LIMIT THEOREMS FOR MARKOV RENEWAL PROCESSES[1]

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0. Summary. This paper is a study of Doeblin Ratio limit laws, the weak and strong laws of large numbers, and the Central Limit theorem for Markov Renewal processes. A general definition of these processes is given in Section 2. The means and variances of random variables associated with recurrence times are computed in Section 4. When restricted to the special case of a Markov chain, certain of the results of Sections 5 and 6 strengthen known results.

1. Introduction. In [14], Pyke defines a Markov Renewal process (MRP) constructively as follows. Let $I^+ = \{0, 1, 2, \cdots\}$. Let $\{Q_{ij}; i, j \,\varepsilon\, I^+\}$ be a family of transition mass functions satisfying $Q_{ij}(x) = 0$ for $x < 0$ and $H_i(+\infty) = 1$ where $H_i = \sum_j Q_{ij}$. (Unless otherwise stated, all summations will be over I^+.) Let $\{(J_n, X_n); n \geqq 0\}$ be a two-dimensional Markov process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ by $X_0 = 0$, $P[J_0 = i] = a_i$ for some $a_i \geqq 0$ with $\sum_i a_i = 1$, and

$$(1.1) \quad P[J_n = j, X_n \leqq x \mid J_0, X_0, \cdots, J_{n-1}, X_{n-1}] = Q_{J_{n-1},j}(x) \quad \text{(a.s.)}$$

for all $n \geqq 1$. Upon setting

$$S_n = X_0 + X_1 + \cdots + X_n, \qquad N(t) = \sup\{n \geqq 0: S_n \leqq t\},$$

and $N_j(t) = \text{card}\,\{k: 0 < k \leqq N(t), J_k = j\}$, one obtains the process $\{\mathbf{N}(t) = (N_0(t), N_1(t), \cdots); t \geqq 0\}$ which is called an MRP determined by $\{Q_{ij}\}$. The process $\{Z_t; t \geqq 0\}$, where $Z_t = J_{N(t)}$, is called a Semi-Markov process (S-MP).

The constructive definition outlined above defines an MRP only up until its first "infinity" or "explosion". In [10], Lévy discusses a class of processes, called by him Semi-Markov processes, which contains processes that may explode. Smith, in [18], describes two ways in which a Semi-Markov process may be defined beyond the first infinity. In Section 2 of this paper a general definition of an MRP is given which allows for explosions. Also a summary of notation to be used in this paper and [16] is given.

2. Definition of an MRP. Let $\{Y_t = (Z_t, U_t); t \geqq 0\}$ be any separable Markov process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$, having state space $I^+ \times [0, +\infty) \equiv \mathfrak{X}$, having the strong Markov property for any stopping time which is almost surely finite, and having the following properties:

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(1) Its transition function $P_t(\cdot, \cdot)$ is such that for each $t \geqq 0$ and $\omega \varepsilon \mathfrak{X}$, $P_t(\omega, \cdot)$ is a probability measure on $\mathcal{B}(\mathfrak{X})$, the natural Borel field for $\mathfrak{X}$, and $P_{(\cdot)}(\cdot, A)$ is jointly measurable as a function of (t, ω) for each $A \varepsilon \mathcal{B}(\mathfrak{X})$.

(2) If $P_t(i, x; j, y) \equiv P_t((i, x), \{j\} \times [0, y])$, then for each fixed $t > 0$, $i, j \varepsilon I^+$, and $x \geqq 0$, $P_t(i, x; j, \cdot)$ is a non-decreasing function which satisfies

$$
\begin{aligned}
P_t(i, x; j, y) &= P_t(i, x; j, t-) && \text{if } t \leqq y < t + x, \\
&= P_t(i, x; j, t-) \\
&\quad + \delta_{ij} P_t((i, x), \{i\} \times \{t + x\}) && \text{if } y \geqq t + x.
\end{aligned} \tag{2.1}
$$

(3) The functions $N_j(t) = \text{card } \{0 < u \leqq t : Z_{u-} \neq Z_u = j\}$ are random variables (possibly infinite) for each $j \varepsilon I^+$ and $t \geqq 0$.

DEFINITION 2.1. *The process $\{Z_t ; t \geqq 0\}$ obtained as the first component of the above described process is called a Semi-Markov process and the process $\{\mathbf{N}(t) = (N_0(t), N_1(t), \cdots); t \geqq 0\}$ is called a Markov Renewal process.*

DEFINITION 2.2. *A Markov Renewal process is said to be regular if*

$$P[N_i(t) < +\infty] = 1$$

for all $i \varepsilon I^+$ and $t \geqq 0$, and is said to be strongly-regular if $P[N(t) < +\infty] = 1$ for all $t \geqq 0$.

(In [14], strong-regularity was simply called regularity.)

