; a, b)\beta(x; a + b) = \gamma(x; a, b)$. The known formula for $P_n(h; \lambda)$ (cf., [10]:

$$P_n(h, \lambda) = [(h + \lambda - n)/\lambda^n] \sum_{j=0}^{[h]} (-1)^j \binom{n}{j} (h - j)^j (h + \lambda - j)^{n-j-1},$$

if $n - \lambda \leqq h \leqq n$, $P_n(h, \lambda) = 0$ if $h + \lambda \leqq n$, $P_n(h, \lambda) = 1$ if $h \geqq n$ yields

$$Q_n(b - 1; a + 1) = (a + b + 1) \sum_{j=0}^{[b]} (-1)^j \frac{(b - 1)^j}{j!} \frac{(n + a + b + 1 - j)^{n-j-1}}{(n - j)!}$$

if $b \leqq n$, $Q_n(b - 1, a + 1) = (n + a + 1)^n/n!$ if $b \geqq n$,

$$Q_n(a + b) = (a + b + 1) \sum_{j=0}^{[a+b+1]} (-1)^j \frac{(a + b + 1 - j)^j}{j!} \cdot \frac{(n + a + b + 1 - j)^{n-j-1}}{(n - j)!}$$

if $a + b + 1 \leqq n$, $Q_n(a + b) = n^n/n!$ if $a + b + 1 \geqq n$. Equation (2.2) now furnishes the basis for an iterative method of determining the probabilities $p_n(a, b)$, hence $p_n(a, b; v)$. Alternatively, $p_n(a, b)$ is the product of $n!/(n + a + 1)^n$ by the coefficient of x^n in the expansion of $\gamma(x; a, b)/\beta(x; a, b)$. By the theorem above we then have

$$\Pr\{K_n \leqq t/(n + 1)\} = (n + 1)p_n(0, t). \tag{2.3}$$

We turn now to the case in which a, b, and v are integers. First, $\rho_n(0, 1) = 1$ satisfies (2.2) with $a = 0$, $b = 1$. Therefore $p_n(0, 1) = n!/(n + 1)^n$, so that $\Pr\{K_n \leqq 1/(n + 1)\} = n!/(n + 1)^{n-1}$. More generally, for integers a, b, and v with $0 \leqq a \leqq v$, $b \geqq 0$, we define

$$A_{ik} = \{U_i < k/(n + v + 1)\} \cap \{U_{i+1} \geqq k/(n + v + 1)\} \cap \bigcap_{j=1}^{i} \{(n + v + 1)U_j - a < j \leqq (n + 1 + v)U_j + b\},$$

$i = 0, 1, 2, \cdots, n - 1$, $k = 1, 2, \cdots, n$; $p_{ik} = P(A_{ik})$, $p_{00} = 1$, $p_{i0} = 0$ for $i = 1, 2, \cdots, n$. Then $p_n(a, b; v) = p_{nn}$; in particular, $p_n(0, t) = p_{nn}$ if $a = v = 0$, $b = t$. One finds that $p_{i,k+1} = 0$ if $i > k + b$,

$$p_{i,k+1} = [(n + v - k)/(n + v + 1 - k)]^{n-i} p_{i,k}$$

if $i \leqq k - a$, and

$$p_{i,k+1} = \sum_{j=\max(0,k-a)}^{i} p_{jk} \binom{n-j}{i-j} \cdot [1/(n+v+1-k)]^{i-j}[(n+v-k)/(n+v+1-k)]^{n-i},$$

if $k - a < i \leqq k + b$. The change of variable

$$\rho_{ik} = [(n+v+1)^n/(n+v+1-k)^{n-i}][(n-i)!/n!]p_{ik},$$

$\rho_{00} = 1$, $\rho_{i0} = 0$ for $i > 0$ yields $\rho_{i,k+1} = 0$ if $i > k + b$,

$$\rho_{i,k+1} = \sum_{j=\max(0,k-a)}^{i} \rho_{jk}/(i-j)!$$

if $k - a < i \leqq k + b$, $\rho_{i,k+1} = \rho_{i,k}$ if $i \leqq k - a$. We note again that the ρ_{ik} depend on a and b but *not on* v. Thus once the matrix ρ_{ik} is computed for particular a and b, the probabilities $p_n(a, b; v)$ can be obtained for arbitrary $v \geqq a$ from the formulas for the change of variable.

The probabilities $\Pr\{K_n \leqq t/(n+1)\}$ were computed[3] for integers t and for $n = 1, 2, \cdots, 20$ using (2.3), and are presented in Table 2.1. They were recomputed, as a check, from formulas developed after the change of variable $u_{i,k} = (i!/k^i)\rho_{i,k}$.

