A WIENER-HOPF TYPE METHOD FOR A GENERAL RANDOM WALK WITH A TWO-SIDED BOUNDARY[1]

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1. Introduction. Let $\{z_n, n \geqq 0\}$ be a process of independent increments such that the increment $z_{n+h} - z_n$ has its distribution independent of n. Here, n and h run either through the nonnegative integers (discrete case) or through the nonnegative real numbers (continuous case). In the continuous case, the process $\{z_n\}$ is assumed to be separable, normalized in such a way that the sample functions are continuous to the right. Thus, in both cases the process is Markovian.

One has

$$E(e^{sz_n}) = E(e^{sz_0})E(e^{s(z_1-z_0)})^n = \hat{\sigma}(s)e^{n\theta(s)},$$

say. Let γ and v be real and *fixed* such that $\theta(\gamma) < v < \infty$. We shall be interested in finding explicit formulae for the generating functions

$$Q_1 = E(\{z_{N+0} \leqq 0\} \exp [sz_{N+0} - vN]), \qquad Q_2 = E(\{z_{N+0} \geqq c\} \exp [sz_{N+0} - vN]).$$

Here, $N = \inf \{n: z_n \not\varepsilon [0, c]\}$, while c denotes a fixed positive number. Further, s denotes a variable complex number with Re $(s) = \gamma$. In the discrete case, we shall usually write $e^{-v} = t$ and $e^{\theta(s)} = \varphi(s)$, thus, $\varphi(\gamma) < t^{-1}$.

As will be shown, such explicit formulae can be found in a large number of important special cases, where the *downward* jumps of the process $\{z_n\}$ are of such a simple type that the function $Q_1(s)$ is a priori known up to a finite number of parameters $a_1, \cdots, a_r$, (each depending on v or t but not on s).

In order to determine these parameters, we consider

$$Q_0(s) = \sum_{n=0}^{\infty} t^n E(e^{sz_n}\{N > n\})$$

in the discrete case, and $Q_0(s) = \int_0^\infty e^{-vn}E(e^{sz_n}\{N > n\})dn$ in the continuous case.

If B is a given Borel subset of the reals we denote by $M(B)$ the class of all complex-valued regular Borel measures μ supported by B such that the integral

$$\hat{\mu}(s) = \int_{-\infty}^{+\infty} e^{sy}\mu(dy), \qquad \text{Re } (s) = \gamma,$$

is absolutely convergent. The corresponding class of transforms $\hat{\mu}$ is denoted as $\hat{M}(B)$. Note that $Q_0 = \hat{\eta}$, where $\eta(D) = \sum_{n=0}^{\infty} t^n \Pr(z_n \varepsilon D, N > n)$ in the discrete case, similarly for the continuous case. Because $N > n$ implies $0 \leqq z_n \leqq c$, we have that the measure η is supported by the interval $[0, c]$, in other words $Q_0 \varepsilon \hat{M}([0, c])$.

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It turns out that $Q_0(s)$ can always be expressed in terms of $Q_1(s)$. Thus, $Q_0 \,\varepsilon\, \hat{M}([0, c])$ is a condition on Q_1 which in many cases leads to a non-singular system of r linear equations in the r unknown parameters a_j mentioned above; see Section 7.

In this approach, the main problem is to derive useful explicit formulae for Q_0 in terms of Q_1. In principle, this is done by means of the following Wiener-Hopf type technique. (As Professor Kesten kindly pointed out, a related method was employed by Widom [9] in handling the symmetric stable process with a two-sided boundary.)

It follows from the Markovian character of the process $\{z_n\}$ that

$$\hat{\sigma}(s)R(s) = Q_0(s) + (Q_1(s) + Q_2(s))R(s) \quad \text{when } \operatorname{Re}(s) = \gamma. \tag{1.1}$$

Here, $R(s) \neq 0$ is defined by $R(s) = (1 - t\varphi(s))^{-1}$ in the discrete case, and by $R(s) = (v - \theta(s))^{-1}$ in the continuous case. Now suppose that we can find functions $U^-(s)$ and $U^+(s)$ (not depending on $\hat{\sigma}$ or c) defined for $\operatorname{Re}(s) = \gamma$, such that

$$U^-(s)U^+(s) = R(s), \qquad (\operatorname{Re}(s) = \gamma), \tag{1.2}$$

and

$$Q_0/U^- \,\varepsilon\, \hat{M}((-\infty, c]), \qquad Q_2U^+ \,\varepsilon\, \hat{M}((c, \infty)). \tag{1.3}$$

Dividing (1.1) by $U^-(s)$ and using (1.2), we have $Q_0/U^- + Q_2U^+ = (\hat{\sigma} - Q_1)U^+$. Invoking (1.3) this gives

$$Q_0/U^- = [(\hat{\sigma} - Q_1)U^+]^{(-\infty, c]}, \tag{1.4}$$

which is the desired expression of Q_0 in terms of Q_1. Here, if $\hat{\mu} = \hat{\mu}(s)$ is the transform of a measure μ then for any Borel set A

$$[\hat{\mu}]^A = [\hat{\mu}(\cdot)]^A(s) = \int_A e^{sy}\mu(dy);$$

in other words, $[\hat{\mu}]^A$ denotes the transform of the A-truncation of μ.

The $U^\pm$ which we shall construct satisfy not only (1.2) and (1.3) but also

$$Q_0/U^+ \,\varepsilon\, \hat{M}([0, \infty)), \qquad Q_1U^- \,\varepsilon\, \hat{M}((-\infty, 0)).$$

Dividing (1.1) by U^+, one obtains $Q_1U^- = [(\hat{\sigma} - Q_2)U^-]^{(-\infty, 0)}$, $Q_0/U^+ = [(\hat{\sigma} - Q_2)U^-]^{[0, \infty)}$. For the case of a one-sided boundary ($c = \infty$, $Q_2 = 0$), this yields the explicit formulae $Q_1 = (1/U^-)[\hat{\sigma}U^-]^{(-\infty, 0)}$, $Q_0 = U^+[\hat{\sigma}U^-]^{[0, \infty)}$, which were already employed in one form or another by Andersen, Baxter, Cramér, Donsker, Pollaczek, Spitzer, Täcklind, Wendel and the author, see [6]. The approach of the present paper is probably closest to the one of Wendel [7].

Let us first consider the discrete case. Then the required pair $U^\pm$ is given by $U^-(s) = e^{L^-(s)}$, $U^+(s) = e^{L^+(s)}$, with $L^\pm$ as in (3.11), (3.12). Invoking the condition $Q_0 \,\varepsilon\, \hat{M}([0, c])$, (1.4) leads to the central Theorem 3.1.

If the distribution function $F(y)$ of the jumps $X_n = z_n - z_{n-1}$ is for $y < 0$

of the exponential type (6.2) or (6.6), (X_n integral valued in the latter case), then a simple formula for $U^{\pm}$ is furnished by the auxiliary results of Section 6. Moreover, in these cases, $Q_1(s)$ is a priori known up to a finite number of parameters. The resulting implications of Theorem 3.1 are discussed in Section 7.

In Section 4, proceeding directly from Theorem 3.1, we derive in full detail the explicit formulae for the Q_i in the special case that $F(y) = de^{\beta y}$ for $y < 0$, (d and β constant, $F(y)$ arbitrary for $y \geqq 0$). This illustration should be a sufficient guide for a reader interested in some of the other cases covered by our method.

As to the continuous-time case, we have restricted ourselves to the Poisson process of independent increments as defined by (5.2). Then $\theta(s) = qs + \lambda(\varphi(s) - 1)$, where q is the so-called trend of the process, while $\varphi(s) = E(e^{sX_k})$ determines the distribution of the jumps X_k of the process $\{z_n, 0 \leqq n < \infty\}$. In this case the required functions $U^-(s)$ and $U^+(s)$ (with $v = \alpha - \lambda$) are given by (5.13), (5.16), (5.19) and (5.20). In this way, (1.4) leads to the central Theorem 5.2 for the continuous case.

As shown in Section 7, exact formulae for the Q_i are easily obtained when the distribution function $F(y)$ of the jumps X_k is for $y < 0$ of the exponential type (6.2). Full details are given at the end of Section 5, for the special case of a downward trend ($q < 0$) and positive jumps X_k.

2. Preliminaries. In this paper, except for Section 5, $\{z_n\}$ will denote a given random walk on the reals with a discrete time parameter $n = 0, 1, \cdots$. Thus, the increment $X_n = z_n - z_{n-1}$ is independent of the z_m with $m \leqq n - 1$ and has its distribution

$$\nu(B) = \Pr(X_n \varepsilon B) \tag{2.1}$$

independent of n, ($n = 1, 2, \cdots$; B denoting any Borel subset of the reals). Here, ν is a probability measure, thus, its Laplace transform

$$\varphi(s) = \int_{-\infty}^{+\infty} e^{sy}\nu(dy) \tag{2.2}$$

is defined at least for Re $(s) = 0$.

In the sequel, t will denote a *fixed* real and positive number. We shall assume that there exists a real number γ such that

$$\varphi(\gamma) < \infty, \qquad t\varphi(\gamma) < 1. \tag{2.3}$$

If $t < 1$ one can always choose $\gamma = 0$. The case $t = 1$ is of particular interest; here, we require that $\varphi(\gamma) < 1$ for some real number γ.

In the sequel, γ will denote a *fixed* real number satisfying (2.3). Further, unless otherwise stated, s will denote a variable complex number satisfying Re $(s) = \gamma$. Finally, M denotes the collection of all complex-valued regular Borel measures μ on the reals such that $\|\mu\| = \int_{-\infty}^{+\infty} e^{\gamma y}|\mu(dy)| < \infty$. One has $\|\nu\| = \varphi(\gamma)$ because ν is nonnegative. Further, $\|\mu_1 * \mu_2\| \leqq \|\mu_1\|\|\mu_2\|$, (a star denoting convolution), thus, M is in fact a Banach algebra.

To each $\mu \,\varepsilon\, M$ we associate the function (transform)

$$\hat{\mu} = \hat{\mu}(s) = \int_{-\infty}^{+\infty} e^{sy}\mu(dy), \qquad (\mathrm{Re}(s) = \gamma). \tag{2.4}$$

Further, we introduce

$$\hat{M} = \{\hat{\mu}: \mu \,\varepsilon\, M\}. \tag{2.5}$$

As is well-known, (2.4) establishes a 1:1 correspondence between M and $\hat{M}$ such that $(\mu_1 * \mu_2)^\wedge = \hat{\mu}_1\hat{\mu}_2$.