Since the Chapman-Kolmogorov equation for any Markov process defined on $\mathfrak{X}$ is

$$P_{t+s}(i, x; j, y) = \sum_k \int_{0-}^{+\infty} P_t(k, z; j, y) P_s(i, x; k, dz),$$

it follows from condition (2) above, that the Chapman-Kolmogorov equation for the Y-process is given by

$$
\begin{aligned}
P_{t+s}(i, x; j, y) &= \sum_k \int_{0-}^{s-} P_t(k, z; j, y) P_s(i, x; k, dz) \\
&\quad + P_t(i, s + x; j, y) P_s((i, x), \{i\} \times \{s + x\}).
\end{aligned} \tag{2.2}
$$

A thorough analysis (à la Chung [5]) could be undertaken of all processes satisfying (2.2) and would undoubtedly show that the above definition includes all the processes Lévy had in mind in [10], including those with instantaneous states. In fact, it would seem that a suitable definition for an instantaneous state is that it is a state, i, for which $P_t(i, x; j, 0) \to \delta_{ij}$ as $t \to 0$ for all $j \varepsilon I^+$ and $x \geqq 0$, while a stable state, j, is one for which $P_t(j, x; k, y) \to \delta_{jk}$ or 0 according as $x < y$ or $x \geqq y$ for all $k \varepsilon I^+$ and $x, y \geqq 0$. An immediate consequence of these definitions of stable and instantaneous states would be that as $t \to 0$, $P_t((i, x), \{i\} \times \{t + x\})$ converges to 1 or 0 according as state i is stable or instantaneous.

The fact that countable state Markov processes are included in this new definition of MRP's is important and is seen as follows. Set

$$P_{ij}(t) = P[Z_t = j \mid Z_0 = i, U_0 = 0],$$

or equivalently, $P_{ij}(t) = P_t(i, 0; j, +\infty)$.

LEMMA 2.1. *If for all* $i, j \in I^+$, $y \geqq 0$ *and* $t > 0$, $P_t(i, x; j, y)$ *is constant in* x *for* $x \geqq 0$, *then* (2.2) *reduces to*

$$P_{t+s}(i, 0; j, y) = \sum_k P_t(k, 0; j, y) P_{ik}(s). \tag{2.3}$$

PROOF. By definition, $P_{ij}(t) = P_t(i, 0; j, t-) + \delta_{ij} P_t((i, 0), \{i\} \times \{t\})$. Thus, the right side of (2.2) becomes $\sum_k P_t(k, 0; j, y) P_s(i, 0; k, s-) + P_t(i, 0; j, y) P_s((i, x), \{i\} \times \{s\}) = \sum_k P_t(k, 0; j, y) P_{ik}(s)$.

COROLLARY 2.1. *If* $P_t(i, x; j, y)$ *is independent of* x *for all* $i, j \in I^+$, $y \geqq 0$ *and* $t > 0$, *then the* Z*-process is a countable state Markov process.*

Proof. This follows by letting $y \to +\infty$ in (2.3) to obtain $P_{ij}(t + s) = \sum_k P_{ik}(s) P_{kj}(t)$.

The main purpose of this paper is to prove limit theorems for a slightly more restricted class of MRP's which arise in the following way. Let $\{Z_t : t \geqq 0\}$ be a separable stochastic process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ having state space I^+, compactified by the addition of ∞, and having the following properties:

(i) Almost all sample functions are right continuous, have left limits on $[0, +\infty)$, and are such that if $Z_t = i (i \neq +\infty)$, there exists $\epsilon(t) > 0$ such that $Z_s = i$ for all $s \in [t, t + \epsilon(t))$, while if $Z_t = +\infty$, there exists no $\epsilon > 0$ such that $Z_s = +\infty$ for all $s \in (t - \epsilon, t + \epsilon)$. Further, $|Z_{u-} - Z_u| < +\infty$ for all $u \geqq 0$.

For each $t \geqq 0$, define $U_t = t - \sup\{u < t : Z_u \neq Z_t\}$, $V_t = \inf\{u > t : Z_u \neq Z_t\} - t$, $X_t = U_t + V_t$, $Z_{t-} = \lim_{u \uparrow t} Z_u$, $Z_t^+ = Z_{t+V_t}$, and $Z_t^- = Z_{(t-U_t)-}$. That these quantities are random variables follows from (i) and the fact that the Z-process is separable. For example $\{U_t \geqq y\} = \bigcup_{k=0}^{\infty} \{U_t \geqq y, Z_t = k\} = \bigcup_{k=0}^{\infty} \{Z_s = k, s \in [t - y, t]\}$ and hence is measurable since, by separability, $A_k = \{Z_s = k, s \in [t - y, t]\}$ is measurable for each k. To show that Z_t^- and Z_t^+ are measurable requires a little more work. For example, the measurability of Z_t^- may be shown as follows. Let $Y_n = (k + 1)/2^n$ if $k/2^n \leqq U_t < (k + 1)/2^n$ for $k = 0, 1, 2, \cdots$ and $n = 1, 2, \cdots$. Let $X_n = Z_{t-Y_n}$. Then $Y_n \downarrow U_t$ and $X_n \to Z_t^-$ as $n \to +\infty$. Hence, it suffices to show that each X_n is measurable. But

$$\begin{aligned} \{X_n = j\} &= \bigcup_k \{X_n = j; k/2^n \leqq U_t < (k + 1)/2^n\} \\ &= \bigcup_k [\{Z_{t-(k+1)/2^n} = j\} \cap \{k/2^n \leqq U_t < (k + 1)/2^n\}], \end{aligned}$$

which is a countable union of measurable sets and hence is measurable. It is clear that Z_t^+ and Z_t^- are finite almost surely.

