LIMITING DISTRIBUTION OF THE MAXIMUM OF A DIFFUSION PROCESS[1]

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1. Introduction. Let $X(t)$, $t \geqq 0$ be a strong homogeneous Markov process on the interval of real numbers (r_1, r_2), $-\infty \leqq r_1 < r_2 \leqq \infty$, with continuous sample functions. For $t > 0$, let $Z(t)$ be the maximum value attained by $X(s)$ on the interval $[0, t]: Z(t) = \max\{X(s); 0 \leqq s \leqq t\}$. In this paper we shall investigate the limiting distribution of $Z(t)$ as $t \to \infty$ for several general types of Markov processes.

First we consider a process having a finite expected first passage time between every pair of points in (r_1, r_2). For this process it is known that a stationary distribution exists [14]; many limit theorems which are valid for sequences of independent random variables also hold for this process [17]. We use the well known renewal principle to show that the asymptotic behavior of $Z(t)$ is similar to that of the maximum in a sequence of independent, identically distributed random variables. Our results are applied to a process whose transition probability function satisfies the classical backward diffusion equation. An analytic method of getting the limiting distribution of $Z(t)$ from asymptotic estimates of the solution to the Fokker-Planck equation has been given by Newell [15]; his results are very close to special cases of our Theorem 5.1.

For certain processes we cannot find the limiting distribution of $Z(t)$ but can assert some form of asymptotic stability such as

$$\lim_{t\to\infty} Z(t)/c(t) = 1$$

in probability for some real function $c(t) \to \infty$. We use the theory developed by Gnedenko [10] and Geffroy [9] for the stability of the maximum in sequences of independent random variables. Similar results have been obtained for stationary normal processes by Cramér [4] and the writer [2].

The above theory is in the spirit of the extension of classical "extreme value" methods [11] to dependent random variables [3], [18]. In the last part of this study, we consider an entirely different type of process, for which "extreme value" methods do not work. We unveil an analogy between the distribution of $Z(t)$ and a distribution arising in renewal theory [8].

In some of the proofs of our results, we shall employ certain fundamental relations for recurrent diffusion processes due to Maruyama and Tanaka [14].

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The methods used in investigating the asymptotic stability of $Z(t)$ can also be used to study the limiting distribution of occupation times. The appendix contains a theorem which characterizes the limiting distributions of a class of occupation times for recurrent diffusion processes. These results contain a general solution of a particular class of problems considered by Khasminski [12].

There is no apparent relation between this study of the maximum functional and other work on the supremum functional for processes with stationary independent increments [1], [16].

2. Extreme Values: independent random variables. In this section, we summarize known results necessary for our investigation, and give some new generalizations.

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with the common distribution function (d.f.) $F(x)$; let $Z_n = \max(X_1, \cdots, X_n)$. The d.f. of Z_n is $F^n(x)$. The limiting d.f. is of one of exactly three types, that is, if there exists a d.f. $\Phi(x)$ and sequences $\{a_n\}$ and $\{b_n\}$, $a_n > 0$, such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \Phi(x) \tag{2.1}$$

on the continuity set of Φ, then the latter is of one of three types [10]. These are known as "extreme value" types [11]. Necessary and sufficient conditions for the convergence (2.1) to each of the types are due to Gnedenko [10]; when (2.1) holds, F is said to be in the domain of attraction of Φ.

Let $\{N_n, n \geqq 1\}$ be a sequence of nonnegative, integer valued random variables defined on the same probability space as $\{X_n\}$, and define:

$$W_n = Z_{N_n} \quad \text{if} \quad N_n > 0 \quad \text{and} \quad W_n = -\infty \quad \text{if} \quad N_n = 0.$$

W_n is the maximum of a random number of independent random variables. A generalization of one of our previous results ([3], Theorem 3.2) is

THEOREM 2.1. *If there is a positive constant c such that*

$$\lim_{n\to\infty} n^{-1} N_n = c \qquad \textit{in probability,} \tag{2.2}$$

then (2.1) *holds if and only if*

$$\lim_{n\to\infty} P\{W_n \leqq a_n x + b_n\} = \Phi^c(x) \tag{2.3}$$

on the continuity set of Φ.