3. The asymptotic distribution of K_n. Doob ([4]; cf., [3]) gives asymptotic formulas for the joint distribution function of $n^{\frac{1}{2}}D_n^+$ and $n^{\frac{1}{2}}D_n^-$, from which the asymptotic distribution of $n^{\frac{1}{2}}V_n$ may readily be determined. The asymptotic distribution of $n^{\frac{1}{2}}K_n$ clearly coincides with that of $n^{\frac{1}{2}}V_n$. Darling and Kuiper [8] give the asymptotic distribution with a higher order term:

$$\Pr\{n^{\frac{1}{2}}V_n \leqq c\} = 1 - \sum_{j=1}^{\infty} 2(4j^2c^2 - 1)\exp(-2j^2c^2) + [(8c)/3(n)^{\frac{1}{2}}] \sum_{j=1}^{\infty} j^2(4j^2c^2 - 3)\exp(-2j^2c^2) + O(1/n). \tag{3.1}$$

In considering the difference $V_n - K_n$, suppose that $C_n^+ = \max_i [i/(n+1) - U_i]$ occurs at $i = k$; we have $(k/n) - k/(n+1) = 1/n(n+1)$, suggesting that on the average $D_n^+ - C_n^+$ is approximately $1/2(n+1)$, and $V_n \doteq K_n + 1/(n+1)$ ("$\doteq$" is to be read: "is approximated by"). For estimating $\Pr\{K_n \leqq t/(n+1)\}$ for large n, the following approximate formula is therefore suggested:

$$\begin{aligned}\Pr\{K_n \leqq t/(n+1)\} &\doteq \Pr\{(n+1)^{\frac{1}{2}}V_n \leqq (t+1)/(n+1)^{\frac{1}{2}}\} \\ &\doteq 1 - 2\sum_{j=1}^{\infty}(4j^2c^2 - 1)\exp(-2j^2c^2) \\ &\quad + [(8c)/3(n+1)^{\frac{1}{2}}] \sum_{j=1}^{\infty} j^2(4j^2c^2 - 3)\exp(-2j^2c^2),\end{aligned} \tag{3.2}$$

[3] The author wishes to thank Mr. C. R. Smith, and the University of Missouri Computer Research Center, under the direction of Mr. Roy F. Keller, for their assistance in preparing the tables.

TABLE 2.1

$\Pr\{K_n \leqq t/(n+1)\}$

t	n = 1	2	3	4	5	6	7	8	9	10
1	1.0000	.6667	.3750	.1920	.0926	.0428	.0192	.0084	.0036	.0015
2		1.0000	.9375	.8160	.6713	.5301	.4062	.3040	.2234	.1617
3			1.0000	.9920	.9645	.9151	.8481	.7700	.6872	.6045
4				1.0000	.9992	.9948	.9836	.9636	.9346	.8972
5					1.0000	.9999	.9994	.9974	.9930	.9850
6						1.0000	1.0000	.9999	.9997	.9988
7								1.0000	1.0000	1.0000

t	n = 11	12	13	14	15	16	17	18	19	20
1	.0006	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.1157	.0819	.0574	.0400	.0277	.0190	.0130	.0089	.0060	.0041
3	.5253	.4518	.3852	.3260	.2740	.2290	.1903	.1575	.1298	.1066
4	.8531	.8041	.7519	.6981	.6440	.5908	.5392	.4899	.4433	.3996
5	.9729	.9564	.9355	.9105	.8819	.8502	.8162	.7803	.7432	.7054
6	.9971	.9940	.9891	.9821	.9729	.9612	.9471	.9306	.9120	.8913
7	.9998	.9995	.9988	.9976	.9957	.9929	.9889	.9838	.9773	.9694
8	1.0000	1.0000	.9999	.9998	.9995	.9991	.9983	.9972	.9956	.9934
9			1.0000	1.0000	1.0000	.9999	.9998	.9996	.9994	.9989
10						1.0000	1.0000	1.0000	.9999	.9999
11						1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 3.1
$\Pr\{K_n \leqq t/(n+1)\}$, $n = 19$.

t	Darling-Kuiper Formula (3.2)	Exact Probability
1	.0000	.0000
2	.004	.006
3	.129	.130
4	.453	.443
5	.753	.743
6	.916	.912
7	.978	.977
8	.9956	.9956

where $c = (t+1)/(n+1)^{\frac{1}{2}}$. For $n = 19$, a comparison is made in Table 3.1 between results using the Darling-Kuiper formula (3.2), and the exact probabilities computed as indicated in Section 2.

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