(ii) There exists a matrix of measurable functions $Q_{ij}(\cdot; \cdot)$ for all $i, j \in I^+$ such that $Q_{ij}(u; \cdot)$ is a mass function, $\sum_j Q_{ij}(u; +\infty) = 1$, $\sum_j Q_{ij}(0; 0) < 1$, and

$$P[Z_t^+ = j, V_t \leqq x \mid Z_t = i, U_t = u, (Z_s, U_s); 0 \leqq s < t] = Q_{ij}(u; x) \quad \text{(a.s.)}. \tag{2.4}$$

(iii) The process $\{Y_t = (Z_t, U_t); t \geqq 0\}$ is a two-dimensional, separable Markov process having state space $I^+ \times [0, +\infty) \equiv \mathfrak{X}$, having the strong Markov property for all stopping times which are almost surely finite and having a transition function $P_t(\cdot; \cdot)$ satisfying (1) above and such that for all $i, j \in I^+$, $x, y \geqq 0$, and $t \geqq 0$:

$$\begin{aligned} P_t(i, x; j, y) &= \sum_k Q_{ik}(x; \cdot) * P_{(\cdot)}(k, 0; j, y)(t) && \text{if } x + t > y \\ &= \sum_k Q_{ik}(x; \cdot) * P_{(\cdot)}(k, 0; j, +\infty)(t) \\ &\quad + \delta_{ij}[1 - H_i(x; t)] && \text{if } x + t \leqq y \end{aligned} \tag{2.5}$$

where $H_i(x; t) = \sum_j Q_{ij}(x; t)$.

($K * L(t) = \int_{0-}^{t} K(t - x)\, dL(x)$ whenever the integration is defined. Also $K^{(0)}(t) = 1$ or 0 as $t \geqq 0$ or $t < 0$ and $K^{(n)}(t) = K^{(n-1)} * K(t)$ for $n \geqq 1$. Since one is usually interested only in t's such that $t \geqq 0$, $K^{(0)}(t)$ isoften replaced by 1.)

It is clear that processes satisfying (i) through (iii) determine regular Markov Renewal processes as defined in Definitions 2.1 and 2.2. Such MRP's will be said to satisfy hypothesis A. A process satisfying hypothesis A has at each time point, t, the property that there is a well-defined state to be visited next by the process and a well-defined state just visited previous to the present state. The former property is related to the property of being conservative in the terminology of continuous parameter Markov chains (see [5]). Also, for such a process, if $Z_t \to +\infty$ as $t \to s$ from one side, then $Z_t \to +\infty$ as $t \to s$ from both sides. Thus, a process satisfying hypothesis A may have explosions but the way in which infinity can be approached is quite restricted.

Before proceeding with the limit theorems, we list the following definitions and notation which are required in this paper and in [16]. For a more complete treatment of some of these quantities, the reader is referred to [14], [15].

$$Q_{ij}(t) = Q_{ij}(0; t),\ p_{ij} = Q_{ij}(+\infty),\ Q_{ij}(t) = p_{ij}F_{ij}(t), \tag{2.6.1}$$

$$P_{ij}(t) = P[Z_t = j \mid Z_0 = i, U_0 = 0], \quad P_{ij}^+(t) = P[Z_t^+ = j \mid Z_0 = i, U_0 = 0], \tag{2.6.2}$$

$$G_{ij}(t) = P[N_j(t) > 0 \mid Z_0 = i, U_0 = 0], \tag{2.6.3}$$

$$M_{ij}(t) = E[N_j(t) \mid Z_0 = i, U_0 = 0] + \delta_{ij}, \tag{2.6.4}$$

$${}_kM_{ij}(t) = E[N_j(T_k) \mid Z_0 = i, U_0 = 0]\, [1 - \delta_{jk}] + \delta_{ij} \tag{2.6.5}$$

where $T_k = \min(t, S_k)$ and $S_k = \inf\{t > V_0 : Z_t = k\}$,

$$ {}_kG_{ij}(t) = P[\text{for some } u \leqq t,\ N_j(u) > 0,\ N_k(u) = 0 \mid Z_0 = i,\ U_0 = 0], \tag{2.6.6} $$

$$ {}_kP_{ij}(t) = P[Z_t = j,\ N_k(t) = 0 \mid Z_0 = i,\ U_0 = 0], \tag{2.6.7} $$

$$ S_i(j, x; t) = P[Z_t = j,\ U_t \leqq x \mid Z_0 = i,\ U_0 = 0], \tag{2.6.8} $$

$$ R_i(j, x; t) = P[Z_t = j,\ V_t \leqq x \mid Z_0 = i,\ U_0 = 0], \tag{2.6.9} $$

$$ S_i(j, k, x: t) = P[Z_t = j,\ Z_t^+ = k,\ U_t \leqq x \mid Z_0 = i,\ U_0 = 0], \tag{2.6.10} $$

$$ R_i(j, k, x: t) = P[Z_t = j,\ Z_t^+ = k,\ V_t \leqq x \mid Z_0 = i,\ U_0 = 0]. \tag{2.6.11} $$

It is clear that $S_i(j, x; t) = P_t(i, 0; j, x)$ and $P_{ij}(t) = S_i(j, +\infty; t)$. Let η_j, b_{ij}, and μ_{ij} denote the first moments of H_j, F_{ij}, and G_{ij} respectively.

A script letter will denote a matrix-valued function whose elements consist of doubly or singly indexed functions which use the same letter. Singly indexed functions are understood to yield diagonal matrices. For example, $\mathcal{G} = (G_{ij})$ and $\mathcal{H} = (\delta_{ij}H_i)$. For any matrix-valued function $\mathcal{K}$, ${}^d\mathcal{K}$ is defined to be the diagonal of $\mathcal{K}$, namely, ${}^d\mathcal{K} = (\delta_{ij}K_{ij})$. For any two matrix-valued functions $\mathcal{K}$ and $\mathcal{L}$, define $\mathcal{K} * \mathcal{L} = (\sum_k K_{ik} * L_{kj})$ whenever the definition is valid.