REMARK. The functional forms of the three types indicate that Φ and Φ^c are of the same type.

PROOF. The necessity of (2.3) is the content of ([3], Theorem 3.2), so that we shall prove only the sufficiency. There exist, by the weak compactness theorem, a monotone function $\bar{\Phi}(x)$ and a sequence of integers $\{n_k\}$ such that

$$\lim_{k\to\infty} F^{n_k}(a_{n_k} x + b_{n_k}) = \bar{\Phi}(x) \tag{2.4}$$

on the continuity set of $\bar{\Phi}$. Using the method of proof of ([3], Theorem 3.2), one can show that (2.2) and (2.4) imply $\lim_{k\to\infty} P\{W_{n_k} \leqq a_{n_k} + b_{n_k}\} = \bar{\Phi}^c(x)$;

this and (2.3) imply that $\Phi = \bar{\Phi}$. Since (2.4) holds for an arbitrary convergent subsequence, (2.1) follows.

Now consider the maximum Z_n of random variables $X_1, \cdots, X_n$ which are unbounded above. $\{Z_n\}$ is said to be relatively stable in probability if there is a sequence $\{A_n\}$ such that $\lim_{n\to\infty} Z_n/A_n = 1$ in probability. A necessary and sufficient condition for this is

$$\lim_{x\to\infty} [1 - F(rx)]/[1 - F(x)] = 0 \tag{2.5}$$

for every $r > 1$; in this case A_n satisfies, by [10],

$$1 - F(A_n) \sim n^{-1}, \qquad \text{as} \quad n \to \infty. \tag{2.6}$$

THEOREM 2.2. *Let $\{\sigma_n\}$ be a nondecreasing sequence of positive numbers which is dense in the set of positive integers for large n. Suppose that N_n/σ_n has a continuous limiting* d.f. *for $n \to \infty$. Then there exists a sequence $\{B_n\}$ such that*

$$\lim_{n\to\infty} W_n/B_n = 1 \qquad \textit{in probability,} \tag{2.7}$$

if and only if there exists a sequence $\{A_n\}$ with respect to which $\{Z_n\}$ is relatively stable in probability; in this case we have $B_n = A_{[\sigma_n]}$, where $[\sigma_n]$ stands for the integral part of σ_n.

PROOF. Let $\epsilon > 0$, $\delta > 0$ be given numbers and $o(1)$ a quantity converging to 0 as $\delta \to 0$. Since N_n/σ_n has a continuous limiting d.f. we have

$$\lim_{n\to\infty} P\{N_n \leqq \delta\sigma_n\} = o(1), \qquad \lim_{n\to\infty} P\{N_n > \sigma_n/\delta\} = o(1). \tag{2.8}$$

On one hand, we have

$$\begin{aligned} P\{W_n \leqq B_n(1 \pm \epsilon)\} &= P\{W_n \leqq B_n(1 \pm \epsilon), N_n \leqq \sigma_n/\delta\} \\ &\quad + o(1) \geqq P\{Z_{[\sigma_n/\delta]} \leqq B_n(1 \pm \epsilon)\} + o(1); \end{aligned} \tag{2.9}$$

on the other hand, we have

$$\begin{aligned} P\{W_n \leqq B_n(1 \pm \epsilon)\} &= P\{W_n \leqq B_n(1 \pm \epsilon), N_n > \delta\sigma_n\} \\ &\quad + o(1) \leqq P\{Z_{[\sigma_n\delta]} \leqq B_n(1 \pm \epsilon)\} + o(1). \end{aligned} \tag{2.10}$$

