A MARKOV PROCESS ON BINARY NUMBERS[1]

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1. Introduction. This paper is a study of a certain discrete parameter Markov process in the interval of real numbers $(0, 1]$. The process is defined by choosing an arbitrary number X_0 in $(0, 1]$, specifying the distribution of a random variable X_1, given X_0, specifying the distribution of a random variable X_2, given X_1, and so on. Let the number X_0 have the binary expansion $X_0 = .\delta_1^{(0)}\delta_2^{(0)} \cdots \delta_n^{(0)} \cdots$, where $\{\delta_n^{(0)}, n = 1, 2 \cdots\}$ is a sequence of zeros and ones. The distribution of a random variable X_1 is determined by the joint distribution of the sequence of digits in its binary expansion, which we now give. Let $\{\delta_n^{(1)}\}$ be a sequence of digits in the binary expansion of X_1. We attribute to the random digit $\delta_1^{(1)}$ a Bernoulli distribution with mean p, that is $\delta_1^{(1)}$ assumes the values 0 and 1 with probabilities $q = 1 - p$ and p, respectively; if $\delta_k^{(0)} = 1$, we set $\delta_{k+1}^{(1)} = 1$, $k = 1, 2 \cdots$; if $\delta_k^{(0)} = 0$, then we give $\delta_{k+1}^{(1)}$ the Bernoulli distribution with mean p, $k = 1, 2, \cdots$. The random digits $\{\delta_n^{(1)}\}$ are assumed to be mutually independent, so that their joint distribution is completed defined. The distribution of X_2, given (the binary expansion of) X_1, is constructed by the same procedure; thus, an entire sequence $\{X_n\}$ is generated. The binary expansion of X_n is written as $.\delta_1^{(n)}\delta_2^{(n)} \cdots$, $n = 1, 2, \cdots$. It is clear that $\{X_n\}$ is a Markov chain with the state space $(0, 1]$. An initial distribution for the chain is introduced by assigning a distribution to (the digits in the binary expansion of) X_0.

In what follows, a binary expansion which terminates after a finite number of digits 1 will always be written in its non-terminating form, i.e., with an uninterrupted sequence of digits 1 after a finite number of digits 0. For example, the number $.100 \cdots$ will be written as $.011 \cdots$. No ambiguities arise in the transition from one random variable to its successor. If X_n has infinitely many digits 1, so must X_{n+1} by its definition; hence, if X_0 is written with infinitely many digits 1, so is every term in the sequence $\{X_n\}$.

In Section 2, a stationary distribution is constructed and shown to be unique. The absolute distribution of X_n converges only weakly to the stationary distribution, but does not converge over every Borel set of the space $(0, 1]$. The strong law of large numbers holds in a restricted form for $\{X_n\}$; it is shown that $(n + 1)^{-1}\sum_{j=0}^{n} f(X_j)$ converges with probability 1 to the integral of f with respect to the stationary distribution for every continuous f, but not for every measurable function f.

In Section 3, the states of the chain are classified; Section 4 treats a first passage time problem; Section 5 treats an absorption problem; and Section 6 contains applications to an epidemiological problem.

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2. The stationary distribution. The stationary distribution will now be constructed. Let $\{\xi_n, n = 1, 2, \cdots\}$ be a sequence of independent Bernoulli random variables with means $E\xi_n = 1 - q^n$, $n = 1, 2, \cdots$, and let the random variable ξ be defined as $\xi = \sum_{n=1}^{\infty} \xi_n \cdot 2^{-n} = .\xi_1\xi_2 \cdots \xi_n \cdots$. Let P_0 be the distribution of ξ.

THEOREM 2.1. *The distribution P_0 is a stationary distribution for $\{X_n\}$. It assigns probability* 1 *to the set R of binary numbers in* $(0, 1]$ *having only a finite number of digits* 0; *each element in R is assigned positive probability.*

PROOF. We show that if X_0 has the distribution P_0, so does X_1. The random digit $\delta_1^{(1)}$ has a Bernoulli distribution with mean p for any value of X_0; the distribution of $\delta_{k+1}^{(1)}$, given X_0, is a Bernoulli distribution with mean $p(1 - \delta_k^{(0)}) + \delta_k^{(0)}$. Under the distribution P_0, $\delta_k^{(0)}$ has the Bernoulli distribution with mean $1 - q^k$, so that by the total probability formula, $\delta_{k+1}^{(1)}$ has the Bernoulli distribution with mean $1 - q^{k+1}$. But this implies that X_1 has the distribution P_0, so that P_0 is stationary.

