ZERO CROSSING PROBABILITIES FOR GAUSSIAN STATIONARY PROCESSES[1]

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1. Introduction. Let $X(t)$ be a separable stationary Gaussian process with mean zero, $EX(t) \equiv 0$, and continuous covariance function

$$\rho(t) = EX(\tau)X(\tau + t)$$

normalized so that $\rho(0) = 1$. Questions relating to the probability

$$H_X(T) = P[X(t) > 0, 0 \leqq t \leqq T] \tag{1}$$

that $X(t)$ does not cross zero during some time interval T arise in various applications [5], [6].

Many of the difficulties that arise in treating such questions are due to the fact that most of the interesting stationary Gaussian processes are not Markovian. Here we shall obtain bounds on the behavior of $H_X(T)$ particularly for large T using an interesting inequality of D. Slepian [6] and estimates of $H_X(T)$ for some simple processes.

Slepian's inequality states the following: if $X(t)$ and $Y(t)$, $0 \leqq t \leqq T$, are two separable Gaussian processes with mean zero and covariance functions $\rho(t, \tau)$ and $r(t, \tau)$ respectively, normalized so that $\rho(t, t) = r(t, t) = 1, 0 \leqq t \leqq T$, and if

$$\rho(t, \tau) \leqq r(t, \tau), \qquad 0 \leqq t, \tau \leqq T, \tag{2}$$

then

$$H_X(T) \leqq H_Y(T). \tag{3}$$

The inequality is true for either continuous or discrete parameter processes.

2. An upper bound.

Theorem 1. *If $X(t)$ is a separable stationary Gaussian process with $EX(t) \equiv 0$ and $\rho(t) \to 0$ for $t \to \infty$, then*

$$H_X(T) = o(T^{-\alpha}) \tag{4}$$

as $T \to \infty$ for every $\alpha > 0$.

To prove this we first note that

$$H_X(T) \leqq P[X(j\Delta) > 0, j = 0, 1, \cdots, n - 1] \tag{5}$$

for any $\Delta > 0$, $n = [T/\Delta]$ where $[x]$ is the greatest integer less than or equal to x.

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We next seek a sequence of random variables Y_j which has a larger covariance sequence than $X(j\Delta)$ and a zero crossing probability that we can easily estimate. Consider the sequence

$$Y_j = (X_j + \sigma Z)(1 + \sigma^2)^{-\frac{1}{2}}, \qquad j = 0, 1, \cdots, n-1 \tag{6}$$

with Z and X_j independent normal random variables with mean zero and variance one. The Y_j are joint normal with mean zero and covariance

$$r_k = EY_jY_{j+k} = (\delta_{0,k} + \sigma^2)(1 + \sigma^2)^{-1}. \tag{7}$$

If we choose

$$\sigma^2/(1 + \sigma^2) \geqq \sup_{t \geqq \Delta} \rho(t) \tag{8}$$

then $r_k \geqq \rho(k\Delta)$. The probability that $Y_j > 0$ for $j < n$ exceeds the probability that $X(j\Delta) > 0$ for $j < n$ by Slepian's inequality, and so by (5)

$$\begin{aligned} H_X(T) &\leqq P[Y_j > 0; j = 0, 1, \cdots, n-1] \\ &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} [1 - \Phi(x)]^n \exp[-x^2/(2\sigma^2)]\, dx. \end{aligned} \tag{9}$$

To estimate this integral, let x_0 be the value of x where

$$g(x) = -n \log [1 - \Phi(x)] + x^2/(2\sigma^2) \tag{10}$$

has a minimum. Since both terms of $g(x)$ are convex and twice differentiable in x, the second derivative of $g(x)$ is everywhere larger than σ^{-2}. Therefore $g(x) \geqq -n \log [1 - \Phi(x_0)] + x_0^2/(2\sigma^2) + (x - x_0)^2/(2\sigma^2) > x_0^2/(2\sigma^2) + (x - x_0)^2/(2\sigma^2)$, and from (9)

$$H_X(T) < \exp[-x_0^2/(2\sigma^2)]. \tag{11}$$

To determine x_0 we set the derivative of $g(x)$ at x_0 equal to zero, i.e.