Let B be a given Borel measurable subset of the reals and let B' denote its complement. Then $M(B)$ will denote the set of all measures $\mu \,\varepsilon\, M$ which are supported by B, (that is, satisfy $\mu(D) = 0$ for each Borel subset D of B'), similarly $M(B')$. Each $\mu \,\varepsilon\, M$ admits a *unique* decomposition as a sum $\mu = [\mu]^B + [\mu]^{B'}$ with $[\mu]^B \,\varepsilon\, M(B)$, $[\mu]^{B'} \,\varepsilon\, M(B')$. Here, $[\mu]^B(A) = \mu(A \cap B)$ is precisely the restriction of μ to B, similarly $[\mu]^{B'}$.

Let further $\hat{M}(B) = \{\hat{\mu}: \mu \,\varepsilon\, M(B)\}$, similarly $\hat{M}(B')$. Then, correspondingly, each $\hat{\mu} \,\varepsilon\, \hat{M}$ admits a *unique* decomposition as a sum

$$\hat{\mu} = [\hat{\mu}]^B + [\hat{\mu}]^{B'} \quad \text{with} \quad [\hat{\mu}]^B \,\varepsilon\, \hat{M}(B),\ [\hat{\mu}]^{B'} \,\varepsilon\, \hat{M}(B'). \tag{2.6}$$

Here, if $\hat{\mu}$ is the transform of the measure $\mu \,\varepsilon\, M$ then

$$[\hat{\mu}]^B = \int_B e^{sy}\mu(dy), \tag{2.7}$$

similarly, $[\hat{\mu}]^{B'}$. The truncation (2.7) will be frequently used in this paper. As an important illustration,

$$[\hat{\mu}_1\, \hat{\mu}_2\, e^{sx}]^B = \iint\limits_{x+y+z \,\varepsilon\, B} e^{s(x+y+z)}\mu_1(dy)\mu_2(dz). \tag{2.8}$$

In the sequel, when we say that a function $f(s)$, defined on a subset H of the complex s-plane, is *analytic* in H, we shall mean that $f(s)$ is continuous at each point $s_0 \,\varepsilon\, H$ and analytic at each interior point $s_0 \,\varepsilon\, H$.

Thus, if $\hat{\mu} \,\varepsilon\, \hat{M}(B)$ and B is bounded on the left then (2.4) defines a unique extension of $\hat{\mu}(s)$ to an analytic function in the closed half plane $\mathrm{Re}\,(s) \leqq \gamma$. This extension is bounded if $B \subset [0, \infty)$ and tends to 0 (as $\mathrm{Re}\,(s) \to -\infty$) if $B \subset (0, \infty)$.

LEMMA 2.1. *Let $\mu \,\varepsilon\, M$ be given. Then $\mu \,\varepsilon\, M([0, \infty))$ as soon as the function $\hat{\mu}(s)$, $\mathrm{Re}(s) = \gamma$, can be continued to a bounded and analytic function in the half plane $\mathrm{Re}(s) \leqq \gamma$. One has $\mu \,\varepsilon\, M((0, \infty))$ if, in addition, $\hat{\mu}(s)$ tends to 0 as $\mathrm{Re}(s)$ tends to $-\infty$.*

PROOF. Consider the function of bounded variation defined by

$$F(z) = \int_{-\infty}^{z-} e^{\gamma y}\mu\,(dy) + e^{\gamma z}\mu(\{z\})/2.$$

It follows from the Lévy inversion formula that

$$F(z) - F(y) = \lim_{T\to\infty} (2\pi i)^{-1} \int_{-iT}^{+iT} \hat{\mu}(\gamma + s)(e^{-ys} - e^{-zs})/s\, ds.$$

On replacing the latter integration path by the arc: $|s| = T$, $\mathrm{Re}(s) \leqq 0$, one easily obtains the stated assertion.

3. A two-sided boundary. Consider the random walk $\{z_n\}$ defined by

$$z_n = z_0 + X_1 + \cdots + X_n, \qquad (n = 0, 1, \cdots), \tag{3.1}$$

where $z_0, X_1, X_2, \cdots$ are independent random variables such that (2.1) holds. The distribution of z_0 will be denoted as

$$\sigma(B) = \Pr(z_0 \,\varepsilon\, B). \tag{3.2}$$

Thus, the random walk $\{z_n\}$ is completely specified by the pair of probability measures ν and σ.

Choose further a fixed positive number c, and let N denote the smallest integer $n = 0, 1, 2, \cdots$, such that $z_n \not\varepsilon [0, c]$; (the following method would work just as well if $[0, c]$ is replaced by $(0, c)$, $[0, c)$ or $(0, c]$). One has $N < \infty$ with probability 1, except for the trivial case $X_n \equiv 0$.

We may and will assume that $z_0 \,\varepsilon\, [0, c]$, that is,

$$\hat{\sigma} \,\varepsilon\, \hat{M}([0, c]). \tag{3.3}$$

In most applications, $z_0 = x$ is non-random, thus, $\hat{\sigma}(s) = e^{sx}$ with $0 \leqq x \leqq c$.

Let $\varphi(s)$, γ and t be as in (2.2) and (2.3). We shall be interested in finding explicit formulae for the generating functions

$$Q_0(s) = \sum_{n=0}^{\infty} t^n E(e^{sz_n}\{N > n, 0 \leqq z_n \leqq c\}), \tag{3.4}$$

(actually, $N > n$ implies $0 \leqq z_n \leqq c$), and

$$Q_1(s) = \sum_{n=1}^{\infty} t^n E(e^{sz_n}\{N = n, z_n < 0\}), \tag{3.5}$$

$$Q_2(s) = \sum_{n=1}^{\infty} t^n E(e^{sz_n}\{N = n, z_n > c\}); \tag{3.6}$$

(in view of (3.3), $N = 0$ does not happen). By (3.1), one has $E(e^{\gamma z_n}) = \hat{\sigma}(\gamma)\varphi(\gamma)^n$, thus, by (2.3), the above series are absolutely convergent for $\mathrm{Re}\,(s) = \gamma$. Moreover, letting

$$B_0 = [0, c], \qquad B_1 = (-\infty, 0), \qquad B_2 = (c, \infty), \tag{3.7}$$

one has

$$Q_i \,\varepsilon\, \hat{M}(B_i) \subset \hat{M}(B_j') \quad \text{if} \quad i \neq j, \ (i, j = 0, 1, 2). \tag{3.8}$$

Finally,

$$(1 - t\varphi)Q_0 = \hat{\sigma} - Q_1 - Q_2 . \tag{3.9}$$

For, if Q_{in} denotes the coefficient of t^n in the expansion of Q_i then, by $z_n = z_{n-1} + X_n$, where X_n is independent of z_{n-1} and the event $N > n - 1$,

$$Q_{0n} + Q_{1n} + Q_{2n} = E(e^{sz_n}\{N > n - 1\}) = \varphi(s)Q_{0,n-1} \quad \text{if} \quad n > 0,$$
$$= \hat{\sigma}(s) \qquad \text{if} \quad n = 0.$$

Multiplying by t^n and summing over $n \geqq 0$, one obtains (3.9).

Let us now show that *the relations* (3.8) *and* (3.9) *together determine the* Q_i *uniquely*.

For, suppose they hold with $\hat{\sigma}$ replaced by 0. Then the right hand side of (3.9) is in $\hat{M}(B_0')$, thus, $t[\varphi Q_0]^{B_0} = [Q_0]^{B_0} = Q_0 = \hat{\mu}$, say. Hence, by (2.2), where ν is nonnegative, $\|\mu\| \leqq t\|\nu * \mu\| \leqq t\varphi(\gamma)\|\mu\|$. But $t\varphi(\gamma) < 1$, thus, $Q_0 = 0$, thus, $Q_1 = Q_2 = 0$ by (3.8) and (3.9).

Returning to the original relation (3.9), note that here Q_0, Q_2 and $\hat{\sigma}$ are in $\hat{M}([0, \infty))$, (see (3.3), (3.7) and (3.8)). It follows that

$$Q_1 = [t\varphi Q_0]^{(-\infty,0)}, \qquad \text{where} \quad Q_0 \,\varepsilon\, \hat{M}([0, \infty)). \tag{3.10}$$

In many cases, see (6.5) and (6.11), this implies that $Q_1(s)$ is known up to a finite number of parameters.

In order to express Q_0 and Q_2 in terms of Q_1, let us introduce, (as we may by $t\varphi = (t\nu)^\wedge$, $\|t\nu\| = t\varphi(\gamma) < 1$),

$$L^- = L^-(s) = \sum_{n=1}^{\infty} (t^n/n)[\varphi^n]^{(-\infty,0)}, \tag{3.11}$$

and

$$L^+ = L^+(s) = \sum_{n=1}^{\infty} (t^n/n)[\varphi^n]^{[0,\infty)}. \tag{3.12}$$

Clearly, $L^- \,\varepsilon\, \hat{M}((-\infty, 0))$. Hence, $(-\infty, 0)$ being a *semigroup* under addition,

$$e^{\pm L^-} - 1 = \sum_{n=1}^{\infty} (\pm L^-)^n/n! \,\varepsilon\, \hat{M}((-\infty, 0)). \tag{3.13}$$

Similarly,

$$e^{\pm L^-} \,\varepsilon\, \hat{M}((-\infty, 0]), \; e^{\pm L^+} \,\varepsilon\, \hat{M}([0, \infty)). \tag{3.14}$$

Finally, adding (3.11) and (3.12), we have the fundamental relation

$$1 - t\varphi(s) = \exp[-L^-(s) - L^+(s)], \qquad \operatorname{Re}(s) = \gamma. \tag{3.15}$$

As a consequence,

$$e^{-L^-} - 1 + t\varphi e^{L^+} = e^{L^+} - 1 \,\varepsilon\, \hat{M}([0, \infty)),$$

by (3.14). Hence, by (3.13) and (3.14),

$$e^{-L^-} - 1 = [-t\varphi e^{L^+}]^{(-\infty,0)}, \qquad \text{where} \quad e^{L^+} \varepsilon \hat{M}([0, \infty)), \tag{3.16}$$

which is of the same type as (3.10) and often permits us to derive a simple formula for e^{-L^-} and hence for

$$e^{-L^+} = (1 - t\varphi)e^{L^-}. \tag{3.17}$$

Multiplying (3.9) by e^{L^+} and using (3.15), one obtains

$$Q_0 e^{-L^-} + Q_2 e^{L^+} = (\hat{\sigma} - Q_1)e^{L^+}. \tag{3.18}$$

Here, by (3.8) and (3.14),

$$Q_0 e^{-L^-} \varepsilon \hat{M}((-\infty, c]), \qquad Q_2 e^{L^+} \varepsilon \hat{M}((c, \infty)).$$

Consequently, (by (2.6) with $B = (-\infty, c]$), these must be equal to

$$[(\hat{\sigma} - Q_1)e^{L^+}]^{(-\infty,c]} \quad \text{and} \quad [(\hat{\sigma} - Q_1)e^{L^+}]^{(c,\infty)},$$

respectively, in other words,

$$Q_0 = e^{L^-}[(\hat{\sigma} - Q_1)e^{L^+}]^{(-\infty,c]} \tag{3.19}$$

and

$$Q_2 = e^{-L^+}[(\hat{\sigma} - Q_1)e^{L^+}]^{(c,\infty)}. \tag{3.20}$$

THEOREM 3.1. *The function* $Q_1 = Q_1(s)$, $(\mathrm{Re}(s) \geqq \gamma)$, *defined by* (3.5), *may be characterized as the unique function* $Q_1(s) \varepsilon \hat{M}((-\infty, 0))$ *with the property that the function* $Q_0(s)$, *defined by* (3.19) *for* $\mathrm{Re}\ (s) \geqq \gamma$, *can be extended to a bounded and analytic function in the half plane* $\mathrm{Re}\ (s) \leqq \gamma$.