By means of the strong Markov property and the usual probabilistic arguments, the following "backward" relationships (that is, relationships based on the first transition) may be derived straightforwardly for any MRP satisfying hypothesis A.

$$ \mathcal{P}(t) = \mathcal{Q} * \mathcal{P}(t) + (I - \mathcal{H})(t), \tag{2.7.1} $$

$$ \mathcal{P}(t) = \mathcal{M} * (I - \mathcal{H})(t) = \mathcal{G} * {}^d\mathcal{P}(t) + (I - \mathcal{H})(t), \tag{2.7.2} $$

$$ \mathcal{G}(t) = \mathcal{Q} * \mathcal{G}(t) + \mathcal{Q} * (I - {}^d\mathcal{G})(t), \tag{2.7.3} $$

$$ \mathcal{M}(t) = \mathcal{Q} * \mathcal{M}(t) + I, \tag{2.7.4} $$

$$ \mathcal{M}(t) = \mathcal{G} * {}^d\mathcal{M}(t) + I, \tag{2.7.5} $$

$$ R_i(j, y; t) = \int_{0-}^{t} H_i(u; y) S_i(j, du; t), \tag{2.7.6} $$

$$ {}_k\mathcal{P}(t) = {}_k\mathcal{M} * (I - \mathcal{H})(t), \tag{2.7.7} $$

$$ P_{ij}(t) = {}_kP_{ij}(t) + G_{ik} * P_{kj}(t),\ M_{ij}(t) = {}_kM_{ij}(t) + G_{ik} * M_{kj}(t), \tag{2.7.8} $$

$$ \mathcal{P}^+(t) = \mathcal{M} * (P - \mathcal{Q})(t) = \mathcal{M}(t)(P - I) + I. \tag{2.7.9} $$

For a strongly-regular MRP, it has been shown (see [15]) that $\mathcal{M}$ is completely determined from $\mathcal{Q}$ by means of the equation $\mathcal{M}(t) = \sum_n \mathcal{Q}^{(n)}(t)$. Hence $\mathcal{M} * \mathcal{Q} = \mathcal{Q} * \mathcal{M}$. This commutativity implies that the "forward" relationship (that is, a relationship based on the last transition before time t) $\mathcal{M}(t) = \mathcal{M} * \mathcal{Q}(t) + I$

holds as well as (2.7.4). The following lemma shows that the same commutativity holds true for MRP's satisfying hypothesis A.

LEMMA 2.2. *For an* MRP *satisfying hypothesis* A,

$$\mathfrak{M}(t) = \mathfrak{M} * \mathcal{Q}(t) + I \qquad \textit{for all } t \geqq 0.$$

PROOF. By conditioning on the last state visited, one obtains

$$P_{ij}(t) = \sum_k M_{ik} * Q_{kj} * (1 - H_j)(t) + \delta_{ij}(1 - H_j(t)).$$

Using p, m, q, and h to represent the Laplace-Stieltjes transforms of $\mathcal{P}$, $\mathfrak{M}$, $\mathcal{Q}$, and $\mathcal{H}$ respectively, one obtains, using (2.7.2), that

$$[\sum_k m_{ik}(s)q_{kj}(s) + \delta_{ij}][1 - h_j(s)] = m_{ij}(s)[1 - h_j(s)].$$

Hence $m_{ij}(s) = \sum_k m_{ik}(s)q_{kj}(s) + \delta_{ij}$. The lemma follows from the uniqueness of the Laplace-Stieltjes transform.

In Sections 4, 5, 6 and 7 below we will be concerned with sums of a function of an MRP satisfying hypothesis A. The specific context for these sections is as follows. Let f be a real valued function defined on $I^+ \times I^+ \times R_1$. Assume that for each $i, j \varepsilon I^+$, $f(i, j, \cdot)$ is Lebesgue measurable. Let X_{ijn} denote the holding time of the nth transition between states i and j, and let $N_{ij}(t)$ equal the number of transitions from i to j which occur during $[0, t]$. For all $t \geqq 0$, define

$$W_f(t) = \sum_{i,j} \sum_{n=1}^{N_{ij}(t)} f(i, j, X_{ijn}) \tag{2.8}$$

whenever the series converges. For a strongly-regular MRP one may write (2.8) equivalently as

$$W_f(t) = \sum_{n=1}^{N(t)} f(J_{n-1}, J_n, X_n). \tag{2.9}$$

Another functional which might be of as much interest as (2.8) is

$$W_g^*(t) = \int_0^t g(X_u, U_u, Z_u, Z_u^+)\, du, \tag{2.10}$$

where g is an integrable function defined on $R_1 \times R_1 \times I^+ \times I^+$. However, if one defines $f(i, j, x) = \int_0^x g(x, y, i, j)\, dy$ then $W_g^*(t) = W_f(t) + \int_0^{U_t} g(X_t, y, Z_t, Z_t^+)\, dy$. We shall study limit theorems for $W_f(t)$ only, since the analogous results for $W_g^*(t)$ may be derived by similar methods.