$$\sigma^2 n \exp[-x_0^2/2] = -[1 - \Phi(x_0)](2\pi)^{\frac{1}{2}} x_0. \tag{12}$$

If for fixed Δ we let $T \to \infty$, then $n = [T/\Delta] \to \infty$; the solution of (12) is $x_0 = x_0(T) = -[1 + o(1)](2 \log T)^{\frac{1}{2}}$; and

$$H_X(T) < T^{-[1+o(1)]/\sigma^2}. \tag{13}$$

Since we assume that $\rho(t) \to 0$ as $t \to \infty$, it follows from (8) that σ^2 can be made arbitrarily small by choosing Δ sufficiently large. Hence we may choose $1/\sigma^2$ larger than any given number α and obtain the result (4).

3. Special bounds. It is also of interest to obtain bounds on $H_X(T)$ if $\rho(t) \to 0$ as $t \to \infty$ at some specified rate.

THEOREM 2. *If $X(t)$ is a separable Gaussian stationary process with $EX(t) \equiv 0$ and $|\rho(t)| < Ct^{-\alpha}$ for some $C > 0$ and all $t > 0$, then*

$$(14)\qquad H_X(T) < \begin{cases} \exp(-KT) & \text{if } 1 < \alpha \\ \exp(-KT/\log T) & \text{if } \alpha = 1 \\ \exp(-KT^{\alpha}) & \text{if } 0 < \alpha < 1 \end{cases}$$

for some $K > 0$.

Let

$$(15)\qquad \rho_{jk} = \rho(|j-k|\Delta)$$

be the covariance matrix of $X(j\Delta)$ and ρ_{jk}^{-1} be its inverse. Since $X(j\Delta)$ are joint normal, the right side of (5) can be written as an n-dimensional integral,

$$(16)\quad H_X(T) \leqq \frac{1}{(2\pi)^{n/2}|\rho_{jk}|^{\frac{1}{2}}} \int_0^{\infty} dx_0 \cdots \int_0^{\infty} dx_{n-1} \exp\left\{-\frac{1}{2}\sum_{j,k=0}^{n-1} \rho_{jk}^{-1} x_j x_k\right\},$$

in which $|\rho_{jk}|$ is the determinant of the matrix (ρ_{jk}).

If (ρ_{jk}) has eigenvalues $0 < \lambda_1 \leqq \cdots \leqq \lambda_n$, then (ρ_{jk}^{-1}) has eigenvalues $0 < \lambda_n^{-1} \leqq \cdots \leqq \lambda_1^{-1}$. By the extremal properties of eigenvalues

$$\sum_{j,k=0}^{n-1} \rho_{jk}^{-1} x_j x_k \geqq \lambda_n^{-1} \sum_{j=0}^{n-1} x_j^2 .$$

If we substitute this into (16) and integrate, we obtain

$$(17)\qquad H_X(T) \leqq \frac{\lambda_n^{\frac{1}{2}}}{2^n |\rho_{jk}|^{\frac{1}{2}}} = \frac{\lambda_n^{n/2}}{2^n(\Pi_1^n \lambda_j)^{\frac{1}{2}}} \leqq \left(\frac{\lambda_n}{4\lambda_1}\right)^{n/2}.$$

The diagonal elements of ρ_{ij} are all 1 and the eigenvalues of $(\rho_{ij} - \delta_{ij})$ are in absolute value less than the maximum row sum of $(|\rho_{ij}| - \delta_{ij})$. From this we conclude that

$$(18)\qquad \lambda_n \leqq 1 + 2\sum_{j=1}^{n-1} |\rho(j\Delta)|$$

and

$$(19)\qquad \lambda_1 \geqq 1 - 2\sum_{j=1}^{n-1} |\rho(j\Delta)|.$$

If $|\rho(t)| < Ct^{-\alpha}$, then

$$2\sum_{j=1}^{n-1} |\rho(j\Delta)| < 2C\Delta^{-\alpha} \sum_{j=1}^{n-1} j^{-\alpha} < 2C\Delta^{-\alpha} \int_{\frac{1}{2}}^{T/\Delta} dj\, j^{-a}$$