PROOF. Necessity. If Q_1 is defined by (3.5) then the function Q_0 defined by (3.4) satisfies (3.19). Moreover, $Q_0 \varepsilon \hat{M}([0, c])$, thus, $Q_0(s)$ is an entire function which is bounded in each left half plane.

Sufficiency. Let $Q_1 \varepsilon \hat{M}(-\infty, 0))$ have the property mentioned. Define further Q_0 by (3.19) and Q_2 by (3.20). Thus, (3.18) holds which, by (3.15), is equivalent to (3.9). By (3.14) and (3.20), one has $Q_2 \varepsilon \hat{M}((c, \infty))$. Further, by (3.14) and (3.19), one has $Q_0 \varepsilon \hat{M}((-\infty, c])$, in fact, $Q_0 \varepsilon \hat{M}([0, c])$ as follows by Lemma 2.1. In other words, the Q_i satisfy (3.8) and (3.9) and must therefore coincide with the functions defined by (3.4), (3.5) and (3.6).

4. An illustration. In this section, we assume that the distribution function $F(y) = \nu\{z: z \leqq y\}$ of the X_n satisfies

$$F(y) = de^{\beta y} \quad \text{if} \quad y < 0, \tag{4.1}$$

with $d \leqq 1$ and β as positive constants; (the behavior of $F(y)$ for $y \geqq 0$ remains unrestricted). As before, let γ real and $t > 0$ be fixed such that

$$-\beta < \gamma, \qquad \varphi(\gamma) < t^{-1} < \infty. \tag{4.2}$$

Here, by (4.1),

$$\varphi(s) = d(1 + s/\beta)^{-1} + \int_{0-}^{\infty} e^{sy}\, dF(y). \tag{4.3}$$

This formula yields an extension of $\varphi(s)$ to an analytic and bounded function in $\mathrm{Re}(s) \leqq \gamma$ with the exception of a simple pole at $s = -\beta$. Also note that the last term in (4.3) is in absolute value smaller than $\varphi(\gamma) < t^{-1}$ throughout $\mathrm{Re}(s) \leqq \gamma$.

Finally, let $\xi = \xi(t)$ denote the unique real number such that

$$\varphi(\xi) = t^{-1}, \qquad -\beta < \xi < \gamma; \tag{4.4}$$

(if $t = 1$ then $-\beta < \xi < \gamma < 0$ when $0 < E(X_n) \leqq \infty$, while $\xi = 0$ and $\gamma > 0$ when $E(X_n) < 0$).

As follows easily by (2.8), if $\hat{\mu} \varepsilon \hat{M}([0, \infty))$ then $[(s + \beta)^{-1}\hat{\mu}]^{(-\infty,0)} = \hat{\mu}(-\beta)(s + \beta)^{-1}$, thus, by (4.3), $[\varphi\hat{\mu}]^{(-\infty,0)} = d\hat{\mu}(-\beta)(1 + s/\beta)^{-1}$. Consequently, by (3.10) and (3.16),

$$Q_1(s) = a(1 + s/\beta)^{-1} \tag{4.5}$$

and $e^{-L^-(s)} = (s + b)(s + \beta)^{-1}$, where

$$a = Q_1(0) = \sum_{n=1}^{\infty} t^n \Pr(N = n, z_n < 0), \tag{4.6}$$

(cf., (3.5)), and b are as yet unknown constants, (depending on t). By (3.15), $(1 - t\varphi)^{-1}e^{-L^-} = e^{L^+}$; here, $e^{L^+(s)}$ is analytic for $\mathrm{Re}(s) \leqq \gamma$. Using (4.4), it follows that

$$e^{-L^-(s)} = (s - \xi)(s + \beta)^{-1} \tag{4.7}$$

and

$$e^{L^+(s)} = [(s - \xi)/(s + \beta)](1 - t\varphi(s))^{-1} \varepsilon \hat{M}([0, \infty)). \tag{4.8}$$

For convenience, let us assume that z_0 is nonrandom, thus,

$$z_0 = x, \quad \text{where} \quad 0 \leqq x \leqq c. \tag{4.9}$$

Further, $\hat{\sigma} = e^{sx}$, thus, by (4.5) and (4.6), (3.19) reduces to

$$Q_0(s) = \frac{s + \beta}{s - \xi}\left[\left\{e^{sx} - \frac{a\beta}{s + \beta}\right\} e^{L^+(s)}\right]^{(-\infty,c]}.$$

By (4.8), this may be written as

$$Q_0(s) = e^{sc}[(s + \beta)/(s - \xi)](\tau_1(c - x, s) - a\beta\tau_2(c, s)), \tag{4.10}$$

where

$$\tau_j(u, s) = e^{-su}\left[\frac{s - \xi}{(s + \beta)^j}\,\frac{1}{1 - t\varphi(s)}\right]^{(-\infty,u]}, \tag{4.11}$$

$$(j = 1, 2).$$

By Theorem 3.1, the constant a is uniquely determined by the condition that $Q_0(s)$, as defined by (4.10), is analytic and bounded in Re $(s) \leqq \gamma$. In view of $[\hat{\mu}]^{(-\infty,u]} = \hat{\mu} - [\hat{\mu}]^{(u,\infty)}$, (4.8) and (4.11), we have that in (4.10) both $e^{sc}\tau_1(c - x, s)$ and $e^{sc}\tau_2(c, s)$ are analytic and bounded for Re $(s) \leqq \gamma$, (except that the latter has a simple pole at $s = -\beta$; however, in the product (4.10) this pole is canceled by the factor $s + \beta$). In view of the factor $(s - \xi)^{-1}$ in (4.10), it follows that the constant a is *uniquely* determined by the condition that $\tau_1(c - x, \xi) - a\beta\tau_2(c, \xi) = 0$. Therefore, $\tau_2(c, \xi) \neq 0$ and, moreover,

$$a = \beta^{-1}\tau_1(c - x, \xi)/\tau_2(c, \xi). \tag{4.12}$$

By the way, in the present case, where t is real and positive, $\tau_2(c, \xi) \neq 0$ also follows from the fact that ξ is real and that τ_2 corresponds to a nonvanishing nonnegative measure, compare (4.8) and (4.11). On the other hand, the above type of reasoning easily carries over to complex values of t, $0 < |t| < \varphi(\gamma)^{-1}$.

It remains to derive more explicit formulae for the $\tau_j(u, s)$. First, as follows easily by (2.8) and Re $(\beta) + \gamma > 0$, we have for each $\hat{\mu} \,\varepsilon\, \hat{M}$ that

$$e^{-su}[\hat{\mu}/(s + \beta)]^{(u,\infty)} = (s + \beta)^{-1}\{e^{-su}[\hat{\mu}]^{(u,\infty)} - e^{\beta u}[\hat{\mu}]_{s=-\beta}^{(u,\infty)}\}, \tag{4.13}$$

thus,

$$e^{-su}[\hat{\mu}/(s + \beta)]^{(-\infty,u]} = (s + \beta)^{-1}\{e^{-su}[\hat{\mu}]^{(-\infty,u]} + e^{\beta u}[\hat{\mu}]_{s=-\beta}^{(u,\infty)}\}. \tag{4.14}$$

These formulae will be used frequently.

Now take in (4.14) $\hat{\mu} = (1 - t\varphi(s))^{-1}$, Re $(s) = \gamma$, (which is the transform of the measure $\mu = \sum_{n=0}^{\infty} t^n \nu^n \,\varepsilon\, M$, with ν^n as the distribution of $X_1 + \cdots + X_n$). Writing $(s - \xi)/(s + \beta) = 1 - (\xi + \beta)/(s + \beta)$, we have from (4.11) that

$$\begin{aligned} \tau_1(u, s) &= e^{-su}\frac{s - \xi}{s + \beta}\left[\frac{1}{1 - t\varphi}\right]^{(-\infty,u]} - \frac{\xi + \beta}{s + \beta}\chi_u(-\beta) \\ &= e^{-su}\frac{s - \xi}{s + \beta}\frac{1}{1 - t\varphi} - \frac{s - \xi}{s + \beta}\chi_u(s) - \frac{\xi + \beta}{s + \beta}\chi_u(-\beta). \end{aligned} \tag{4.15}$$

Here,

$$\chi_u(s) = e^{-su}\left[\frac{1}{1 - t\varphi}\right]^{(u,\infty)} = \sum_{n=0}^{\infty} t^n \int_{u+}^{\infty} e^{s(y-u)}\nu^n(dy). \tag{4.16}$$

Combining (4.11), (4.14) and (4.15), one further obtains

$$\tau_2(u, s) = e^{-su}\frac{s - \xi}{(s + \beta)^2}\left[\frac{1}{1 - t\varphi}\right]^{(-\infty,u]} + \frac{s - \xi}{(s + \beta)^2}\chi_u(-\beta) - \frac{\xi + \beta}{s + \beta}\chi_u'(-\beta),$$

where

$$\chi_u'(-\beta) = \sum_{n=0}^{\infty} t^n \int_{u+}^{\infty} (y - u)e^{-\beta(y-u)}\nu^n(dy).$$

It follows, cf., (4.4), that

$$\tau_1(u, \xi) = \{-t\varphi'(\xi)(\xi + \beta)e^{\xi u}\}^{-1} - \chi_u(-\beta),$$
$$\tau_2(u, \xi) = \{-t\varphi'(\xi)(\xi + \beta)^2 e^{\xi u}\}^{-1} - \chi_u'(-\beta).$$

Together with (4.12), these yield the desired explicit formula for the quantity $Q_1(0)$ defined by (4.6).