If $\{Z_n\}$ is relatively stable in probability, then (2.9) implies, for $B_n = A_{[\sigma_n]}$, that

$$\begin{aligned} P\{W_n \leqq B_n(1 + \epsilon)\} &\geqq P\{Z_{[\sigma_n/\delta]} \leqq A_{[\sigma_n]}(1 + \epsilon)\} + o(1) \\ &= \{F^{\sigma_n}(A_{[\sigma_n]}(1 + \epsilon))\}^{1/\delta} + o(1) \to 1 + o(1); \end{aligned}$$

(2.10) in turn implies that

$$\begin{aligned} P\{W_n \leqq B_n(1 - \epsilon)\} &\leqq P\{Z_{[\sigma_n\delta]} \leqq A_{[\sigma_n]}(1 - \epsilon)\} + o(1) \\ &= \{F^{\sigma_n}(A_{[\sigma_n]}(1 - \epsilon))\}^{\delta} + o(1) \to o(1). \end{aligned}$$

Since δ is arbitrary, (2.7) follows.

Suppose that (2.7) is given; then the result follows by defining the sequence $\{A_n\}$ implicity as $A_{[\sigma_n]} \sim B_n$, and using (2.9) and (2.10) with $1 \mp \epsilon$ instead of $1 \pm \epsilon$.

3. Description of the diffusion process. Let $X(t)$ be a homogenous Markov process on the real interval (r_1, r_2), $-\infty \leqq r_1 < r_2 \leqq \infty$, with the (stationary) transition probability function $P(t, x, E)$, $t \geqq 0$, $r_1 < x < r_2$, defined for every Borel set E in (r_1, r_2). We shall assume that:

(a) $X(t), t \geqq 0$, is continuous with probability 1.
(b) The strong Markov property for first passage times holds [14].
(c) The boundary points r_1 and r_2 are inaccessible.
(d) All points of (r_1, r_2) are mutually accessible.

A process with these properties will be called a diffusion process on (r_1, r_2). If, in addition, the first passage times among all points of (r_1, r_2) are finite with probability 1, the process will be called a recurrent diffusion process. These definitions coincide with those of Maruyama and Tanaka [14].

Let $P_x(\cdot)$ denote the probability measure on the σ-field of events generated by $X(t)$, $t \geqq 0$, when the initial distribution of $X(0)$ assigns probability 1 to the point x, $r_1 < x < r_2$. The expectation of a random variable ξ with respect to $P_x(\cdot)$ is written as $E_x\xi$. Any probability $P_x(\cdot)$ or expectation $E_x\xi$ which is independent of x will be written without the subscript x; a relation which holds in probability or with probability 1 under all $P_x(\cdot)$, $r_1 < x < r_2$, will be said to hold simply in probability or with probability 1.

4. Fundamental representation of $Z(t)$. Let $X(t)$ be a recurrent diffusion process on (r_1, r_2); for any x, $r_1 < x < r_2$, we define

$$\tau(x) = \inf\{t : t \geqq 0, X(t) = x\};$$

$\tau(x)$ is the first passage time to x. Now fix two arbitrary points x_1 and x_2, $r_1 < x_1 < x_2 < r_2$, and define:

$$\begin{aligned} T_0 &= \tau(x_2) \\ T_1' &= \inf\{t : t \geqq T_0, X(t) = x_1\} \\ T_1 &= \inf\{t : t \geqq T_1', X(t) = x_2\} \\ &\cdots \\ T_n' &= \inf\{t : t \geqq T_{n-1}, X(t) = x_1\} \\ T_n &= \inf\{t : t \geqq T_n', X(t) = x_2\} \\ &\cdots \end{aligned} \tag{4.1}$$

$$\begin{aligned} &X_0 = \max\nolimits_{0<s\leqq T_0} X(s), \qquad X_n = \max\nolimits_{T_{n-1}<s\leqq T_n} X(s), \qquad n \geqq 1, \\ &Z_n = \max(X_1, \cdots, X_n), \qquad N(t) = \max\{k : T_k \leqq t\}. \end{aligned} \tag{4.2}$$