$$(20)\qquad < \begin{cases} C'\Delta^{-\alpha} & \text{for } 1 < \alpha \\ C'\Delta^{-1}\log(T/\Delta) & \text{for } \alpha = 1 \\ C'\Delta^{-1}T^{1-\alpha} & \text{for } 0 < \alpha < 1 \end{cases}$$

for some constant $C' > 0$. For each T we choose $\Delta = \Delta(T)$ so that the right hand side of (20) is $(4 - e)/(4 + e)$. Then (18) gives $\lambda_n < 8/(4 + e)$ and $\lambda_1 \geqq 2e/(4 + e)$ while (17) gives

$$(21)\qquad H_X(T) < \exp(-\tfrac{1}{2}[T/\Delta]).$$

The theorem now follows because this choice of $\Delta(T)$ implies that

$$\begin{array}{ll} \Delta^{-1} \text{ is independent of } T & \text{for } \; 1 < \alpha \\ \Delta^{-1} > K/\log T & \text{for } \; \alpha = 1 \\ \Delta^{-1} = KT^{\alpha-1} & \text{for } \; 0 < \alpha < 1 \end{array}$$

4. Lower bounds. Slepian's inequality implies that for a stationary Gaussian process with $\rho(t) \geqq 0$ for all t, $H_X(t)$ can not decrease faster than exponentially, i.e. $e^{KT}H_X(T) \to \infty$ as $T \to \infty$ for some $K > 0$. If $\rho(t)$ can take negative values, $H_X(T)$ can decrease faster than exponentially. The following theorem provides a lower bound for $H_X(T)$ that will enable us to give examples of Gaussian processes which are mixing and for which $H_X(T)$ decreases slower than an exponential.

THEOREM 3. *If $X(t)$ is a separable stationary Gaussian process with $EX(t) \equiv 0$ and a covariance function $\rho(t)$, $\rho(0) = 1$, which satisfies the conditions*
(i) *for some $\lambda > 0$ and all t, $0 < t < \infty$, $\rho(t) \geqq e^{-\lambda t}$*
(ii) *for some α, $0 < \alpha < 1$, $\liminf t^\alpha \rho(t) > 0$ for $t \to \infty$, then*

$$H_X(T) > \exp\{-[K + o(1)]T^\alpha \log T\} \tag{22}$$

for some $K > 0$ as $T \to \infty$.

Our object is to construct a Gaussian process $Z(t)$, with a covariance function $r(t)$, $r(0) = 1$, that is a lower bound for $\rho(t)$, i.e.,

$$0 \leqq r(t) \leqq \rho(t), \qquad 0 \leqq t \leqq T. \tag{23}$$

If we can estimate $H_Z(T)$, Slepian's inequality (3) tells us that

$$H_Z(T) \leqq H_X(T). \tag{24}$$

Let $U(t)$ be the Ornstein-Uhlenbeck process, [8], a Gaussian process with zero mean and covariance function $\exp(-\mu|t|)$. Let

$$Z(t) = [U(t) + \sigma Z](1 + \sigma^2)^{-\frac{1}{2}} \tag{25}$$

where Z is a Gaussian random variable with mean zero and variance one independent of the process $U(t)$. The covariance function $r(t)$ of $Z(t)$ is

$$r(t) = [\sigma^2 + \exp(-\mu|t|)](1 + \sigma^2)^{-1}. \tag{26}$$

For sufficiently large T, condition (23) will be satisfied if

$$\sigma^2(T) = AT^{-\alpha} \quad \text{with} \quad 0 < A < \liminf_{t \to \infty} t^\alpha \rho(t) \tag{27}$$

and μ is independent of T with $\mu > \lambda$.

We must estimate

$$\begin{aligned} H_Z(T) &= P[U(t) > -\sigma Z, 0 \leqq t \leqq T] \\ &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} P[U(t) > y, 0 \leqq t \leqq T] \exp[-y^2/(2\sigma^2)]\, dy. \end{aligned} \tag{28}$$

The quantity

$$G(y, T) = P[U(t) > y, 0 \leqq t \leqq T] \tag{29}$$

is the probability of survival for the Ornstein-Uhlenbeck process with an absorbing barrier at y or equivalently the probability that the first passage time for the coordinate y is larger than T.