Finite expressions for the $\tau_j(u, s)$ can be given in the important special case that (4.1) is strengthened to

$$F(y) = de^{\beta y} \quad \text{if} \quad y < \epsilon, \tag{4.17}$$

where ϵ is a *positive* constant. We are only interested in the cases $u = c - x$ or $u = c$, thus we may assume that $u \geqq 0$. Now write τ_j as defined by (4.11) in the form

$$\tau_j(u, s) = \sum_{n=0}^{N(u)} t^n e^{-su}\{[(s - \xi)/(s + \beta)^j]\varphi(s)^n\}^{(-\infty,u]} + R_j, \tag{4.18}$$

where

$$R_j = e^{-su}\left[\frac{s - \xi}{(s + \beta)^j}\,\frac{(t\varphi)^{N(u)+1}}{1 - t\varphi}\right]^{(-\infty,u]} \tag{4.19}$$

and $N(u) = [u\epsilon^{-1}]$. Notice further that (4.17) implies $|\varphi(s)| \leqq K|e^{\epsilon s}|$ if $\mathrm{Re}\ (s) \leqq \gamma$, $|s + \beta| \geqq \delta > 0$, K denoting a constant depending on δ. Hence, by $u < (N(u) + 1)\epsilon$, the following lemma will yield a finite expression for R_j.

LEMMA 4.1. *Let $\mu \in M$ be such that the integral (2.4) is absolutely convergent for $-\beta < \mathrm{Re}(s) \leqq \gamma$ and defines a function $\hat{\mu}(s)$ which can be extended to a single valued and analytic function in* Re $(s) \leqq \gamma$ *except for a possible singularity at $-\beta$.*

Suppose further that there exist real constants q and $\delta > 0$ such that

$$\hat{\mu}(s) = O(e^{qs}) \quad \text{for} \quad \mathrm{Re}\ (s) \leqq \gamma,\ |s + \beta| \geqq \delta. \tag{4.20}$$

Then

$$e^{-su}[\hat{\mu}]^{(-\infty,u]} = -\mathrm{Res}\,\{\hat{\mu}(w)e^{-wu}/(w - s)\}_{w=-\beta}, \tag{4.21}$$

whenever $u < q$, (Res = *residue*).

PROOF. Let s be fixed, $-\beta < \mathrm{Re}\ (s) \leqq \gamma$, and put $G(z) = \int_{-\infty}^{z-} e^{sy}\mu\ (dy) + e^{sz}\mu(\{z\})/2$. Then $\int_{-\infty}^{\infty} e^{vz}\, dG(z) = \int_{-\infty}^{\infty} e^{(v+s)z}\mu\ (dz) = \hat{\mu}(v + s)$, when $-\beta < \mathrm{Re}\ (v + s) \leqq \gamma$. Hence, cf., Widder [8] p. 242,

$$G(z) = -\lim_{T\to\infty} (2\pi i)^{-1} \int_{v_0-iT}^{v_0+iT} \hat{\mu}(v + s)e^{-vz}\frac{dv}{v},$$

whenever Re $(v_0) < 0$, $-\beta < \mathrm{Re}\ (v_0 + s) \leqq \gamma$. Substituting $v + s = w$, this yields

$$G(z)e^{-sz} = -\lim_{T\to\infty} (2\pi i)^{-1} \int_{w_0-iT}^{w_0+iT} \hat{\mu}(w)e^{-wz}\frac{dw}{w - s}$$

whenever Re $(w_0) <$ Re (s), $-\beta <$ Re $(w_0) \leqq \gamma$. On the other hand, using (4.20), the latter right hand side is equal to zero whenever Re $(w_0) < -\beta - \delta$ and $z < q$. Consequently, if $z < q$ then

$$G(z)e^{-sz} = -\text{Res}\,\{\hat{\mu}(w)e^{-wz}(w - s)^{-1}\}_{w=-\beta}\,.$$

Here, the right hand side is a continuous (in fact entire) function of z. Observing that the left hand side of (4.21) is equal to $G(u + 0)e^{-su}$, one obtains (4.21).

Combining (4.19) and Lemma 4.1, one obtains

$$R_j = -\text{Res}\left\{\frac{w - \xi}{(w + \beta)^j}\,\frac{(t\varphi(w))^{N(u)+1}}{1 - t\varphi(w)}\,\frac{e^{-wu}}{w - s}\right\}_{w=-\beta}.$$

Introducing

$$f(w) = (w + \beta)\varphi(w) = d\beta e^{(\beta+w)\epsilon} + (w + \beta)\int_{\epsilon-}^{\infty} e^{wy}\,dF(y), \tag{4.22}$$

and writing

$$-\frac{(t\varphi)^{N(u)+1}}{1 - t\varphi} = \sum_{n=0}^{N(u)} t^n f(w)^n (w + \beta)^{-n} + \frac{w + \beta}{tf(w) - (w + \beta)},$$

this in turn yields

$$R_j = R_j(u, s) = R_j'(u) + (s - \xi)R_j''(u, s), \qquad (j = 1, 2), \tag{4.23}$$

where

$$R_j'(u) = \sum_{m=0}^{N(u)+j-1} (t^{m-j+1}/m!)\{(d/dw)^m[e^{-wu}f(w)^{m-j+1}]\}_{w=-\beta} \tag{4.24}$$

and

$$R_j''(u, s) = \sum_{m=0}^{N(u)+j-1} (t^{m-j+1}/m!)\{(d/dw)^m[(e^{-wu}/(w - s))f(w)^{m-j+1}]\}_{w=-\beta}. \tag{4.25}$$

Here, $N(u) = [u\epsilon^{-1}]$.

The sum in (4.18) can be handled by means of (4.14), (compare (4.15) and subsequent formulae). Using (4.23), one obtains

$$\tau_1(u, s) = A(u) + (s - \xi)C(u, s) \tag{4.26}$$

and

$$\tau_2(u, s) = B(u) + (s - \xi)D(u, s), \tag{4.27}$$

where

$$\begin{aligned} A(u) &= R_1'(u) - G(u), \\ B(u) &= R_2'(u) - H(u). \end{aligned} \tag{4.28}$$

Further,

$$(4.29)\qquad \begin{aligned} C(u, s) &= R_1''(u, s) + (s+\beta)^{-1}M(u, s), \\ D(u, s) &= R_2''(u, s) + (s+\beta)^{-2}M(u, s) + (s+\beta)^{-1}H(u). \end{aligned}$$

Here,

$$(4.30)\qquad \begin{aligned} G(u) &= \sum_{n=0}^{N(u)} t^n \int_{u+}^{\infty} e^{-\beta(y-u)}\nu^n(dy), \\ H(u) &= \sum_{n=0}^{N(u)} t^n \int_{u+}^{\infty} (y-u)e^{-\beta(y-u)}\nu^n(dy), \\ M(u, s) &= G(u) + \sum_{n=0}^{N(u)} t^n \int_{-\infty}^{u+} e^{s(y-u)}\nu^n(dy). \end{aligned}$$

Finally, $N(u) = [u\epsilon^{-1}]$.

Using (4.6), (4.10), (4.12), (4.26), and (4.27), it now follows (assuming (4.17)) that

$$(4.31)\qquad a = \sum_{n=1}^{\infty} t^n \Pr(N = n, z_n < 0) = \beta^{-1}A(c-x)/B(c),$$

while, for all $s \neq -\beta$,

$$(4.32)\qquad Q_0(s) = e^{sc}(s+\beta)\{[B(c)C(c-x, s) - A(c-x)D(c, s)]/B(c)\}.$$

Here, $B(c) \neq 0$. Notice that these expressions for a and Q_0 are *rational* functions in t, which *do not involve* ξ at all. In view of (3.9), Q_2 may be obtained from $Q_1 + Q_2 = e^{sx} - (1 - t\varphi)Q_0$. Also note that, by (3.5) and (3.6),

$$(4.33)\qquad E(N) = [(d/dt)(Q_1 + Q_2)]_{t=1, s=0} = [Q_0(0)]_{t=1}.$$

More precisely, one must have that, for $t = 1$, $B(c) = \tau_2(c, \xi(t)) \neq 0$, which is true provided there exists a real number γ such that $\varphi(\gamma) < 1 = t^{-1}$. In most applications, the latter holds if and only if $E(X_n) \neq 0$.

Considerably simpler is the special case of (4.17) where

$$(4.34)\qquad \begin{aligned} F(y) &= e^{\beta(y-\epsilon)} \quad \text{if} \quad y \leqq \epsilon, \\ &= 1 \qquad\quad \text{if} \quad y \geqq \epsilon. \end{aligned}$$

Here, $\varphi(s) = \beta e^{\epsilon s}(s+\beta)^{-1}$, $f(w) = \beta e^{\epsilon w}$. Further, if $n \leqq N(u) = [u\epsilon^{-1}]$ then $\varphi^n \varepsilon \hat{M}((-\infty, u])$, thus, $G(u) = H(u) = 0$ and

$$(4.35)\qquad M(u, s) = \sum_{n=0}^{N(u)} (\beta t)^n e^{(n\epsilon - u)s}(\beta + s)^{-n}.$$

Finally, by (4.24), (4.25) and (4.28),

$$(4.36)\qquad A(u) = \sum_{n=0}^{N(u)} [(\beta t)^n/n!](n\epsilon - u)^n e^{(u-n\epsilon)\beta},$$

$$(4.37)\qquad R_1''(u, s) = -\sum_{n=0}^{N(u)} A(u - n\epsilon)(\beta t)^n(\beta + s)^{-n-1},$$

and $B(u) = (\beta t)^{-1} A(u + \epsilon)$, $R_2''(u, s) = (\beta t)^{-1} R_1''(u + \epsilon, s)$. Thus, (4.31) becomes

$$(4.38) \qquad a = \sum_{n=0}^{\infty} t^n \Pr(N = n, z_n < 0) = tA(c - x)/A(c + \epsilon).$$

Further, the above formulae together with (4.29) and (4.32) yield a rather explicit formula for $Q_0(s)$ and for $E(N)$, cf., (4.33).

Remark. As is to be expected, a formula as simple as (4.38) can also be obtained in a more straightforward fashion. Namely, let the function $A(u)$ be defined by (4.36) when $u \geqq 0$, while $A(u) = 0$ for $u < 0$. Then $A(u)$ is continuous for $u \neq 0$, differentiable for $u \neq 0$, $u \neq \epsilon$ and satisfies

$$(4.39) \qquad \beta t A(u) e^{-\beta u} = -(d/du)[e^{-\beta u} A(u + \epsilon)] \quad (u \neq 0, u \neq -\epsilon).$$

On the other hand, the sum in (4.38) defines a function $a = a(x)$ satisfying $a(x) = 1$ for $x < 0$, $a(x) = 0$ for $x > c$ and further $a(x) = t\int_{-\infty}^{\epsilon} a(x + y) e^{\beta(y-\epsilon)} \beta \, dy$, thus, $a(x) = \beta t \int_0^{x+\epsilon} a(z) e^{\beta(z-x-\epsilon)} \, dz + te^{-\beta(x+\epsilon)}$, for $0 \leqq x \leqq c$. The latter relation together with $a(x) = 0$ for $x > c$ determines $a(x)$ uniquely, $0 \leqq x < \infty$. Finally, the above properties of $A(u)$, in particular (4.39), imply that these same relations are satisfied by the right hand side of (4.38).