It is known that the strong Markov property implies that $\{X_n, n \geqq 1\}$ is a sequence of mutually independent random variables with a common d.f. $G(x; x_1, x_2)$ defined as

$$
\begin{aligned}
G(x; x_1, x_2) &= 0, && x \leqq x_2 \\
&= P_{x_2}\{\tau(x_1) < \tau(x)\}, && x_2 < x < r_2 \\
&= 1 && x \geqq r_2 .
\end{aligned} \tag{4.3}
$$

From the monotonicity of $Z(t)$ and from the relations

$$T_{N(t)} \leqq t < T_{N(t)+1}, \qquad Z(T_{N(t)}) = \max(X_0, Z_{N(t)})$$

we get $\max(X_0, Z_{N(t)}) \leqq Z(t) \leqq \max(X_0, Z_{N(t)+1})$; from the inaccessibility of r_2 we get $P\{X_0 \leqq r_2\} = 1$; from the finiteness of all first passage times it follows that $Z(t) \to r_2$ in probability as $t \to \infty$; therefore, for any $\epsilon > 0$, $r_1 < x < r_2$, the inequality

$$P_x\{Z_{N(t)} \leqq Z(t) \leqq Z_{N(t)+1}\} \geqq 1 - \epsilon \tag{4.4}$$

will hold for all sufficiently large t.

If $X(t)$ has a transition probability function $P(t, x, E)$ satisfying the backward diffusion equation

$$\frac{\partial P}{\partial t} = a(x)\frac{\partial^2 P}{\partial x^2} + b(x)\frac{\partial P}{\partial x}, \qquad t > 0, \quad r_1 < x < r_2, \tag{4.5}$$

it is known that $G(x; x_1, x_2)$, $x_2 < x < r_2$, is the solution of

$$a(x)(\partial^2 G/\partial x^2) + b(x)(\partial G/\partial x) = 0$$

with the boundary conditions $G(x_2; x_1, x_2) = 0$, $G(r_2; x_1, x_2) = 1$. The solution for $x_2 < x < r_2$ is, following [13],

$$G(x; x_1, x_2) = \frac{\int_{x_2}^{x} \exp\left\{-\int_{x_2}^{s} (b(u)/a(u))\,du\right\} ds}{\int_{x_1}^{x} \exp\left\{-\int_{x_2}^{s} (b(u)/a(u))\,du\right\} ds}. \tag{4.6}$$

5. Finite expected first passage times. We now characterize the limiting d.f. of $Z(t)$ for a process satisfying

$$\mathbf{E}(T_1 - T_0) = m(x_1, x_2) < \infty. \tag{5.1}$$

This is equivalent to the finiteness of all expected first passage times [14].

THEOREM 5.1. *Let $X(t)$ be a recurrent diffusion process on (r_1, r_2). There exist functions $\alpha(t) > 0$ and $\beta(t)$ and a* d.f. $\Phi(x)$ *such that*

$$\lim_{t\to\infty} P\{Z(t) \leqq \alpha(t)x + \beta(t)\} = \Phi(x) \tag{5.2}$$

on the continuity set of Φ if and only if Φ is an extreme d.f. *and for some x_1, x_2 the* d.f. *$G(x; x_1, x_2)$ is in the domain of attraction of $[\Phi(x)]^{m(x_1, x_2)}$. In this case, the same will be true for all pairs x_1, x_2.*

PROOF. We know from renewal theory that $\lim_{t\to\infty} t^{-1}N(t) = (m(x_1, x_2))^{-1}$

with probability 1, and, therefore, in probability. We also see, from (4.4), that $Z(t)$ is distributed, for large t, essentially as the maximum of a random number $N(t)$ of independent random variables with the common d.f. $G(x; x_1, x_2)$. The proof is completed by a direct application of Theorem 2.1, after we define the functions $\alpha(t)$ and $\beta(t)$ in terms of the sequences $\{a_n\}$ and $\{b_n\}$ as $\alpha(t) = a_{[t]}$, $\beta(t) = b_{[t]}$.