By considering the related MRP with new two-dimensional states (i, j), $i, j \varepsilon I^+$, and the new associated S-MP $Z_t^* = (Z_t, Z_t^+)$ with transition distributions of the form

$$Q_{(i,j),(k,r)}(x; t) = \delta_{jk} p_{kr} Q_{ij}(x; t)/Q_{ij}(x; \infty) \tag{2.11}$$

one may always, without loss of generality, restrict oneself to functions, f, of the two variables (Z_u, X_u). Therefore, when studying the limiting behavior of $W_f(t)$ the functions f are thus restricted and hence (2.8) is replaced with

$$W_f(t) = \sum_i \sum_{n=1}^{N_i(t)} f(i, X_{in}). \tag{2.12}$$

In Section 4, where explicit computations of moments are made, the results are given for the more general functions of three variables, since the reader would find it prohibitive to carry out the transformation given by (2.11) each time he applies them to a particular situation.

Most of the methods of proof used in this paper are similar to those used by Chung [5] when dealing with the analogous problems of Markov chains.

3. Doeblin Ratio limit theorems. In this section, one is led to consider the expected amount of time which an MRP spends in certain states. We will therefore use the convention that a bar over a symbol will denote the integral over $[0, t]$ of the same symbol without the bar. For example, $\bar{P}_{ij}(t) = \int_0^t P_{ij}(u)\,du$.

The following relationships are straightforwardly derived.

$$G_{ik}(t) = \sum_{n=0}^{\infty} {}_iG_{ik} * {}_kG_{ii}^{(n)}(t) = {}_iG_{ik} * {}_kM_{ii}(t), \tag{3.1}$$

$$P_{ij}(t) = G_{ij} * P_{jj}(t) + \delta_{ij}[1 - H_i(t)] = {}_iP_{ij} * M_{ii}(t), \tag{3.2}$$

$$G_{jj}(t) = {}_iG_{jj}(t) + G_{ij} * {}_jG_{ji}(t), \qquad (i \neq j), \tag{3.3}$$

$${}_iM_{ij}(t) = {}_iM_{jj} * {}_iG_{ij}(t), \qquad (i \neq j), \tag{3.4}$$

$$\bar{P}_{ij}(t) = \int_0^t \int_0^u P_{jj}(u - v)\,dG_{ij}(v)\,du = \bar{P}_{jj} * G_{ij}(t), \qquad (i \neq j). \tag{3.5}$$

In deriving the limit theorems below, one uses Abelian arguments. The form of Abelian theorem most applicable to these discussions is given now, the proof of which is standard and therefore omitted.

LEMMA 3.1. *Let K be a mass function (that is, a non-decreasing right-continuous function) for which $K(t) = 0$ if $t < 0$ and $K(t + 1) - K(t) < c$ for all $t \geq 0$ and some constant c. Then for any mass function F satisfying $F(t) = 0$ if $t < 0$, one has*

$$\lim_{t\to\infty}[K * F(t)]/K(t) = F(\infty).$$

In Markov Chain theory, a Doeblin Ratio limit theorem gives the limit as $n \to \infty$ of the ratio of the expected number of visits to one state, to that of another state under various initial conditions (cf. [5]). Analogous theorems for abstract state space discrete parameter Markov processes are also possible (cf. [6] and [12]). For continuous parameter Markov chains, Doeblin Ratio limit theorems are obtained for the expected amount of time spent in a state rather than the expected number of visits to a state. Of course, for discrete parameter processes the "number of visits" and the "time spent" are equivalent. It is the purpose of this section to derive the limits for both of these types of ratios, as well as for some more general ones. Notice that $\bar{P}_{ij}(t)$ is the expected *amount of time* spent in state j during $[0, t]$ given that $Z_0 = i$, $U_0 = 0$, while $M_{ij}(t)$ is the expected *number of visits* to state j during $[0, t]$ given that $Z_0 = i$, $U_0 = 0$.

LEMMA 3.2. *For any two states i and j, $(i \neq j)$,*

$$\lim_{t\to\infty} \bar{P}_{ij}(t)/\bar{P}_{jj}(t) = G_{ij}(\infty) = \lim_{t\to\infty} M_{ij}(t)/M_{jj}(t). \tag{3.6}$$

PROOF. That the ratios are both well defined follows since $M_{jj}(t) \geqq M_{jj}(0) = 1$ and $\bar{P}_{jj}(t) \geqq E[\min(V_0, t) \mid Z_0 = j, U_0 = 0] > 0$. The existence and evaluation of the limits follows by application of Lemma 3.1 to (2.7.5) and (3.5).

LEMMA 3.3. *For any two states i and j,*

$$\lim_{t\to\infty} \bar{P}_{ij}(t)/M_{ii}(t) = {}_i\bar{P}_{ij}(\infty). \tag{3.7}$$

PROOF. It follows from (3.2) that

$$\bar{P}_{ij}(t) = {}_i\bar{P}_{ij} * M_{ii}(t). \tag{3.8}$$

Therefore (3.7) follows from Lemma 3.1. The quantity ${}_i\bar{P}_{ij}(\infty)$ is the expected time that the S-MP spends in state j before the first transition into state i is made, given that $Z_0 = i$, $U_0 = 0$. The limit in (3.7) may then take on any value in the closed interval $[0, \infty]$. It may be shown directly, or by application of Lemma 4.1 below that ${}_i\bar{P}_{ij}(\infty) = \eta_j \, {}_iM_{ij}(\infty)$ whenever $G_{ij}(\infty) > 0$.