As is well-known, the special case (4.34) is of importance in sequential analysis. Namely, let U be a random variable whose distribution function has a derivative equal to

$$(4.40) \qquad \begin{aligned} g_\rho(u) &= \rho e^{-\rho u} \quad \text{if} \quad u > 0, \\ &= 0 \qquad\quad \text{if} \quad u < 0, \end{aligned}$$

with $\rho > 0$ unknown. One wants to test the simple hypothesis $H_0 : \rho = \rho_0$ against the simple hypothesis $H_1 : \rho = \rho_1$, where $\rho_0 < \rho_1$ are given. In doing this, one takes independent observations $U_1, U_2, \cdots$ and considers

$$X_n = \log [g_{\rho_1}(U_n)/g_{\rho_0}(U_n)] = \log \rho_1/\rho_0 - (\rho_1 - \rho_0) U_n .$$

Its distribution is of the form (4.34) with $\epsilon = \log \rho_1/\rho_0$, $\beta = \rho/(\rho_1 - \rho_0)$. Next, one considers the random walk $z_n = x + X_1 + \cdots + X_n$ with absorbing boundaries $\{z < 0\}$ and $\{z > c\}$. Here,

$$x = \log B^{-1}, \; c - x = \log A, \quad \text{where} \quad B = \beta_0/(1 - \alpha_0), \; A = (1 - \beta_0)/\alpha_0,$$

with $\alpha_0 = \beta_0 = .05$, say. If $z_N < 0$ one accepts H_0; if $z_N > c$ one accepts H_1.

The above formulae now yield an explicit finite expression for the probability $Q_1(0)$ of accepting H_0 and for the duration $E(N)$ of the sequential test, both depending on ρ.

5. The continuous case. A random walk analogous to the case (4.34), but with a continuous time parameter n, arises in the so-called sequential life test, see [2], [3], [4] and [10].

Here, we consider items (say, light bulbs) whose random life time has the exponential distribution (4.40) with ρ unknown. At time $n = 0$ one draws m_0 items at random and places them on a life test; every time an item fails it is immediately replaced by a new item.

Let $r(n)$ denote the (random) number of failures up to time n and consider the stochastic process defined by

$$z(n) = x + r(n) \log \rho_1/\rho_0 - m_0(\rho_1 - \rho_0)n. \tag{5.1}$$

As soon as $z(n) < 0$ one accepts H_0, as soon as $z(n) > c$ one accepts H_1; here, $x = \log B^{-1}$ and $c - x = \log A$ with A and B as before.

Exact formulae for this absorption problem were first given in [2] pp. 261–263. Let us now show how this and more general problems can be handled by our Wiener-Hopf type method.

Consider the more general process defined by

$$z(n) = x + qn + \sum_{1 \leqq k \leqq M(n)} X_k, \tag{5.2}$$

$(k = 1, 2, \cdots)$. Here, $\{M(n)\}$ is a simple Poisson process of independent increments taking integral values $j = 0, 1, 2, \cdots$, with $M(0) = 0$ and such that, conditional on $M(n) = j$, one has $M(n + h) = j$ with probability $1 - \lambda h + o(h)$ and $M(n + h) = j + 1$ with probability $\lambda h + o(h)$, (h small and positive, λ a positive constant). Further, the $X_k (k = 1, 2, \cdots)$ denote real-valued random variables, independent of each other and independent of the process $\{M(n)\}$, and with a common distribution $\nu(B) = \Pr (X_k \varepsilon B)$. Finally, $0 \leqq x \leqq c$ and $q \neq 0$ denote given real constants; (the case $q = 0$ would be largely *equivalent* to the discrete-time problem considered in Section 3).

As usual, let $\hat{\nu} = \varphi$, and let γ be a *fixed* real number with $\varphi(\gamma) < \infty$. For $\operatorname{Re}(s) = \gamma$, one has

$$E(e^{sz(n)}) = \sum_{k=0}^{\infty} e^{-\lambda n}[(\lambda n)^k/k!]e^{(x+qn)s}\varphi(s)^k = \exp(sx + n(qs + \lambda\varphi(s) - \lambda)).$$

Thus,

$$\int_0^\infty E(e^{sz(n)})e^{(\lambda-\alpha)n}\, dn = e^{sx}/(\alpha - qs - \lambda\varphi(s)). \tag{5.3}$$

Here, and in the sequel, α denotes a fixed real number such that

$$\alpha > q\gamma + \lambda\varphi(\gamma), \qquad \alpha > 0. \tag{5.4}$$

In what follows, many quantities depend on α even if this is not shown by the notation used.

Now, consider the random variable

$$N = \inf \{n: n > 0, z(n) \notin [0, c]\}. \tag{5.5}$$

Then either $z(N+) \leqq 0$ or $z(N+) \geqq c$. In fact, $z(N+) = 0$ happens with a positive probability if and only if $q < 0$. Similarly, $z(N+) = c$ happens with a positive probability if and only if $q > 0$.

We shall be interested in finding explicit formulae for the quantities

$$Q_0 = \int_0^\infty E(e^{sz(n)}\{N > n\})e^{(\lambda-\alpha)n}\, dn \tag{5.6}$$

and

$$Q_1 = E(e^{sz(N+)}\{z(N+) \leqq 0\}e^{(\lambda-\alpha)N}), \tag{5.7}$$

$$Q_2 = E(e^{sz(N+)}\{z(N+) \geqq c\}e^{(\lambda-\alpha)N}). \tag{5.8}$$

Each of these belongs to $\hat{M}$, in fact,

$$Q_0 \,\varepsilon\, \hat{M}((0, c)), \tag{5.9}$$

and

$$\begin{aligned} Q_1 \,\varepsilon\, \hat{M}((-\infty, 0]) &\quad \text{if} \quad q < 0, \\ \varepsilon\, \hat{M}((-\infty, 0)) &\quad \text{if} \quad q > 0, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} Q_2 \,\varepsilon\, \hat{M}((c, \infty)) &\quad \text{if} \quad q < 0, \\ \varepsilon\, \hat{M}([c, \infty)) &\quad \text{if} \quad q > 0, \end{aligned} \tag{5.11}$$

compare the remark following (5.5).

Since (5.2) implies (5.3), (whatever the real number x), it follows easily from the Markovian character of the process $\{z(n)\}$ that, for $\mathrm{Re}(s) = \gamma$, $e^{sx}/(\alpha - qs - \lambda\varphi(s)) = Q_0(s) + (Q_1(s) + Q_2(s))/(\alpha - qs - \lambda\varphi(s))$. This is equivalent to

$$[1 - (\lambda/\alpha)\psi]Q_0 = (\alpha - qs)^{-1}(e^{sx} - Q_1 - Q_2), \tag{5.12}$$

where

$$\psi = \psi_\alpha = [\alpha/(\alpha - qs)]\varphi(s) \,\varepsilon\, \hat{M}. \tag{5.13}$$

By (5.4), $\alpha/(\alpha - qs)$ is the transform of a nonnegative measure (with density $|\alpha/q|\, e^{-\alpha y/q}$) carried by $(0, \infty)$ or $(-\infty, 0)$ depending on whether $q > 0$ or $q < 0$. Thus, also ψ is the transform of a *nonnegative* measure such that

$$|\psi(\gamma)| < \alpha/\lambda, \tag{5.14}$$

hence,

$$(1 - (\lambda/\alpha)\psi)^{-1} = \sum_{n=0}^{\infty} ((\lambda/\alpha)\psi)^n \,\varepsilon\, \hat{M}.$$

We further assert that

$$(\alpha - qs)Q_0 \,\varepsilon\, \hat{M}([0, c]). \tag{5.15}$$

For, let the left hand side of (5.15) be denoted as $\hat{\mu}_0$; this is a legitimate notation, since $\hat{\mu}_0 \varepsilon \hat{M}$ by (5.12). Further, by (4.14) and (5.9),

$$0 = (\alpha - qs)[Q_0]^{(-\infty,0)} = (\alpha - qs)[(\alpha - qs)^{-1}\hat{\mu}_0]^{(-\infty,0)} = [\hat{\mu}_0]^{(-\infty,0)} + K,$$

where K denotes a constant. Clearly, $K = 0$ thus $\hat{\mu}_0 \varepsilon \hat{M}([0, \infty))$. Similarly, from a (c, ∞)-truncation, $\hat{\mu}_0 \varepsilon \hat{M}((-\infty, c])$. This proves (5.15).

Because of (5.14), we are allowed to introduce

$$\text{(5.16)} \qquad \begin{aligned} L^- &= L_\alpha^- = \sum_{n=1}^{\infty} (1/n)(\lambda/\alpha)^n[\psi^n]^{(-\infty,0)}, \\ L^+ &= L_\alpha^+ = \sum_{n=1}^{\infty} (1/n)(\lambda/\alpha)^n[\psi^n]^{[0,\infty)}. \end{aligned}$$

They satisfy (3.14) and

$$\text{(5.17)} \qquad 1 - (\lambda/\alpha)\psi = e^{-L^- - L^+}.$$

Consequently,

$$\text{(5.18)} \quad [\alpha - qs - \lambda\varphi(s)]^{-1} = (\alpha - qs)^{-1}[1 - (\lambda/\alpha)\psi(s)]^{-1} = U^-(s)U^+(s),$$

when $\operatorname{Re}(s) = \gamma$, where

$$\text{(5.19)} \qquad \begin{aligned} U^+(s) &= (\alpha - qs)^{-1}e^{L^+(s)} && \text{if } q > 0, \\ &= e^{L^+(s)} && \text{if } q < 0, \end{aligned}$$

and

$$\text{(5.20)} \qquad \begin{aligned} U^-(s) &= e^{L^-(s)} && \text{if } q > 0, \\ &= (\alpha - qs)^{-1}e^{L^-(s)} && \text{if } q < 0. \end{aligned}$$

It follows by (3.14) that

$$\text{(5.21)} \qquad U^+ \varepsilon \hat{M}([0, \infty)), \; U^- \varepsilon \hat{M}((-\infty, 0]).$$

Also note that U^+ corresponds to an absolutely continuous measure when $q > 0$, similarly, U^- when $q < 0$. Using (3.14), (5.9), (5.10), (5.11) and (5.15), it follows that

$$\text{(5.22)} \qquad \begin{aligned} &Q_1U^- \varepsilon \hat{M}((-\infty, 0)), \; Q_2U^+ \varepsilon \hat{M}((c, \infty)), \\ &Q_0/U^+ \varepsilon \hat{M}([0, \infty)), \; Q_0/U^- \varepsilon \hat{M}((-\infty, c]). \end{aligned}$$

In view of (5.18), (5.12) can be written as $Q_1U^- + Q_0/U^+ = (e^{sx} - Q_2)U^-$ and also as $Q_2U^+ + Q_0/U^- = (e^{sx} - Q_1)U^+$. Using (2.6) and (5.22), this yields the following fundamental result.