Let $P_0\{A\}$ denote the probability of the event A under the distribution P_0; then

$$\sum_{n=1}^{\infty} P_0\{X_0 \text{ has a 0 for its } n\text{th binary digit}\}$$
$$= \sum_{n=1}^{\infty} P_0\{\delta_n^{(0)} = 0\} = \sum_{n=1}^{\infty} q^n = q/(1 - q) < \infty,$$

and it follows from the Borel-Cantelli Lemma that with probability 1 $\delta_n^{(0)}$ has the value 0 for only a finite number of values of n. This affirms the second assertion of the theorem.

Let x be an element of R; there is then an integer $m \geqq 1$ such that the $(m - 1)$st digit in the binary expansion of x is 0, while digits from the mth on are all 1. It is easily seen that $P_0\{X_0 = x\}$ is a positive multiple of $\prod_{j=m}^{\infty} (1 - q^j)$, which is a convergent infinite product. This proves that every element of R is assigned positive probability by P_0.

Let $G(x)$ be the distribution function on the real line corresponding to P_0. Since R is dense in $(0, 1]$, $G(x)$ has an infinite number of discontinuities in every open subinterval of $(0, 1]$, and is a pure step function.

THEOREM 2.2. *Let $F_n(x)$ be the absolute distribution function of X_n, i.e., $F_n(x) = P\{X_n \leqq x\}$, $n = 1, 2, \cdots$. Then, for each x in the continuity set of $G(x)$, the relation $\lim_{n\to\infty} F_n(x) = G(x)$ holds for any initial distribution of X_0.*

PROOF. The random variable X_n is of the form

$$X_n = \sum_{j=1}^{n} \delta_j^{(n)} \cdot 2^{-j} + \sum_{j=n+1}^{\infty} \delta_j^{(n)} \cdot 2^{-j}. \tag{1}$$

The first sum on the right side is distributed independently of X_0, and $\delta_j^{(n)}$ are independent Bernoulli random variables with means $1 - q^j$, $j = 1, 2, \cdots, n$; hence, the first sum differs by at most 2^{-n} from a random variable having the stationary distribution P_0. On the other hand, the second sum is at most 2^{-n}; therefore, our theorem follows from a well-known convergence theorem ([1], p. 254).

THEOREM 2.3. *The distribution P_0 is the unique stationary distribution for $\{X_n\}$.*

PROOF. Let P' be any stationary distribution and $G'(x)$ the corresponding distribution function: then by Theorem 2.2, $G'(x) = \lim_{n\to\infty} P'\{X_n \leqq x\} = G(x)$ for each x in the continuity set of $G(x)$. Since the continuity set is dense, and since $G'(x)$ is determined by its values on a dense set, $G(x)$ and $G'(x)$ must be identical, and, therefore, P_0 and P' must be the same.

Theorem 2.2. assures only the weak convergence of the absolute distribution to P_0, not convergence over every Borel set in (0, 1] for every initial distribution. We give a counterexample to the latter. Let R' be the complement of R, so that R' is the set of numbers in (0, 1] having infinitely many digits 0 and infinitely many digits 1 in their binary expansions. Now the strong law of large numbers implies that almost every sequence of independent repeated Bernoulli trials results in infinitely many successes and infinitely many failures. From this we conclude that if X_0 is in R', so is X_1, and, therefore, so is X_n for every $n \geqq 1$; hence, if the initial distribution assigns probability 1 to R', so does the absolute distribution of X_n for every n. On the other hand, the stationary distribution P_0 assigns probability 0 to R'; therefore, the absolute distributions are all singular with respect to the stationary one.

THEOREM 2.4. *Let $f(x)$ be any continuous function on* [0, 1]; then with probability 1,

$$\lim_{n\to\infty} (n+1)^{-1} \sum_{j=0}^{n} f(X_j) = \int_0^1 f(x)\, dG(x), \tag{2}$$

for any initial distribution of X_0.

PROOF. The conclusion of the theorem is known if X_0 has the stationary distribution ([2], p. 465). To prove the theorem for any initial distribution, we show that the limit in (2) exists and is independent of the initial distribution.