THEOREM 3.1. *For any two states i and j,*

$$\lim_{t\to\infty} M_{ij}(t)/M_{ii}(t) = {}_iM_{ij}(\infty). \tag{3.9}$$

PROOF. Assume first that state i is recurrent and that $i \neq j$. Define V_n to be the number of transitions into state j between the $(n-1)$th and nth consecutive visit to state i $(n = 1, 2, \cdots)$. Define W_t to be the number of transitions into state j during the interval $(t, V_i(t)]$ where $V_i(t) = \inf\{u > V_t : Z_u = i\}$. Then

$$\begin{aligned} M_{ij}(t) &= E\left(\sum_{n=1}^{N_i(t)+1} V_n \mid Z_0 = i, U_0 = 0\right) - E(W_t \mid Z_0 = i, U_0 = 0) \\ &= {}_iM_{ij}(\infty)M_{ii}(t) - E(W_t \mid Z_0 = i, U_0 = 0) \end{aligned} \tag{3.10}$$

by Wald's Fundamental Identity. If $G_{ij}(\infty) = 0$, then $V_1 = W_t = 0$ and (3.9) is vacuously true. Otherwise $G_{ij}(\infty) = 1$ and state j is also recurrent. That ${}_iM_{ij}(+\infty) < +\infty$ in this case follows since ${}_iM_{ij}(+\infty) = {}_iG_{ij}(+\infty)/[1 - {}_iG_{jj}(+\infty)]$. Moreover, one may show that

$$\begin{aligned} E(W_t \mid Z_0 = i, U_0 = 0) = {}_iM_{jj}(\infty)\{&[{}_iG_{jj}(\infty) - {}_iG_{jj}] * {}_iM_{ij} \\ &+ [{}_iG_{ij}(\infty) - {}_iG_{ij}]\} * M_{ii}(t) \end{aligned}$$

which is finite. Hence by repeated application of Lemma 3.1 one obtains $\lim_{t\to\infty} E(W_t \mid Z_0 = i, U_0 = 0)/M_{ii}(t) = 0$. This together with (3.10) proves (3.9) for recurrent states i. Assume now that state i is not recurrent. Then

$M_{ii}(\infty) < \infty$ and the limit in (3.11) is equal to $M_{ij}(\infty)/M_{ii}(\infty)$. It remains to prove that this limit equals ${}_iM_{ij}(\infty)$. But (2.7.8) with $k = i$ yields $M_{ij}(\infty) = {}_iM_{ij}(\infty)/[1 - G_{ii}(\infty)]$ and $M_{ii}(t) = 1 + G_{ii} * M_{ii}(t)$ yields $M_{ii}(\infty) = 1/[1 - G_{ii}(\infty)]$ so that $M_{ij}(\infty)/M_{ii}(\infty) = {}_iM_{ij}(\infty)$.

COROLLARY 3.1. *For any two states i and j $(i \neq j)$ for which $G_{ij}(\infty) > 0$,*

$$\lim_{t\to\infty} M_{jj}(t)/M_{ii}(t) = {}_iM_{jj}(\infty)/{}_jM_{ii}(\infty). \tag{3.11}$$

If, moreover, $\eta_i < \infty$, then

$$\lim_{t\to\infty} \bar{P}_{ij}(t)/\bar{P}_{ii}(t) = (\eta_j/\eta_i)\,{}_iM_{ij}(\infty). \tag{3.12}$$

PROOF. From the above theorem and Lemma 3.2,

$$\lim_{t\to\infty} M_{jj}(t)/M_{ii}(t) = {}_iM_{ij}(\infty)/G_{ij}(\infty).$$

But from (3.1) and (3.4), ${}_iM_{ij}(\infty)\,{}_jM_{ii}(\infty) = G_{ij}(\infty)\,{}_iM_{jj}(\infty)$, as desired, since the positivity of $G_{ij}(\infty)$ implies the finiteness of ${}_jM_{ii}(\infty)$. The second statement of the corollary is a consequence of the foregoing results applied to the relationship

$$\bar{P}_{ij}/\bar{P}_{ii} = (\bar{P}_{ij}/M_{ii})(M_{ii}/M_{jj})(M_{jj}/\bar{P}_{ji})(\bar{P}_{ji}/\bar{P}_{ii})$$

in which the dependence upon t has been omitted.

It is remarked that in the case of finite mean recurrence times μ_{ii} and μ_{jj} the limit in (3.12) is equal to $G_{ij}(\infty)\eta_j\mu_{ii}/\eta_i\mu_{jj}$. This follows from Lemma 4.1 below.

For an MRP it turns out that the most important Doeblin Ratios are those involving the functions $R_i(j, k, x: t)$ or $S_i(j, k, x: t)$. In particular, it is the limit of these ratios which provide the stationary measures for null-recurrent MRP's (cf. [16]). Nevertheless the proofs of these results follow exactly the same lines of those given above for the simpler ratios. Therefore, only the outlines for the proofs of the following theorems are given.