The probability density $p(x, t)$ of $U(t)$ satisfies the Fokker-Planck equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}(xp) = \frac{1}{\mu}\frac{\partial p}{\partial t}. \tag{30}$$

If we let

$$p(x, y, t) = (\partial/\partial x)P[U(t) < x, U(\tau) > y \quad \text{for} \quad 0 \leqq \tau \leqq t], \tag{31}$$

then $p(x, y, t)$ satisfies (30) for $x > y$ subject to the boundary condition

$$p(y, y, t) = 0. \tag{32}$$

Since we are concerned with positivity for the stationary process $Z(t)$ in $(0, T)$, the initial conditions for (30) must be the stationary solution of (30)

$$p(x, y, 0) = p(x, 0) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \qquad \text{for} \quad x > y \tag{33}$$

and because of absorption at y,

$$p(x, y, 0) = 0 \qquad \text{for} \quad x \leqq y. \tag{34}$$

Clearly

$$G(y, t) = \int_y^\infty p(x, y, t)\, dx. \tag{35}$$

Equations (30) and (32) can be solved by separation of variables and expansion in terms of orthogonal functions. The solution has the form

$$p(x, y, t) = \sum_{j=1}^{\infty} a_j(y) \exp[-\alpha_j(y)t] g_j(x, y) \tag{36}$$

where $\alpha_j(y)$, $g_j(x, y)$ are the eigenvalues and corresponding eigenfunctions of the Sturm-Liouville system

$$\frac{d^2 g_j}{dx^2} + \frac{d}{dx}(xg_j) + \frac{\alpha_j(y)}{\mu} g_j = 0 \tag{37}$$

with $g_j(\infty, y) = g_j(y, y) = 0$, and the $a_j(y)$ are Fourier coefficients to be taken so that (36) converges to (33), (34) for $t \to 0$.

Equation (37) has the solution

$$g_j(x, y) = \exp(-x^2/4)\, D_{\alpha_j(y)/\mu}(x) \tag{38}$$

where $D_\nu(x)$ is the Weber-Hermite function (see [1] p. 116 or [9] p. 347). The eigenvalues which are ordered so that $\alpha_1(y) < \alpha_2(y) < \cdots$ must be determined so that

$$D_{\alpha_j(y)/\mu}(y) = 0. \tag{39}$$

The $D_\nu(x)$ are themselves solutions of a singular Sturm-Liouville equation of the type discussed by Titchmarsh [7]. From Titchmarsh's work we conclude that the $g_j(x, y)$ are orthogonal and complete in the L_2 space with inner product

$$(g, h) = \int_y^\infty g(x)h(x) \exp(x^2/2)\, dx. \tag{40}$$

Since the function given by (33), (34) is in this space, it follows that the expansion (36) converges in mean square with weight $\exp(x^2/2)$ if

$$a_j(y) = \int_y^\infty g_j(x, y)\, dx \Big/ \left\{(2\pi)^{\frac{1}{2}} \int_y^\infty \exp(x^2/2)[g_j(x, y)]^2\, dx\right\}. \tag{41}$$

From (35), (36) and (41) we obtain

$$G(y, t) = \sum_{j=1}^\infty b_j(y) \exp[-\alpha_j(y)t] \tag{42}$$

with

$$b_j(y) = \left[\int_y^\infty g_j(x, y)\, dx\right]^2 \Big/ \left\{(2\pi)^{\frac{1}{2}} \int_y^\infty \exp(x^2/2)[g_j(x, y)]^2\, dx\right\}. \tag{43}$$

Since $b_j(y) \geqq 0$, (42) implies

$$b_1(y) \exp[-\alpha_1(y)t] \leqq G(y, t) \tag{44}$$

while (44), (28), and (29) give

$$\begin{aligned} H_Z(T) &\geqq (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} b_1(y) \exp[-y^2/(2\sigma^2) - \alpha_1(y)T]\, dy \\ &\geqq (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{y_0} b_1(y) \exp[-y^2/(2\sigma^2) - \alpha_1(y)T]\, dy \end{aligned} \tag{45}$$

for any real y_0.