Now X_n depends on the initial distribution only through the second sum in (1), which is bounded by 2^{-n}. If n is large, then $(n+1)^{-1}\sum_{j=0}^{n} f(X_j)$ varies by only a small amount for different initial distributions since $f(x)$ is uniformly continuous and bounded; therefore, the limit exists and is the same as when the initial distribution is the stationary one.

Theorem 2.4 is not true for every measurable function $f(x)$ and every initial distribution. Let $f(x)$ be the indicator function of the set R' and let the initial distribution assign probability 1 to R'. It follows from the discussion of the counterexample to Theorem 2.2 that $f(X_n) = 1$ with probability 1 for every $n \geqq 0$, so that $\lim_{n\to\infty} (n+1)^{-1}\sum_{j=0}^{n} f(X_j) = 1$. On the other hand, $f(x)$ is equal to 0 almost everywhere with respect to P_0, so that $\int_0^1 f(x)\, dG(x) = 0$.

3. Classification of the states.

THEOREM 3.1. *The sets R and R' are closed sets of states. Every state in R' is transient, and every state in R is recurrent with a finite mean recurrence time.*

PROOF. The argument in Section 2 shows that X_{n+1} is in R if and only if X_n is in R; hence, R and R' are closed sets.

We shall prove that every state in R' is transient. For a given X_0 in R' consider the sequence of the ratios of the number of digits 0 in the binary expansions of X_1 and X_0,

$$\sum_{i=1}^{n} (1 - \delta_i^{(1)}) / \sum_{i=1}^{n} (1 - \delta_i^{(0)}).$$

The strong law of large numbers implies that, with probability 1, the sequence converges to q. The corresponding sequence of ratios for X_m and X_0, for $m \geqq 1$, converges to q^m with probability 1. Since the binary expansion is unique, X_m is equal to none of the X_j for $j \neq m$.

It is not hard to see that R is irreducible, so that $\{X_n\}$, restricted to R, is an irreducible Markov chain with a countable number of states; furthermore, this chain is aperiodic. The distribution P_0 is the stationary distribution for this chain; therefore, a well-known result on Markov chains asserts that each state in R is recurrent with a finite mean recurrence time ([3], p. 356).

4. A first passage time problem. Let B_k, $k \geqq 1$, denote the set of real numbers in $(0, 1]$ having a binary expansion with the digits 0 in the first $k - 1$ places, and a digit 1 in the kth place, i.e., numbers of the form $.0\,0 \cdots 0\,1\,\delta_{k+1}, \delta_{k+2}, \cdots$. Every number in $(0, 1]$ belongs to exactly one set B_k so that $\{B_k\}$ forms a decomposition of $(0, 1]$ into mutually exclusive sets. In this section we consider the distribution of the waiting time for the process to enter the set B_k, given that the process starts from a point of B_j, $j < k$. Explicit forms of the generating function and expected value of the waiting time are given.

LEMMA 4.1. *Let Q_M be a set in $(0, 1]$ determined by the first M digits in the binary expansion of its elements. Then the conditional probability*

$$P\{X_1 \not\in Q_M, X_2 \not\in Q_M, \cdots X_n \not\in Q_M, X_{n+1} \in Q_M \mid X_0 = x\}$$

depends only on the first $M - 1$ digits in the binary expansion of x for any $n \geqq 0$.

PROOF. The random digit $\delta_{k+1}^{(n+1)}$ is independent of all digits in X_n except $\delta_k^{(n)}$, $k \geqq 1$.

LEMMA 4.2. *Let $\{Y_n^k\}$ be the process derived from $\{X_n\}$ by considering only the first k digits in the expansion of each element of $\{X_n\}$, that is, $Y_n^k = \sum_{j=1}^{k} \delta_j^{(n)} \cdot 2^{-j}$, $n \geqq 0$. Then $\{Y_n^k, n \geqq 0\}$ is a Markov process.*

PROOF. This has the same proof as Lemma 4.1.

LEMMA 4.3. *For any state x in the set B_k, let T_k denote the first passage time into the set B_{k+1} from the state x. Then the distribution of T_k depends only on B_k but not on x.*

PROOF. The set B_{k+1} is determined by the first $k + 1$ digits of the binary expansion so that Lemma 4.1 applies.