THEOREM 5.1. *One has the following relations*:

$$\text{(5.23)} \qquad Q_1U^- = [(e^{sx} - Q_2)U^-]^{(-\infty,0)},$$

$$\text{(5.24)} \qquad Q_0 = U^+[(e^{sx} - Q_2)U^-]^{[0,\infty)},$$

$$Q_2 U^+ = [(e^{sx} - Q_1) U^+]^{(c,\infty)}, \tag{5.25}$$

and

$$Q_0 = U^-[(e^{sx} - Q_1) U^+]^{(-\infty,c]}. \tag{5.26}$$

If $c = \infty$ (thus $Q_2 = 0$) then (5.23) and (5.24) yield explicit expressions for Q_0 and Q_1. A different proof and further applications of this special case may be found in [6], chapter 19.

THEOREM 5.2. *The function $Q_1(s)$ defined by* (5.7) *may be characterized as the unique function Q_1 satisfying* (5.10) *with the property that the function $Q_0(s)$ defined by* (5.26), *for* $\text{Re}(s) \geqq \gamma$, *can be extended to a bounded and analytic function in the half plane* $\text{Re}(s) \leqq \gamma$.

For the proof of Theorem 5.2 we shall need:

LEMMA 5.3. *The Q_i ($i = 0, 1, 2$) are uniquely determined by the relations* (5.9), (5.10), (5.11) *and* (5.12).

PROOF OF LEMMA 5.3. It suffices to show that

$$(1 - (\lambda/\alpha)\psi)Q_0 = (\alpha - qs)^{-1}(-Q_1 - Q_2), \tag{5.27}$$

together with (5.9), (5.10) and (5.11), imply that the Q_i are all zero. Here, the Q_i do *not* necessarily correspond to nonnegative measures.

Putting

$$\hat{\mu}_0(s) = (\alpha - qs)Q_0(s) = -(1 - (\lambda/\alpha)\psi)^{-1}(Q_1 + Q_2), \tag{5.28}$$

by (5.27), it follows from (5.9) that $\hat{\mu}_0 \,\varepsilon\, \hat{M}([0, c])$, compare the proof of (5.15). By (5.13), ψ corresponds to an absolutely continuous measure, hence, by (5.10), (5.11) and (5.28), we have that $\hat{\mu}_0$ is of the form $\hat{\mu}_0 = \theta e^{sx_0} + \hat{\mu}_1$, where $\hat{\mu}_1 \,\varepsilon\, \hat{M}((0, c))$, while θ is a constant. Further, $x_0 = 0$ if $q < 0$, $x_0 = c$ if $q > 0$.

Using (5.10), (5.11) and (5.28), we have

$$\begin{aligned} 0 = -[Q_1 + Q_2]^{(0,c)} &= [(1 - (\lambda/\alpha)\psi)\hat{\mu}_0]^{(0,c)} \\ &= \hat{\mu}_1 - (\lambda/\alpha)[\psi\hat{\mu}_1]^{(0,c)} - \theta(\lambda/\alpha)[e^{sx_0}\psi]^{(0,c)}. \end{aligned}$$

It follows that $\hat{\mu}_1 = \theta\hat{\mu}_2$, thus,

$$Q_0 = (\alpha - qs)^{-1}\hat{\mu}_0 = \theta(\alpha - qs)^{-1}(e^{sx_0} + \hat{\mu}_2), \tag{5.29}$$

where $\hat{\mu}_2$ denotes the unique solution of the "integral equation"

$$\hat{\mu}_2 - (\lambda/\alpha)[\psi\hat{\mu}_2]^{(0,c)} = (\lambda/\alpha)[e^{sx_0}\psi]^{(0,c)},$$

($\hat{\mu}_2 \,\varepsilon\, \hat{M}((0, c))$). The uniqueness of $\hat{\mu}_2$ follows by (5.14) and the fact that $\psi = \hat{\nu}_1$ is the transform of a nonnegative measure ν_1. Representing $\hat{\mu}_2$ as the usual Neumann series, and using $\alpha > 0$, we see that $\hat{\mu}_2$ and hence $e^{sx_0} + \hat{\mu}_2$ are transforms of *nonnegative* measures, the latter being non-vanishing. Invoking (5.9), it follows from (5.29) that $\theta = 0$, thus, $Q_0 = 0$, thus, $Q_1 = Q_2 = 0$, by (5.10), (5.11) and (5.27).

PROOF OF THEOREM 5.2. The necessity of the stated conditions follows by (5.9), (5.10) and (5.26).

Sufficiency. By (5.26) and Lemma 2.1, the stated conditions imply that $Q_0 \varepsilon \hat{\mathfrak{M}}([0, c])$. Because of the factor $(\alpha - qs)^{-1}$ in U^- or U^+, Q_0 corresponds to an absolutely continuous measure, thus, $Q_0 \varepsilon \hat{\mathfrak{M}}((0, c))$. Defining Q_2 by (5.25), we obtain a triplet $\{Q_0, Q_1, Q_2\}$ satisfying (5.9), (5.10), (5.11) and (5.12); (in verifying (5.11) for $q > 0$ one uses (4.13) and $e^{-L^+} \varepsilon \hat{\mathfrak{M}}([0, \infty)))$. Now apply Lemma 5.3.

In a number of important applications, Q_1 or Q_2 are a priori known up to a finite number of parameters. More precisely, if the distribution function $F(y)$ of the jumps X_k satisfies (6.2) then Q_1 is of the form (6.5), except for an additive constant when $q < 0$. The resulting implications of Theorem 5.2 are discussed in Section 7.

Let us here restrict ourselves to the important special case that

$$X_k \geqq 0 \quad \text{and} \quad q < 0. \tag{5.30}$$

Then Q_1 is a constant, that is, independent of s. Further, $\varphi(s) = \int_{0-}^{\infty} e^{sy}\nu(dy)$ is a bounded and analytic function for $\mathrm{Re}(s) \leqq \gamma$. Moreover, by (5.13), the distribution function corresponding to ψ satisfies (4.1) with

$$\beta = -\alpha/q > -\gamma, \qquad \beta > 0, \tag{5.31}$$

(see (5.4)). Further, by (4.7), the function e^{-L^-} given by (5.16) is of the form

$$e^{-L^-(s)} = (s - \xi)(s + \beta)^{-1}. \tag{5.32}$$

Here, ξ denotes the unique real number satisfying (4.4), with φ and t replaced by ψ and λ/α, respectively; that is,

$$q\xi + \lambda\varphi(\xi) = \alpha \text{ and } \alpha/q < \xi < \gamma. \tag{5.33}$$

By (5.17) and (5.19),

$$U^+ = e^{L^+} = \frac{s - \xi}{s + \beta}\,\frac{1}{1 - (\lambda/\alpha)\psi} = \frac{-q(s - \xi)}{\alpha - qs - \lambda\varphi(s)}. \tag{5.34}$$

Finally, by (5.20), (5.26) and (5.32),

$$Q_0 = |q|^{-1}(s - \xi)^{-1}\{[e^{sx}e^{L^+}]^{(-\infty,c]} - Q_1[e^{L^+}]^{(-\infty,c]}\}.$$

Thus,

$$Q_0(s) = |q|^{-1}[e^{cs}/(s - \xi)]\{\tau_1(c - x, s) - Q_1\tau_1(c, s)\}, \tag{5.35}$$

where

$$\tau_1(u, s) = e^{-su}[e^{L^+}]^{(-\infty,u]} \tag{5.36}$$

is precisely the same function as the one defined by (4.11), provided that in (4.11) we replace t by λ/α and φ by ψ, see (5.34).

By $e^{L^+} \varepsilon \hat{\mathfrak{M}}([0, \infty))$ and (5.36), $\tau_1(u, s)$ is clearly an entire function of s. It follows from (5.35) and Theorem 5.2 that $\tau_1(c, \xi) \neq 0$ and

$$Q_1 = E(\{z(N+) = 0\}e^{(\lambda-\alpha)N}) = \tau_1(c - x, \xi)/\tau_1(c, \xi). \tag{5.37}$$

A somewhat more explicit formula than (5.36) for τ_1 is given by (4.15), (4.16), with t and φ replaced by λ/α and ψ, respectively. Using (5.13), (5.33) and (5.37), this yields $Q_1 = \{e^{\xi x} - K(c - x)e^{\xi c}\}\{1 - K(c)e^{\xi c}\}^{-1}$, where

$$K(u) = (1 + (\lambda/q)\varphi'(\xi)) \sum_{n=0}^{\infty} (\lambda/\alpha)^n \int_{u+}^{\infty} e^{-\beta(y-u)}\nu_1^n\,(dy).$$

Here, $\beta = -\alpha/q$, while ν_1 is such that $\hat{\nu}_1 = \psi$.

Finite expressions can be obtained when

$$X_k \geqq \epsilon > 0, \qquad q < 0, \tag{5.38}$$

where ϵ denotes a constant. For, then we have by (4.26) that

$$\tau_1(u, s) = A(u) + (s - \xi)C(u, s). \tag{5.39}$$

Here, $A(u)$ and $C(u, s)$ are defined by (4.24), (4.25), (with $j = 1$), (4.28), (4.29) and (4.30), where $t, f(w)$ and ν are to be replaced by λ/α, $(-\alpha/q)\varphi(w)$ and ν_1, respectively.

It follows by (5.35), (5.37) and (5.39) that

$$E(\{z(N+) = 0\}e^{(\lambda-\alpha)N}) = Q_1 = A(c - x)/A(c) \tag{5.40}$$

and

$$Q_0(s) = \frac{e^{cs}}{|q|}\,\frac{A(c)C(c - x, s) - A(c - x)C(c, s)}{A(c)}, \tag{5.41}$$

where $A(c) \neq 0$. Each of these expressions is a *rational* function in t, not involving ξ.

Finally, consider the still more special case that $X_k = \epsilon$ is constant (and $q < 0$). In fact, (5.1) is of this type with $\lambda = m_0\rho$; $q = -m_0(\rho_1 - \rho_0) < 0$; $\epsilon = \log \rho_1/\rho_0$. Now, $A(u)$ is given by (4.36) and $C(u, s)$ is given by (4.29), (4.35) and (4.37), (with β and βt replaced by $-\alpha/q$ and $-\lambda/q$, respectively).

For this special case, Dvoretzky, Kiefer and Wolfowitz [2] already proved (using (4.39)) the case $\alpha = \lambda$ of (5.40) and the case $s = 0$ of (5.41), while Woodall and Kurkjian [10] proved (5.40); Baxter and Donsker [1] p. 83 discussed the case $X_k = \epsilon$, $c = \infty$. Note that, even in the general case (5.2), $E(e^{(\lambda-\alpha)N}) = Q_1(0) + Q_2(0) = 1 - (\alpha - \lambda)Q_0(0)$, by (5.7), (5.8) and (5.12).