With condition (27), one can show that for large T the main contribution to (45) comes from large negative y. In any case, however, if we let $y_0 \to -\infty$ as $T \to \infty$ we can use asymptotic estimates of $\alpha_1(y)$ and $b_1(y)$ to evaluate (45). From the asymptotic properties of the Weber-Hermite functions and (39) one finds

$$\alpha_1(y) = -(2\pi)^{-\frac{1}{2}}\mu y \exp(-y^2/2)[1 + o(1)] \quad \text{as} \quad y \to -\infty. \tag{46}$$

One can show that

$$b_1(y) \to 1 \quad \text{as} \quad y \to -\infty \tag{47}$$

either by substituting (38) and (46) into (43) and taking the limit $y \to -\infty$ or by using the fact that the Ornstein-Uhlenbeck process is continuous with probability 1. The latter implies that $G(y, t) \to 1$ as $y \to -\infty$ for all finite $t > 0$. Since $b_j(y) \geqq 0$

$$1 \geqq G(y, 0) \geqq \sum_1^\infty b_j(y) \tag{48}$$

and

$$\begin{aligned} G(y, t) &\leqq b_1(y) \exp[-\alpha_1(y)t] + \exp[-\alpha_2(y)t] \\ &\leqq b_1(y) + \exp[-\alpha_2(y)t]. \end{aligned} \tag{49}$$

From (48) and (49), plus the fact that $\alpha_2(y) \to \mu$ for $y \to -\infty$ (which one can also obtain from (38) and the asymptotic properties of $D_\nu(x)$) we have $1 - e^{-\mu t} \leqq b_1(y) \leqq 1$ as $y \to -\infty$. This can be true for arbitrary finite positive t only if (47) holds.

From (46) we see that $0 \leqq \alpha_1(y) \leqq \alpha_1(y_0)[1 + o(1)]$ for $y < y_0$, $y_0 \to -\infty$ therefore (45) and (47) give

$$H_Z(T) \geqq \exp\{-\alpha_1(y_0)[1 + o(1)]T\}\Phi(y_0/\sigma)[1 + o(1)] \tag{50}$$

as $y_0 \to -\infty$.

The theorem now follows from (50), (27), and (46) by a choice of y_0 that maximizes the bound in (50), or a choice $y_0 = -(B \log T)^{\frac{1}{2}}$ with $B \geqq 2(1 - \alpha)$.

One can easily give examples of Gaussian stationary processes that are mixing and for which $H_X(T)$ decreases slower than an exponential. Consider the process $X(t)$ with mean zero and covariance function

$$\rho(t) = (1 + |t|)^{-\alpha}, \qquad 0 < \alpha < 1. \tag{51}$$

The fact that $\rho(t)$ is convex for $t \geqq 0$ and approaches zero as $t \to \infty$ implies that it is a covariance function [3]. A theorem of Maruyama [4] states that a Gaussian process is mixing if and only if its covariance function vanishes as $t \to \infty$, therefore $X(t)$ is mixing. Theorem 3 implies that $H_X(T)$ decreases slower than an exponential.

The process $X(t)$ is also purely nondeterministic. Its spectral density is given by

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(t\lambda)\rho(t)\,dt = \frac{1}{\pi}\int_0^\infty \cos(t\lambda)(1 + t)^{-\alpha}\,dt \\ &= \frac{1}{\pi|\lambda|^{1-\alpha}}\int_0^\infty \frac{\cos z\,dz}{(|\lambda| + z)^\alpha} = \frac{\alpha(\alpha+1)}{\pi|\lambda|^{1-\alpha}}\int_0^\infty \frac{(1 - \cos z)\,dz}{(|\lambda| + z)^{\alpha+2}}. \end{aligned}$$

Since the last integral is positive definite, $f(\lambda)$ is monotone decreasing in $|\lambda|$, varies as $|\lambda|^{-1+\alpha}$ as $|\lambda| \to 0$ and as $|\lambda|^{-2}$ as $|\lambda| \to \infty$. Therefore

$$\int_{-\infty}^{+\infty} |\log f(\lambda)|\,d\lambda/(1 + \lambda^2) < \infty$$

and the process is purely nondeterministic [2].

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