Consider a recurrent diffusion process on (r_1, r_2) satisfying (4.5). For a fixed x' in (r_1, r_2), define $f(x) = \exp\{-\int_{x'}^{x} (b(s)/a(s))\, ds\}$. Doob [6] has shown that (5.1) holds if

$$(5.3) \qquad \int_{r_1}^{x'} (f(s))^{-1}\, ds = \int_{x'}^{r_2} (f(s))^{-1}\, ds = \infty; \qquad \int_{r_1}^{r_2} (f(s)/a(s))\, ds < \infty.$$

Theorem 5.1 asserts that if (5.3) is given, then $Z(t)$ has a limiting d.f. $\Phi(x)$ (necessarily an extreme value d.f.) if and only if the d.f. G, given by (4.6), is in the domain of attraction of the same type as Φ. Newell [15] found conditions on $a(x)$ and $b(x)$ which are sufficient for (5.2) in the case where $P(t, x, E)$ has a density which satisfies the forward (Fokker-Planck) equation; his conditions can be shown to coincide with ours in the case where $P(t, x, E)$ satisfies the backward equation. It is interesting that the standard probabilistic arguments used to prove Theorem 5.1 provide more general results than those obtained by Newell's analytic method.

6. Some properties of the return times T_n with infinite expectation. In this section, we shall give necessary and sufficient conditions on $P(t, x, E)$ for the existence of a function $\sigma(t)$ such that $N(t)/\sigma(t)$ has a limiting d.f.; $\mathbf{E}(T_1 - T_0)$ is then necessarily infinite. The results will be applied in the next section to the stability of $Z(t)$ and in the appendix to an occupation time theorem.

From the familiar relation for the random variables T_k and $N(t)$ (defined in (4.1) and (4.2)) $\{N(t) \geqq k\} \equiv \{T_k \leqq t\}$ it follows that $N(t)$ has a limiting d.f. if and only if T_k does [8]. The strong Markov property implies that $T_k - T_0$ is distributed as the sum of k independent random variables, each distributed as $T_1 - T_0$ Let $m_1(t)$ and $m_2(t)$ be positive functions with the property

$$m_1(t) = P\{T_1 - T_0 > t\}, \qquad P\{T_1 - T_0 > m_2(t)\} \sim t^{-1}, \qquad t \to \infty.$$

Feller has shown that

$$(6.1) \qquad \lim_{t\to\infty} P\{T_{[t]} - T_0 \leqq um_2(t)\} = \lim_{t\to\infty} P\{T_{[t]} \leqq um_2(t)\} = G_\alpha(u)$$

if and only if

$$(6.2) \qquad \lim_{t\to\infty} P\{m_1(t)N(t) \geqq u\} = G_\alpha(u^{-1/\alpha}),$$

where $G_\alpha(u)$, $0 < \alpha < 1$, is the d.f. of the positive stable law of index α and $G_\alpha(u^{-1/\alpha})$ is the Mittag-Leffler d.f. of index α; furthermore, this is the only possible limiting d.f. for $N(t)$ that can be obtained by a scale normalization.

A function $h(x)$ is said to be slowly varying as $x \to \infty$ if for every constant

$c > 0$, $h(cx) \sim h(x)$. We define $\varphi(s) = \int_0^\infty e^{-st}\, dP\{T_1 - T_0 \leqq t\}$ and record the known

LEMMA 6.1 [7]. *The existence of a slowly varying $h(x)$, $x \to \infty$, such that*

$$1 - \varphi(s) \sim s^\alpha h(s^{-1}), \qquad s \to 0, \tag{6.3}$$

is necessary and sufficient for (6.1).