From the definitions in (2.6), the following relationships are immediate:

$$\begin{aligned} R_i(j, k, x: t) &= G_{ij} * R_j(j, k, x: \cdot)(t), && (i \neq j), \\ \bar{R}_i(j, k, x: t) &= G_{ij} * \bar{R}_j(j, k, x: \cdot)(t), && (i \neq j), \end{aligned} \tag{3.13}$$

$$\begin{aligned} R_i(j, k, x: t) &= {}_iR_i(j, k, x: \cdot) * M_{ii}(t), \\ \bar{R}_i(j, k, x: t) &= {}_i\bar{R}_i(j, k, x: \cdot) * M_{ii}(t). \end{aligned} \tag{3.14}$$

By direct application of Lemma 3.1 to (3.13) and (3.14) one obtains

THEOREM 3.2. *For any states i, j, k and any $x > 0$*

$$\lim_{t\to\infty} \bar{R}_i(j, k, x: t)/\bar{R}_j(j, k, x: t) = G_{ij}(\infty), \qquad (i \neq j,\ p_{jk} > 0)$$

and

$$\lim_{t\to\infty} \bar{R}_i(j, k, x: t)/M_{ii}(t) = {}_i\bar{R}_i(j, k, x: \infty).$$

LEMMA 3.4. *${}_i\bar{R}_i(j, k, x: \infty) = +\infty$ if and only if $p_{jk} > 0$ and ${}_iM_{ij}(\infty) = +\infty$. More precisely,*

$$(3.15) \qquad {}_i\bar{R}_i(j, k, x: \infty) = {}_iM_{ij}(\infty) \int_0^x [Q_{jk}(\infty) - Q_{jk}(u)]\, du.$$

PROOF. Since ${}_iR_i(j, k, x: t) = [Q_{jk}(x + \cdot) - Q_{jk}(\cdot)] * {}_iM_{ij}(t)$, one obtains

$$\begin{aligned} {}_i\bar{R}_i(j, k, x: t) = {}_iM_{ij}(t) \int_0^x [Q_{jk}(\infty) - Q_{jk}(u)]\, du \\ - \int_0^t \int_{t-u}^{t-u+x} [Q_{jk}(\infty) - Q_{jk}(v)]\, dv\, d_iM_{ij}(u). \end{aligned}$$

This expression may be shown to converge to the right hand side of (3.15) as $t \to \infty$ whenever $p_{jk} > 0$. To do this, one uses (3.4) and the Key Renewal theorem in the case of ${}_iM_{ij}(\infty) = +\infty$.

As a consequence of the above results, one obtains the following corollary which is used in [16] in studying the stationary measures for a null-recurrent MRP.

COROLLARY 3.2. *If i, j, k, m, r, h are in the same recurrent class and if $p_{rh} > 0$, then*

$$\lim_{t\to\infty} \frac{\bar{R}_i(j, k, x: t)}{\bar{R}_m(r, h, y: t)} = {}_mM_{mi}(\infty)\, \frac{{}_iM_{ij}(\infty) \int_0^x [Q_{jk}(\infty) - Q_{jk}(u)]\, du}{{}_mM_{mr}(\infty) \int_0^y [Q_{rh}(\infty) - Q_{rh}(u)]\, du}.$$

4. Computation of moments. Throughout the remainder of this paper it is assumed that we are working with an irreducible recurrent, (i.e., $G_{ij}(+\infty) = 1$ for all $i, j \in I^+$), regular MRP satisfying hypothesis A. Let f be a real-valued function of the type described in Section 2. Let $\{T_{jn} : n \geqq 1\}$ be the successive occurrence times of state j, with the understanding that $T_{j1} > 0$ even if $Z_0 = j$. For convenience set $T_{j0} = 0$. For each j, define the sequence of r.v.'s

$$(4.1) \qquad Y_n^{(j)} = W_f(T_{j,n+1}) - W_f(T_{jn}) \qquad (n \geqq 1).$$

Thus $Y_n^{(j)}$ denotes the part of the summation in (2.9) which occurred during $(T_{jn}, T_{j,n+1}]$. These r.v.'s are independent and identically distributed by the assumptions of Section 2.

Throughout the remainder of this paper, we will focus our attention on a fixed state j, say $j = 0$. It will then be convenient to write simply Y_n and T_n for $Y_n^{(0)}$ and T_{0n}, respectively.

Whenever the quantities are defined, set

$$(4.2) \qquad \xi_{ik} = \int_0^\infty f(i, k, x)\, dQ_{ik}(x), \qquad \xi_{ik}^{(2)} = \int_0^\infty [f(i, k, x)]^2\, dQ_{ik}(x)$$

and

$$\zeta_i = \sum_k \xi_{ik}, \qquad \zeta_i^{(2)} = \sum_k \xi_{ik}^{(2)}.$$

The results to follow will require one of the conditions,

$$\sum_j \int_0^\infty |f(k, j, x)| \, dQ_{kj}(x) < \infty \tag{4.3}$$

or

$$\sum_j \int_0^\infty [f(k, j, x)]^2 \, dQ_{kj}(x) < \infty. \tag{4.4}$$

For convenience, set $m_i = {}_0M_{0i}(+\infty)$.

LEMMA 4.1. *If* (4.3) *is satisfied and*

$$\sum_i \sum_k |\xi_{ik}| m_i < \infty, \tag{4.5}$$

then $E|Y_n| < \infty$ *and*

$$E(Y_n) = \sum_i m_i \zeta_i . \tag{4.6}$$

If also $\mu_{00} < \infty$, then $\mu_{00} = \sum_i \eta_i m_i$ *and* $E(Y_n) = \mu_{00} A_f$ *where*

$$A_f = \sum_i m_i \zeta_i / \mu_{00} \tag{4.7}$$

does not depend on the state 0.

PROOF. The proof of (4.6) is an extension of that given by Chung (Section I. 14 of [5]) for the corresponding result for Markov Chains. In order to compute $E(Y_n)$, it suffices to compute $E(Y_1)$.