6. Auxiliary results. In the Sections 4 and 5, we derived useful explicit formulae for the Q_i, in certain special cases where Q_1 is a priori known up to a single parameter. An analogous procedure yields explicit formulae for the Q_i, see Section 7, whenever $\varphi(s) = E(e^{sX_k})$ is such that there exist *finitely many* $\hat{\lambda}_1, \cdots, \hat{\lambda}_r$ in $\hat{M}((-\infty, 0))$ spanning the manifold $[\varphi\hat{M}([0, \infty))]^{(-\infty,0)}$. In other words, if for each $\hat{\chi} \in \hat{M}([0, \infty))$

$$[\varphi\hat{\chi}]^{(-\infty,0)} = \sum_{j=1}^{r} b_j\{\hat{\chi}\}\hat{\lambda}_j(s), \tag{6.1}$$

where the $b_j\{\hat{\chi}\}$ are independent of s.

Moreover, in the discrete-time case of Section 3, if the X_k are integer-valued we need to require (6.1) only for the transforms $\hat{\chi}$ of the measures $\chi \in M$ carried by the nonnegative integers.

As usual, γ real and $t > 0$ will be fixed and such that (2.3) holds. Further, M, $\hat{M}$ and $\hat{M}(B)$ will be as in Section 2, L^- and L^+ as in Section 3.

As was shown in [5], Section 9, a relation (6.1) holds (for *all* $\hat{\chi} \in \hat{M}([0, \infty))$) if and only if the distribution function $F(y)$ of the X_k is such that, for $y < 0$, the derivative $F'(y)$ of $F(y)$ exists and is of the special form

$$F'(y) = \sum_{h=1}^{p} e^{\beta_h y} \sum_{k=1}^{k_h} c_{h,k}|y|^{k-1}, \qquad (y < 0). \tag{6.2}$$

Here, the $c_{h,k}$ and β_h denote complex constants with $c_{h,k_h} \neq 0$ $(1 \leqq h \leqq p)$, $\beta_1, \cdots, \beta_p$ distinct, $\mathrm{Re}(\beta_h) > \max(0, -\gamma)$. We shall allow $p = 0$, that is, $F(y) = 0$ for $y < 0$.

Note that (6.2) implies

$$\varphi(s) = \sum_{h=1}^{p} \sum_{k=1}^{k_h} (k-1)!\,c_{h,k}(\beta_h + s)^{-k} + \int_{0-}^{\infty} e^{sy}\, dF(y), \tag{6.3}$$

which defines $\varphi(s)$ as an analytic function in the half plane $\mathrm{Re}(s) \leqq \max(0, \gamma)$, except for the poles $-\beta_h$ of order k_h, $(h = 1, \cdots, p)$.

LEMMA 6.1. *Given* (2.3) *and* (6.2), *the equation* $\varphi(s) = t^{-1}$ *has precisely* $r = \sum k_h$ *roots* $\eta_1, \cdots, \eta_r$ *in the half plane* $\mathrm{Re}(s) < \gamma$, (*a root of multiplicity* m *counted* m *times*). *Moreover,*

$$e^{L^-(s)} = \prod_{h=1}^{p} (\beta_h + s)^{k_h} \prod_{j=1}^{r} (s - \eta_j)^{-1}. \tag{6.4}$$

Finally, if $\hat{\chi} \in \hat{M}([0, \infty))$ *then*

$$[\varphi\hat{\chi}]^{(-\infty,0)} = \sum_{h=1}^{p} \sum_{k=1}^{k_h} a_{h,k}(\beta_h + s)^{-k}, \tag{6.5}$$

where the $a_{h,k}$ *denote complex constants depending on* $\hat{\chi}$.

The proof of this lemma may be found in [5] and [6]. The assertion (6.5) is easily proved directly.

Next, let us turn to the case that the distribution ν of the X_k is carried by the integers. Let p_j denote the mass of ν at $j = 0, \pm 1, \pm 2, \cdots$, thus, $p_j \geqq 0$ and $\sum p_j = 1$.

As was shown in [5], a relation (6.1) holds for all $\chi \in M$ carried by the nonnegative integers if and only if there exists a nonnegative integer m such that, for $j < -m$, p_j is of the special form

$$p_j = \sum_{h=1}^{p} \theta_h^{|j|} \sum_{k=1}^{k_h} c_{h,k} |j|^{k-1}, \qquad (j < -m). \tag{6.6}$$

Here, the $c_{h,k}$ and θ_h denote constants with $c_{h,k_h} \neq 0$ $(1 \leqq h \leqq p)$, $\theta_1, \cdots, \theta_p$ distinct and $0 < |\theta_h| < \min(1, \gamma_0)$, where $\gamma_0 = e^{\gamma} > 0$. We shall allow that $p = 0$, that is, $p_j = 0$ for $j < -m$.

In the case on hand, $\varphi(s) = \psi(e^s)$, where

$$\psi(w) = \sum_j p_j w^j, \qquad (|w| = \gamma_0). \tag{6.7}$$

Thus, (2.3) is equivalent to

$$\psi(\gamma_0) < t^{-1} < \infty. \tag{6.8}$$

Also note that (6.6) implies

$$\psi(w) = \sum_{h=1}^{p} \sum_{k=1}^{k_h} d_{h,k}(w - \theta_h)^{-k} - \sum_{j=-m}^{-1} e_j w^j + \sum_{j=0}^{\infty} p_j w^j, \tag{6.9}$$

with $d_{h,k}$ and e_j constants, $d_{h,k_h} \neq 0$ for $1 \leqq h \leqq p$.Thus, $\psi(w)$ is analytic for $|w| \leqq \gamma_0$, except for poles θ_h of order k_h, $h = 1, \cdots, p$, and for a pole at $w = 0$ if $m > 0$. The proof of the following lemma may be found in [5] p. 303.

LEMMA 6.2. *Suppose that* (6.8) *holds and further* (6.6) *for* $j < -m$, *where* $m \geqq 0$ *is minimal. Then the equation* $\psi(w) = t^{-1}$ *has precisely* $r = m + \sum k_h$ *roots* $\xi_1, \cdots, \xi_r$ *in the circle* $|w| < \gamma_0$. *Moreover,*

$$e^{L^-} = w^m \prod_{h=1}^{p} (w - \theta_h)^{k_h} \prod_{j=1}^{r} (w - \xi_j)^{-1}, \tag{6.10}$$

where $w = e^s$.

Finally, if $\chi \varepsilon M([0, \infty))$ *is carried by the nonnegative integers then*

$$[\varphi\hat{\chi}]^{(-\infty,0)} = \sum_{h=1}^{p} \sum_{k=1}^{k_h} a_{h,k}(w - \theta_h)^{-k} + \sum_{j=-m}^{-1} b_j w^j, \tag{6.11}$$

where the $a_{h,k}$ *and* b_j *denote constants depending on* χ; *further* $w = e^s$.

7. Explicit formulae. In this section, we assume that the distribution function of the X_k satisfies (6.2), except that, in the discrete-time situation of Section 3, we shall also allow that, in stead, X_k is integral-valued in such a way that (6.6) holds. In the latter case, as is natural, we shall further assume that c and z_0 are integers, to the effect that $\exp(\pm L^{\pm}(s))$, $\hat{\sigma}(s)$ and the $Q_i(s)$ are all series $\sum b_k w^k$ in integral powers of $w = e^s$, $(|w| = \gamma_0)$.

We claim that, under these assumptions, useful explicit formulae for the Q_i can be derived by using either Theorem 3.1 or Theorem 5.2, and further the auxiliary results of Section 6.

Let us first consider the discrete-time problem of Section 3. If (6.2) holds then

$$\{\hat{\lambda}_j(s), j = 1, \cdots, r\} \tag{7.1}$$

will denote the collection of the $r = \sum k_h$ functions

$$\{(\beta_h + s)^{-k};\ h = 1, \cdots, p;\ k = 1, \cdots, k_h\}. \tag{7.2}$$

If in stead of (6.2), X_k is integral-valued and such that (6.6) holds (with $m \geqq 0$ minimal), then (7.1) will denote the collection of the $r = m + \sum k_h$ functions

$$\{(w - \theta_h)^{-k}, w^{-g};\ h = 1, \cdots, p;\ k = 1, \cdots, k_h\ ;\ g = 1, \cdots, m\}. \tag{7.3}$$

Here, $w = e^s$.

Let t and γ be as in (2.3) and let Q_1 be defined by (3.5). It follows by (3.10), and by (6.5) or (6.11), that $Q_1(s)$ is necessarily of the form

$$Q_1(s) = \sum_{j=1}^{r} a_j \hat{\lambda}_j(s). \tag{7.4}$$

Here, the a_j are as yet unknown constants, (that is, independent of s). Because the functions (7.1) are linearly independent, the constants a_j in (7.4) are uniquely determined by Q_1.

Let L^- and L^+ be as in (3.11) and (3.12), and define

$$\hat{\chi}_j(s) = [\hat{\lambda}_j e^{L^+}]^{(-\infty, c]}, \qquad j = 1, \cdots, r, \tag{7.5}$$

and

$$\hat{\tau}(s) = [\hat{\sigma} e^{L^+}]^{(-\infty, c]} = [\hat{\sigma} e^{L^+}]^{[0, c]}, \tag{7.6}$$

where $\hat{\sigma} \varepsilon \hat{\mathfrak{M}}([0, c])$ is given by $\hat{\sigma}(s) = E(e^{sz_0})$.

THEOREM 7.1. *The constants a_j in (7.4) are uniquely determined by the condition that each finite zero of the function $1 - t\varphi(s)$ in the half plane $\operatorname{Re}(s) < \gamma$, (that is, each pole of $e^{L^-(s)}$), must also be a zero of at least the same multiplicity of the function*

$$\hat{\tau}(s) - \sum_{j=1}^{r} a_j \hat{\chi}_j(s). \tag{7.7}$$

PROOF. By Theorem 3.1, $Q_1(s)$ and thus the constants a_j are *uniquely* determined by the condition that the function $Q_0(s)$, as defined by (3.19), that is,

$$Q_0(s) = e^{L^-(s)}\{\hat{\tau}(s) - \sum_{j=1}^{r} a_j \hat{\chi}_j(s)\}, \tag{7.8}$$

(by (7.4), (7.5) and (7.6)), can be continued to a bounded and analytic function in the half plane $\operatorname{Re}(s) \leqq \gamma$.