We now relate (6.3) to $P(t, x, E)$; we define

$$p(s, x, E) = \int_0^\infty e^{-st}\, P(t, x, E)\, dt, \qquad s > 0,$$

LEMMA 6.2. *The existence of a nondegenerate interval E whose closure is contained in (r_1, r_2), a slowly varying function h and a positive number δ such that*

$$\lim_{s\to 0} h(s^{-1}) s^\alpha p(s, x_2, E) = \delta \tag{6.4}$$

is necessary and sufficient for (6.3).

PROOF. Define, for $i, j = 1, 2$,

$$\varphi_{ij}(s) = \int_0^\infty e^{-st}\, dP_{x_i}\{\tau(x_j) \leqq t\}, \qquad s > 0$$

$$p(x_j\,; s, x_i, E) = \int_0^\infty e^{-st}\, P_{x_i}\, \{X(t) \,\varepsilon\, E, \tau(x_j) > t\}\, dt, \qquad s > 0;$$

then, by the strong Markov property [14], we get

$$\begin{aligned} p(s, x_j, E)\varphi_{ij}(s) - p(s, x_i, E) &= -p(x_j\,; s, x_i, E) \\ \varphi(s) &= \varphi_{ij}(s)\varphi_{ji}(s), \quad i, j = 1, 2. \end{aligned} \tag{6.5}$$

In the equation with the factor $\varphi_{ij}(s)$, we multiply each side by $\varphi_{ji}(s)$, and then substitute for $\varphi_{ji}(s)p(s, x_i, E)$ from the dual equation; this yields

$$p(s, x_j, E)(1 - \varphi(s)) = p(x_i, s, x_j, E) + \varphi_{ji}(s)p(x_j\,; s, x_i, E),$$

which is equivalent to an identity in ([14], Formula (3.19)). Now let $s \to 0{:}\varphi_{ji}(s) \to 1$, and the right side converges to $p(x_i\,; 0, x_j, E) + p(x_j\,; 0, x_i, E)$, which, by ([14], p. 131), is positive and finite. The equivalence of (6.3) and (6.4) follows by letting $s \to 0$ in the last equation.

We remark that (6.4) resembles a condition of Darling and Kac [5] which they used in occupation time theory (see appendix).

The results of this section are summarized as follows:

THEOREM 6.1. *There exists a function $\sigma(t)$ such that $N(t)/\sigma(t)$ has a limiting* d.f. *if and only if the conditions of Lemma* 6.2 *are satisfied; in this case the limiting* d.f. *is the Mittag-Leffler* d.f. *of index α* (6.2).

7. Stability of $Z(t)$. We now suppose $r_2 = \infty$ and give conditions under which $Z(t)$ is relatively stable in probability, that is, there exists a function $c(t) \to \infty$ such that

(7.1) $$\lim_{t\to\infty} Z(t)/c(t) = 1 \qquad \text{in probability.}$$

In view of (4.4), this problem, too, is reducible to that of a random number of independent random variables. Under the Condition (5.1), the methods used to prove Theorem 5.1 would enable us to prove an analogous result for (7.1). But it would be more interesting to obtain (7.1) without (5.1). We shall replace the latter by the conditions of Theorem 6.1.

THEOREM 7.1. *Let $X(t)$ be a recurrent diffusion process on (r_1, ∞), such that the conditions of Theorem 6.1 are satisfied. Then there exists a function $c(t)$ such that (7.1) holds if and only if for some x_1, x_2 satisfying $r_1 < x_1 < x_2 < \infty$,*

(7.2) $$\lim_{x\to\infty} [1 - G(rx; x_1, x_2)]/[1 - G(x; x_1, x_2)] = 0,$$

for every $r > 1$.

PROOF. This follows from Theorem 2.2 just as Theorem 5.1 follows from Theorem 2.1. $Z(t)$ is distributed approximately as the maximum of $N(t)$ independent random variables with the common d.f. $G(x; x_1, x_2)$; (7.2) relates to (2.5). The function $c(t)$ is taken to be $A_{[\sigma(t)]}$; $\sigma(t)$ is asymptotic to t^α times a slowly varying function of t [8], so that $\sigma_{[t]} = \sigma(t)$ satisfies the conditions of Theorem 2.2.