LEMMA 4.4. *Let T_k denote the first passage time from (any state in the set) B_k to the set B_{k+1}. Let T_k' denote the recurrence time of the state 0 for the chain Y_n^k defined in Lemma 4.2, that is the value N such that $\{Y_0^k = 0;\ Y_i^k \neq 0, i = 1, \cdots, N - 1;\ Y_N^k = 0\}$. Then T_k and T_k' have the same distribution.*

PROOF. The probability that T_k assumes the value N is the conditional probability that there is at least one digit 1 among $\delta_1^{(n)}, \cdots, \delta_k^{(n)}$ for $n = 1, 2, \cdots,$

$N - 1$ and $\delta_j^{(N)} = 0$ for $j = 1, 2, \cdots k$, given that $\delta_1^{(0)} = 0, \delta_2^{(0)} = 0, \cdots \delta_{k-1}^{(0)} = 0, \delta_k^{(0)} = 1$. The probability that T_k' assumes the value N is the conditional probability of the same event given that $\delta_1^{(0)} = 0, \delta_2^{(0)} = 0, \cdots \delta_{k-1}^{(0)} = 0, \delta_k^{(0)} = 0$. But Lemma 4.1 implies that the two conditional probabilities are independent of $\delta_k^{(0)}$, so that they are identical.

LEMMA 4.5. *The first passage time from (any state in the set) B_j to B_k, $j < k$, is distributed as $T_j + T_{j+1} + \cdots + T_{k-1}$ where T_i, $i = j, \cdots k - 1$, are independently distributed.*

PROOF. If X_0 is in B_j and X_N is in B_k, then there exist integers N_i, $i = j + 1, \cdots k - 1$, such that $0 < N_{j+1} < N_{j+2} < \cdots < N_{k-1} < N$, where N_i is the smallest index n for which X_n is in B_i. We see that $N_{i+1} - N_i$ is distributed as T_i; we have to show the mutual independence of $N_{j+1}, N_{j+2} - N_{j+1}, \cdots, N - N_{k-1}$. It will be sufficient to show that $N_{j+2} - N_{j+1}$ and N_{j+1} are independent, as the proof of the general case is similar. Now we have

$$\begin{aligned} P\{N_{j+1} = h, N_{j+2} - N_{j+1} = m\} \\ &= P\{X_i \not\in B_{j+1}, i = 1, \cdots, h - 1; X_h \in B_{j+1}; X_i \not\in B_{j+2}, \\ &\qquad i = h + 1, \cdots, h + m - 1; X_{h+m} \in B_{j+2} \mid X_0 \in B_j\} \\ &= \int_{\{X_h \in B_{j+1}\}} P\{X_i \not\in B_{j+1}, i = 1, \cdots, h - 1; X_i \not\in B_{j+2}, \\ &\qquad\qquad i = h + 1, \cdots, h + m - 1; X_{h+m} \in B_{j+2} \mid X_h\}\, dP, \end{aligned}$$

where the last integration is performed with respect to the conditional distribution of X_h given X_0. By using the Markov property, we may write the integral as

$$\int_{\{X_h \in B_{j+1}\}} P\{X_i \not\in B_{j+1}, i = 1, \cdots, h - 1 \mid X_h\}.$$

$$P\{X_i \not\in B_{j+2}, i = h + 1, \cdots, h + m - 1; X_{h+m} \in B_{j+2} \mid X_h\}\, dP.$$

It follows from Lemma 4.3 that the second factor in the integrand is a constant equal to

$$P\{X_i \not\in B_{j+2}, i = h + 1, \cdots, h + m - 1; X_{h+m} \in B_{j+2} \mid X_h \in B_{j+1}\}$$

which, by the stationarity of the transition probabilities of $\{X_n\}$ is equal to

$$P\{X_i \not\in B_{j+2}, i = 1, \cdots, m - 1; X_m \in B_{j+2} \mid X_0 \in B_{j+1}\} = P\{T_{j+1} = m\}.$$

The first factor in the integrand is integrated to get the expression $P\{T_j = h\}$.