Here, by (6.4) or (6.10), $e^{L^-(s)}$ is bounded as $\operatorname{Re}(s) \to -\infty$. Further, by (7.6), $\hat{\tau}(s)$ is clearly an entire function bounded in each left half plane. Moreover, by (7.5),

$$\hat{\chi}_j(s) = \hat{\lambda}_j(s) e^{L^+(s)} - [\hat{\lambda}_j e^{L^+}]^{(c, \infty)},$$

which is analytic and bounded in $\operatorname{Re}(s) \leqq \gamma$, except for singularities of the same type as those of $\hat{\lambda}_j(s)$, (that is, compare (7.2) and (7.3), at the points s_0 with $s_0 = -\beta_h$ or $e^{s_0} = \theta_h$ or $e^{s_0} = 0$). But in the product (7.8) these singularities are completely canceled by the zeros of the factor

$$e^{L^-(s)} = (1 - t\varphi(s))^{-1}e^{-L^+(s)}, \tag{7.9}$$

(cf., (3.17)), as follows by (6.3), (6.9) and also by (6.4), (6.10).

Consequently, the a_j are uniquely determined by the condition that, in the product (7.8), the finite poles of (7.9) in $\operatorname{Re}(s) < \gamma$ are to be canceled by the zeros of (7.7). This proves Theorem 7.1.

If (6.2) holds then, by Lemma 6.1, Theorem 7.1 leads to a *nonsingular* system of r linear equations in the r unknowns $a_1, \cdots, a_r$. More precisely, let $\zeta_1, \cdots, \zeta_g$ denote the distinct roots of $\varphi(s) = t^{-1}$ in $\operatorname{Re}(s) < \gamma$ and let $m_1, \cdots, m_g$ denote their multiplicities. By Lemma 6.1, $m_1 + \cdots + m_g = r$. Put

$$\Gamma_{ij} = \{(d/ds)^k \hat{\chi}_j(s)\}_{s=\zeta_h}, \qquad (i, j = 1, \cdots, r), \tag{7.10}$$

and

$$\Lambda_i = \{(d/ds)^k \hat{\tau}(s)\}_{s=\zeta_h} \qquad (i = 1, \cdots, r), \tag{7.11}$$

where $k = k(i)$ and $h = h(i)$ are defined by $i = m_1 + \cdots + m_{h-1} + k + 1$, $0 \leqq k \leqq m_h - 1$. It follows by Theorem 7.1 that the a_j are uniquely determined by the system

$$\sum_{j=1}^{r} \Gamma_{ij} a_j = \Lambda_i, \qquad i = 1, \cdots, r. \tag{7.12}$$

In particular, $|\Gamma_{ij}| = \det(\Gamma_{ij}) \neq 0$. Moreover, using (7.4), it follows that

$$Q_1(s) = -|\Gamma_{ij}|^{-1} \begin{vmatrix} 0 & \hat{\lambda}_1(s) & \hat{\lambda}_2(s) & \cdots & \hat{\lambda}_r(s) \\ \Lambda_1 & \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1r} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & \\ \Lambda_r & \Gamma_{r1} & \Gamma_{r2} & \cdots & \Gamma_{rr} \end{vmatrix} \tag{7.13}$$

is an explicit formula for the function $Q_1(s)$ defined by (3.5). Having found the a_j, Q_0 and Q_2 may be obtained from (3.20), (6.4), (7.8) and (7.9). The case $t = 1$ is presumably of most interest.

As in Section 4, the major problem in this procedure is to derive useful expressions for the functions $\hat{\chi}_j(s)$ and $\hat{\tau}(s)$ defined by (7.5) and (7.6). Finite expressions can be obtained if (6.2) holds for $y < \epsilon$, ϵ denoting a positive constant, namely be employing the immediate extension of Lemma 4.1 to the case of several singularities, compare the proof of (4.26) and (4.27).

An analogous procedure holds if X_k is integral-valued and (6.6) is assumed. Here, $\{\hat{\lambda}_j\}$ stands for the system (7.3). Further, $\hat{\chi}_j(s)$, $\hat{\tau}(s)$ etc. are series in $w = e^s$ and, thus, have period $2\pi(-1)^{\frac{1}{2}}$ in s. Let $\zeta_1, \cdots, \zeta_g$ denote the roots of $\psi(w) = t^{-1}$ in the circle $|w| < \gamma_0 = e^{\gamma}$ and let $m_1, \cdots, m_g$ denote their multiplicities. By Lemma 6.2, $m_1 + \cdots + m_g = r$. Replacing (7.10) and (7.11) by

$$\Gamma_{ij} = \{(d/dw)^k \hat{\chi}_j(s)\}_{w=\zeta_h}, \qquad (i, j = 1, \cdots, r) \tag{7.14}$$

and

$$\Lambda_i = \{(d/dw)^k \hat{\tau}(s)\}_{w=\zeta_h}, \qquad (i = 1, \cdots, r), \tag{7.15}$$

one again has (7.12) and (7.13). Having found the a_j from (7.12), Q_0 and Q_2 may be obtained from (3.20), (6.10), (7.8) and (7.9).

Let us now turn to the continuous-time problem of Section 5. Here, we assume (for definiteness) that $z_0 = x$ is nonrandom, $(0 \leqq x \leqq c)$, and further that the distribution function of the jumps X_k satisfies (6.2).

Let α and γ be real and fixed such that (5.4) holds, $(\gamma > -\mathrm{Re}(\beta_h), h = 1, \cdots, p)$. Let $Q_i(s)$ $(i = 0, 1, 2; \mathrm{Re}(s) = \gamma)$ be defined by (5.6), (5.7), (5.8), and let ψ, $L^\pm$, $U^\pm$ be defined by (5.13), (5.16), (5.19) and (5.20).

Explicit formulae for $U^\pm$ can be obtained from Lemma 6.1. If $q > 0$ then, by (5.13), a relation of the type (6.3) holds for ψ, therefore,

$$U^- = e^{L^-} = \prod_{i=1}^{r} (s - \eta_i)^{-1} \prod_{h=1}^{p} (\beta_h + s)^{k_h}, \tag{7.16}$$

where $\{\eta_1, \cdots, \eta_r\}$ denotes the set of all numbers η such that

$$q\eta + \lambda\varphi(\eta) = \alpha, \qquad \mathrm{Re}(\eta) < \gamma, \tag{7.17}$$

(a root of multiplicity m being counted m times).

If $q < 0$, we have in a similar fashion, by (5.13) and (6.3), that

$$U^- = (\alpha - qs)^{-1} e^{L^-} = |q|^{-1} \prod_{i=1}^{r+1} (s - \eta_i)^{-1} \prod_{h=1}^{p} (\beta_h + s)^{k_h}, \tag{7.18}$$

where $\{\eta_1, \cdots, \eta_r, \eta_{r+1}\}$ denotes the set of all numbers η satisfying (7.17), (one of them $\approx \alpha/q$ if q is small and negative). Using (5.18), we have from (7.16) and (7.18) that

$$U^+(s) = K/(\alpha - qs - \lambda\varphi(s)) \prod_{i=1}^{r'} (s - \eta_i) \prod_{h=1}^{p} (\beta_h + s)^{-k_h}. \tag{7.19}$$

Here, $K = 1$ if $q > 0$, $K = |q|$ if $q < 0$. Further,

$$r' = r \quad \text{if} \quad q > 0; \qquad r' = r + 1 \quad \text{if} \quad q < 0. \tag{7.20}$$

By (5.9), (5.10), (5.11), (5.12) and (6.2), (compare (6.5)), it follows easily that $Q_1(s)$ is of the form

$$Q_1(s) = \sum_{j=1}^{r'} a_j \hat{\lambda}_j(s). \tag{7.21}$$

Here, $\{\hat{\lambda}_1, \cdots, \hat{\lambda}_r\}$ denotes the system (7.2), while $\hat{\lambda}_{r+1}(s) = 1$ (in the case $q < 0$). Further, the a_j denote as yet unknown constants.

By Theorem 5.2, $Q_1(s)$ and thus the a_j are uniquely determined by the condition that

$$Q_0(s) = U^-\{[e^{sx}U^+]^{(-\infty,c]} - \sum_{j=1}^{r'} a_j[\hat{\lambda}_j U^+]^{(-\infty,c]}\} \tag{7.22}$$

defines a function which can be extended to a bounded and analytic function in the half plane $\mathrm{Re}(s) \leqq \gamma$. Exactly as in the proof of Theorem 7.1, using (7.16), (7.18) and (7.19), it follows that *the* a_j *are uniquely determined by the condition that each root* η *of* (7.17) *is also a root of at least the same multiplicity of the equation*

$$\sum_{j=1}^{r'} a_j[\hat{\lambda}_j U^+]^{(-\infty,c]} = [e^{sx}U^+]^{(-\infty,c]},$$

(where both the left and right hand sides are functions of s).

In the usual way, compare (7.10) and (7.11), this condition leads to a non-singular system

$$\sum_{j=1}^{r'} \Gamma_{ij}a_j = \Lambda_i, \qquad (i = 1, \cdots, r'), \tag{7.23}$$

of r' linear equations in the r' unknowns a_j. For instance, if all the r' roots of (7.17) are distinct, one has

$$\begin{aligned} \Gamma_{ij} &= [\hat{\lambda}_j U^+]_{s=\eta_i}^{(-\infty,c]}, \\ \Lambda_i &= [e^{sx}U^+]_{s=\eta_i}^{(-\infty,c]}, \end{aligned} \tag{7.24}$$

$(i, j = 1, \cdots, r')$. Substituting the solution of (7.23) into (7.21), one obtains the explicit formula (7.13) for $Q_1(s)$, (where r is to be replaced by r'). Similarly, (7.22) yields an explicit formula for $Q_0(s)$.

Finite expressions for the quantities (7.24) can be obtained when (6.2) holds for $y < \epsilon$, with ϵ a positive constant, namely, by using (7.19) and the obvious extension of Lemma 4.1 to the case of several singularities.

REFERENCES

[1] Baxter, G. and Donsker, M. D. (1957). On the distribution of the supremum functional for processes with stationary independent increments. *Trans. Amer. Math. Soc.* **85** 73–87.

[2] Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1953). Sequential decision problems for processes with continuous time parameter. *Ann. Math. Statist.* **24** 254–264.

[3] Epstein, B. (1954). Truncated life tests in the exponential case. *Ann. Math. Statist.* **25** 555–564.

[4] Epstein, B. and Sobel, M. (1955). Sequential life tests in the exponential case. *Ann. Math. Statist.* **26** 82–93.

[5] Kemperman, J. H. B. (1961). Changes of sign in cumulative sums. *Indag. Math.* **23** 291–313.

[6] Kemperman, J. H. B. (1961). *The passage problem for a stationary Markov chain.* Statist. Res. Monographs **1**, Univ. of Chicago Press.

[7] Wendel, J. (1958). Spitzer's formula: a short proof. *Proc. Amer. Math. Soc.* **9** 905–908.

[8] WIDDER, D. V. (1946). *The Laplace transform.* Princeton Univ. Press.
[9] WIDOM, H. (1961). Stable processes and integral equations. *Trans. Amer. Math. Soc.* **98** 430–449.
[10] WOODALL, R. C. and KURKJIAN, B. M. (1962). Exact operating characteristic for truncated sequential life tests in the exponential case. *Ann. Math. Statist* **33** 1403–1412.