We give an example to show the consistency of the conditions of Theorem 7.1. Let $Y(t)$ be a separable linear Brownian motion process with $\mathbf{E}Y^2(t) = t$, and $Z'(t)$ the maximum up to time t. Let $f(y)$ be an increasing, continuous function on the real line, mapping it onto the semi-infinite interval (r_1, ∞). $X(t) = f(Y(t))$ is a recurrent diffusion process; let $Z(t) = f(Z'(t))$. The distribution of the return time $T_1 - T_0$ is the same for both the $X(t)$ and $Y(t)$ processes, so that the return time has the positive stable distribution of index $\frac{1}{2}$. $Z'(t)t^{-\frac{1}{2}}$ has the half-normal d.f. for every t. Suppose that $f(x) \sim \log x$, $x \to \infty$; then,

$$\frac{Z(t)}{\frac{1}{2}\log t} \sim \frac{\log Z'(t)}{\frac{1}{2}\log t} = \frac{\log(Z'(t)\cdot t^{-\frac{1}{2}}) + \frac{1}{2}\log t}{\frac{1}{2}\log t} \to 1$$

in probability.

8. Processes with periodic transition functions. We consider a diffusion process on (r_1, ∞) having not necessarily the recurrence property—all first passage times are finite with probability 1—but the weaker property that the first passage time from x to y, $r_1 < x < y < r_2$, is finite with probability 1:

(8.1) $$P_x\{\tau(y) < \infty\} = 1, \qquad r_1 < x < y < \infty.$$

From the relation

(8.2) $$P_x\{Z(t) \leqq y\} = P_x\{\tau(y) \geqq t\}, \qquad r_1 < x < y < \infty$$

we see, in analogy to Section 6, that $Z(t)$ has a limiting d.f. if and only if $\tau(t)$ has one.

For any linear Borel set E, let E' denote the translate of E by a unit amount: $E' = \{x : x - 1 \varepsilon E\}$. We now assume the following periodicity property for the transition function:

$$P(t, x, E) = P(t, x + 1, E'), \qquad t \geqq 0, \quad r_1 < x < \infty. \tag{8.3}$$

If $P(t, x, E)$ satisfies the diffusion equation (4.5), then it has the property (8.3) if the diffusion coefficients $a(x)$ and $b(x)$ are periodic with unit period.

Suppose first that y is an integer, $r_1 < x < y$. Under $P_x(\cdot)$, the random variable $\tau(y)$ may be written as a sum of $y - [x]$ independent random variables, as

$$\begin{aligned} \tau(y) = (\tau(y) - \tau(y-1)) + (\tau(y-1) - \tau(y-2)) + \cdots \\ + (\tau([x]+2) - \tau([x]+1)) + \tau([x]+1). \end{aligned} \tag{8.4}$$

Property (8.3) implies that all the summands, except for $\tau([x] + 1)$, have a *common* d.f. Fixing our attention on Formula (8.2), and going back to the discussion at the beginning of Section 6, we can see that $Z(t)$ has the same limiting distribution as the "renewal" random variable $\max\{y : \tau(y) \leqq t\}$. The class of limiting d.f.'s for this random variable has been described by Feller [8], so that $Z(t)$ has the same class of limiting d.f.'s. We shall not enumerate all the corresponding limit theorems, but only one.

THEOREM 8.1. *Let $X(t)$ be a diffusion process on (r_1, ∞) satisfying* (8.1) *and* (8.3). *There exists a positive function $\sigma(t)$ and a* d.f. *$H(x)$ such that*

$$\lim_{t\to\infty} P\{Z(t) \leqq \sigma(t)x\} = H(x) \tag{8.5}$$

on the continuity set of H if and only if the Laplace transform $p(s, x, E)$ of the transition function satisfies the following conditions: There exist a slowly varying function $h(x)$, a number α, $0 < \alpha < 1$, an integer $k > r_1$, and a nondegenerate interval E whose closure is contained in $(k + 1, \infty)$ such that

$$1 - p(s, k, E)/p(s, k + 1, E) \sim s^{\alpha} h(s^{-1}), \qquad s \to 0. \tag{8.6}$$

In such a case H is the Mittag-Leffler d.f. *of index α, given by* (6.2).