THEOREM 4.1. *The first passage time T_k from (any state in the set) B_k to the set B_{k+1} has the generating function*

$$F_k(s) = \frac{\sum_{n=1}^{k-1} q^{n(n-1)/2+n(k-n+1)} s^n + q^{k(k+1)/2} s^k (1 - s)^{-1}}{1 + \sum_{n=1}^{k-1} q^{n(n-1)/2+n(k-n+1)} s^n + q^{k(k+1)/2} s^k (1 - s)^{-1}} \tag{3}$$

and the expected value

$$ET_k = q^{-k(k+1)/2}. \tag{4}$$

The generating function of the first passage time from B_j to B_k, $j < k$, is $\prod_{i=j}^{k-1} F_i(s)$.

Proof. By Lemma 4.4, it is sufficient to find the generating function of T_k' for the chain Y_n^k. Let the sequence $\{u_n^{(k)}\}$ be defined as $u_n^{(k)} = P\{Y_n^k = 0 \mid Y_0^k = 0\}$, $n \geqq 1$; $u_0^{(k)} = 1$; let the sequence $\{f_n^{(k)}\}$ be defined as $f_n^{(k)} = P\{T_k' = n\}$, $n \geqq 1$; $f_0^{(k)} = 0$; and let the generating function $U_k(s)$ and $F_k(s)$ be defined as

$$U_k(s) = \sum_{n=0}^{\infty} U_n^{(k)} s^n; \qquad F_k(s) = \sum_{n=0}^{\infty} f_n^{(k)} s^n. \tag{5}$$

Now $u_1^{(k)}$ is the conditional probability that $\delta_i^{(1)} = 0$ for $i = 1, 2, \cdots k$, given that $\delta_i^{(0)} = 0$ for $i = 1, 2, \cdots k$; therefore, $u_1^{(k)} = q^k$. We note that $u_2^{(k)}$ is equal to

$$P\{\delta_i^{(2)} = 0, i = 1, 2, \cdots, k \mid \delta_i^{(0)} = 0, i = 1, 2, \cdots, k\} = q\cdot(q^2)^{k-1}.$$

In general, we have

$$\begin{aligned} u_n^{(k)} &= q^{n(n-1)/2+n(k-n+1)}, & 1 \leqq n \leqq k-1 \\ &= q^{k(k+1)/2}, & n \geqq k. \end{aligned} \tag{6}$$

The last relation implies that $\lim_{n\to\infty} u_n^{(k)} = q^{k(k+1)/2}$; application of a theorem in the theory of recurrent events gives us (4) ([3], p. 286).

It is clear from (5) and (6) that

$$U_k(s) = 1 + \sum_{n=1}^{k-1} q^{n(n-1)/2+n(k-n+1)} s^n + q^{k(k+1)/2} s^k (1-s)^{-1};$$

the formula for $F_k(s)$ follows from the known relation ([3], p. 285) $U_k(s) = [1 - F_k(s)]^{-1}$. The last assertion of the theorem follows from Lemma 4.5.

5. An absorption problem. Let A_N be the set of numbers in (0, 1] whose binary expansions have digits 0 in the first N places, that is, A_N is the interval $(0, 2^{-N}]$. Let D_N be the set of numbers in (0, 1] whose binary expansions have digits 1 in the first N places, that is, D_N is the interval $(1 - 2^{-N}, 1]$. Let E_N denote the event that the process $\{X_n\}$ enters the set A_N before entering D_N; and α_k is defined as

$$\alpha_k = \sup_{x\varepsilon B_k} P\{E_N \mid X_0 = x\}, \qquad k = 1, 2, \cdots N.$$

In this section, upper bounds are obtained for the numbers α_k.

Theorem 5.1. *For a fixed $N > 0$, let $C_1, C_2, \cdots, C_N$ be real numbers recursively defined as*

$$\begin{aligned} C_1 &= q[q + p^{N-1}]^{-1} \\ C_k &= q^k[1 - pq^{k-1} - C_{k-1}(1 - q^{k-1} - p^{N-1})]^{-1}, \quad k = 2, \cdots, N. \end{aligned}$$

Then α_k satisfies the inequality

$$\alpha_k \leqq \prod_{j=k}^{N} C_j. \tag{7}$$

PROOF. The following inequalities, which imply (7), will be derived:

$$(8) \qquad \alpha_k \leqq C_k\alpha_{k+1}, \qquad k = 1, \cdots, N-1; \qquad \alpha_N \leqq C_N .$$

We prove these by induction, starting with $k = 1$. If X_0 is in B_1, then the event E_N can occur in two different ways: either X_1 is in B_2, i.e., $\delta_1^{(1)} = 0$ and $\delta_2^{(1)} = 1$, or X_1 is in $B_1 - D_N$, i.e., $\delta_1^{(1)} = 1$ and $\delta_i^{(1)} = 0$ for some i, $3 \leqq i \leqq N$. The former event has probability q and the latter has probability at most equal to $p(1 - p^{N-2})$. These considerations lead to the inequality $\alpha_1 \leqq p(1 - p^{N-2})\alpha_1 + q\alpha_2$, and, therefore, $\alpha_1 \leqq q(q + p^{N-1})^{-1}\alpha_2$.

Suppose that (8) holds for an integer $k - 1$, $1 \leqq k - 1 \leqq N - 2$. If X_0 is in B_k, then E_N can occur in $k + 1$ mutually exclusive ways: X_1 is in $B_1 - D_N$, or X_1 is in B_2, $\cdots$, or X_1 is in B_{k+1}. It is easily seen, as in the case $k = 1$, that $P\{X_1 \varepsilon B_1 - D_N \mid X_0 \varepsilon B_k\} \leqq p(1 - p^{N-2})$, $P\{X_1 \varepsilon B_j \mid X_0 \varepsilon B_k\} = pq^{j-1}$, $j = 2, \cdots k$, $P\{X_1 \varepsilon B_{k+1} \mid X_0 \varepsilon B_k\} = q^k$. From these follows the inequality $\alpha_k \leqq p(1 - p^{N-2})\alpha_1 + p\sum_{j=2}^{k} q^{j-1}\alpha_j + q^k\alpha_{k+1}$.

Each number C_k is evidently less than 1; hence, by (8) and the induction hypothesis for $k - 1$,

$$(9) \qquad \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq C_{k-1}\alpha_k .$$

From (8) and (9) we get $\alpha_k \leqq [p(1 - p^{N-2}) + p\sum_{j=2}^{k-1} q^{j-1}]C_{k-1}\alpha_k + pq^{k-1}\alpha_k + q^k\alpha_{k+1}$, so that $\alpha_k \leqq q^k[1 - pq^{k-1} - C_{k-1}(1 - q^{k-1} - p^{N-1})]^{-1}\alpha_{k+1}$ and (8) holds for k. The inequality for α_N is proved in the same way.

THEOREM 5.2. *If $p > q^2$, then* $\lim_{n\to\infty} \max(\alpha_1, \alpha_2, \cdots, \alpha_N) = 0$.

PROOF. Since (9) holds, it suffices to show that $\alpha_N \to 0$, or $C_N \to 0$. Since $C_k \leqq 1$, we have $C_k \leqq q^k(q^k + p^{N-1})^{-1}$, $k = 1, \cdots N$. From this and the recursive relation for C_N, the inequality

$$C_N \leqq q^N/\{1 - pq^{N-1} - [q^{N-1}(1 - q^{N-1} - p^{N-1})/(p^{N-1} + q^{N-1})]\}$$

follows. The conclusion of the theorem follows by a simple passage to the limit.

Theorem 5.2 indicates that the process is highly likely to visit D_N before A_N if N is large and $p > q^2$.

6. Applications. This paper was motivated by the study of a simple model for the age distribution of a chronic disease in a population. Suppose there is a population of organisms, such that one new organism is born at each time unit, one dies at each time unit, and each organism lives a fixed number N of time units. At each time unit each living organism, including the one just born, has the probability p of contracting a certain incurable disease. If a given organism contracts the disease, then it has the disease during the remainder of its lifetime. If it does not contract the disease, then it again has probability p of contracting it at the next time unit, and so on until it dies. The contractions of the disease are stochastically independent for the different individuals.

Let $\delta_k^{(n)}$, $k = 1, \cdots N$, $n = 1, 2, \cdots$ be the indicator of the event that the organism of age k at time n has the disease. The sequence of random variables $Y_n^N = \sum_{k=1}^{N} \delta_k^{(n)}2^{-k}$ forms a Markov chain with a finite number of states. It is the process obtained by considering the first N digits in the binary expansion of each element of $\{X_n\}$. The first passage problem of Section 4 can be applied to determining the distribution of the waiting time until the population is free of disease. The absorption problem of Section 5 gives conditions under which it is very probable that the population will consist entirely of diseased individuals at some point before it is free of disease.

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