PROOF. It follows from the results of Feller on renewal theory [8], in particular, from our relations (6.1) and (6.2), that (8.5) holds if and only if the d.f. of $\tau(k + 1) - \tau(k)$, the first passage time between successive integer points, is in the domain of attraction of G_α (6.1); in this case H is the d.f. given by the theorem.

Define:

$$\psi(s) = \int_0^\infty e^{-st}\, dP\{\tau(k + 1) - \tau(k) \leqq t\}, \qquad s > 0;$$

by (8.3), ψ is independent of k. According to Lemma 6.1, the d.f. of $\tau(k + 1) - \tau(k)$ is in the domain of attraction of G_α if and only if (6.3) holds, with φ replaced by ψ.

In Formula (6.5), let us replace x_i, x_j and $\varphi_{ij}(s)$ by k, $k + 1$, and $\psi(s)$, respectively, and let E denote a nondegenerate interval whose closure is contained in $(k + 1, \infty)$. It follows from the definition of $p(k + 1; s, k, E)$ that it vanishes; thus, from (6.5) we obtain

$$\psi(s) = p(s, k, E)/p(s, k + 1, E).$$

By comparing (6.3) and (8.6), we now complete the proof.

A special case of this theorem is illustrated by the Brownian motion process: the first passage times have the positive stable distribution of index $\frac{1}{2}$ and the maximum has the Mittag-Leffler distribution of index $\frac{1}{2}$ (half-normal).

Using the method of proof of Theorem 8.1, we can prove corresponding limit theorems for $Z(t)$ in cases where the distribution of normed sums of the first passage times converges to the stable law of index α, $1 \leqq \alpha \leqq 2$.

9. Appendix: An occupation time theorem. Using the methods and notation of Section 6, we can prove a stronger form of a general theorem of Darling and Kac [5] for the particular case of a recurrent diffusion process:

THEOREM 9.1. *Let $X(t)$ be a recurrent diffusion process on (r_1, r_2), E a nondegenerate interval whose closure is contained in (r_1, r_2), $V(x)$ the indicator function of E, and $S(t) = \int_0^t V(X(s))\, ds$. There exists a function $\sigma(t) > 0$ and a* d.f. *$G(y)$ such that*

$$\lim_{t\to\infty} P\{S(t)/\sigma(t) \leqq y\} = G(y) \tag{9.1}$$

on the continuity set of G, if and only if there exists a function $h(x)$ which is slowly varying as $x \to \infty$, and numbers δ, α, x, $\delta > 0$, $0 < \alpha < 1$, $r_1 < x < r_2$ such that

$$\lim_{s\to 0} h(s^{-1}) s^{\alpha} p(s, x, E) = \delta. \tag{9.2}$$

In this case $G(y)$ is necessarily the Mittag-Leffler distribution of index α.

PROOF. We apply the strong law of large numbers for independent random variables, as in the proof of the ergodic theorem in [14], and obtain:

$$\lim_{t\to\infty} S(t)/N(t) = \mu(x_1, x_2, E) \tag{9.3}$$

with probability 1, where the right side is positive and finite. The relation (9.3) implies that $S(t)/\sigma(t)$ and $N(t)/\sigma(t)$ have a limiting d.f. of the same type, if any. Theorem 6.1 now provides the rest of the proof.

Condition (9.2) (in a stronger form) was shown in [5] to be *sufficient* for $G(y)$ to be a Mittag-Leffler d.f.

There is an inherent relation between Theorem 9.1 and some results of Khasminski on processes satisfying diffusion equations, which were given without proof [12]. Theorem 9.1 applies to more general diffusion processes than those considered by Khasminski.

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