For each $u \geqq 0$, define the random variable $G(u)$ as

$$G(u) = X_u^{-1} f(Z_u, Z_u^+, X_u) I_{[T_2 > u > T_1]} . \tag{4.8}$$

It is easy to deduce from (4.1) that

$$Y_1 = \int_{T_1}^{T_2} G(u) \, du.$$

Therefore,

$$E(Y_1) = E\left[\int_0^\infty G(u) \, du\right] = \int_0^\infty E[G(u)] \, du.$$

But

$$E[G(u)] = \sum_{i \neq 0} \sum_k \sum_{n=1}^\infty \int_0^u \int_0^{u-v} \int_{u-v-s}^\infty x^{-1} f(i, k, x) \, dQ_{ik}(x) \, d_0 G_{0i} * {}_0G_{ii}^{(n-1)}(v) \, dG_{00}(s)$$

$$+ \sum_k \int_0^u \int_{u-v}^\infty x^{-1} f(0, k, x) \, dQ_{0k}(x) \, dG_{00}(v)$$

and

$$\int_0^\infty E[G(u)]\,du = \sum_{i\neq 0}\sum_k \sum_{n=1}^\infty \int_0^\infty \int_0^u \int_0^{u-v}$$

$$\cdot \int_{u-v-s}^\infty x^{-1}f(i, k, x)\,dQ_{ik}(x)\,d_0G_{i0} * {}_0G_{ii}^{(n-1)}(v)\,dG_{00}(s)\,du$$

$$+ \sum_k \int_0^\infty \int_0^u \int_{u-v}^\infty x^{-1}f(0, k, x)\,dQ_{0k}(x)\,dG_{00}(v)\,du$$

$$= \sum_i m_i \sum_k \int_0^\infty f(i, k, x)\,dQ_{ik}(x) = \sum_i m_i \zeta_i$$

since $m_0 = 1$. All interchanges of summation and integration in the above computations are valid because of assumptions (4.3) and (4.5).

Consider now the special function $f^*(i, j, x) = x$ for all i and j. Then $G(u) = 1$ for $T_1 < u < T_2$, $G(u) = 0$ otherwise, and $\zeta_i = \eta_i$. Hence (4.6) yields

$$\mu_{00} = E(T_2 - T_1) = \sum_i \eta_i m_i. \tag{4.9}$$

Therefore, the proof is complete since one may show, using Corollary 3.1, that $m_i/m_k = {}_kM_{ki}(+\infty)$, a constant depending on i and k but *not* on the fixed state 0.

To compute the second moment of Y_1, and hence its variance, assume that (4.4) is satisfied. Whenever $\mu_{00} < \infty$ set

$$B_f = \left(\sum_i \eta_i m_i\right)^{-1}\left\{\sum_i \zeta_i^{(2)} m_i + 2\sum_i \sum_{k\neq 0} \sum_{r\neq 0} (\xi_{ik}\zeta_r m_i)\,{}_jM_{kr}(\infty)\right\} \tag{4.10}$$

and assume that each series on the right hand side is absolutely convergent when ζ_i is replaced by $\sum_k |\xi_{ik}|$. Then one may prove, by the same method as for Lemma 4.1,

LEMMA 4.2. *Under the above assumptions*

$$E(Y_1^2) = \mu_{00}B_f < \infty. \tag{4.11}$$

Again, by using the specific function f^* defined above, one could deduce from Lemma 4.2, an expression for the second moment $\mu_{00}^{(2)}$, and hence the variance, of the recurrence time of state 0. One should note also that because of the form of $E(Y_1^2)$ it is not necessary that $\mu_{00}^{(2)} < \infty$ in order that $E(Y_1^2) < \infty$.

A more general result than that given in Lemma 4.2 may be derived in a similar manner. Namely, let f and g be two arbitrary functions such that f, g and fg each satisfies (4.3). Then one may obtain

LEMMA 4.3. *Whenever the indicated series are absolutely convergent, one has*

$$\begin{aligned} E[Y_1(f)Y_1(g)] &= \sum_i \zeta_i(fg)m_i \\ &\quad + \sum_i \sum_{k\neq 0} \sum_{r\neq 0} [\xi_{ik}(f)\zeta_r(g) + \xi_{ik}(g)\zeta_r(f)]m_i[{}_jM_{kr}(\infty)] \end{aligned} \tag{4.12}$$

where, for example, $\zeta_i(fg)$, $\xi_{ik}(f)$ and $Y_1(g)$ are simply the quantities defined by (4.2) *and* (4.1), *but for the functions fg, f and g, respectively.*

Clearly Lemma 4.2 is the special case of Lemma 4.3 in which $g = f$.

By considering (without loss of generality) the restricted class of 2-dimensional functions f discussed at the end of Section 2, and the restricted class of strongly regular MRP's which satisfy (2.11), it is possible to consider Lemmas 4.1 and 4.2 as consequences of the corresponding results for Markov Chains, by using the conditional independence of $X_1, X_2, \cdots$ given $J_0, J_1, \cdots$. A third method for obtaining these moment computations is to employ a result of Orey (Theorem 6.1 of [12]) for abstract state Markov Chains which are recurrent in the sense of Harris [6]. As is pointed out in [14], the process $\{(J_n, X_n); n \geqq 0\}$ is a Markov process. Clearly this is a recurrent process (in the sense of Harris) whenever the c.M.C. $\{J_n; n \geqq 0\}$ is recurrent. To see this, define a sigma-finite measure μ on Borel sets of $I^+ \times [0, \infty)$ by $\mu(\{k\} \times [0, x)) = m_k H_k(x)$. It is then possible to apply the stationarity results of [16] to Orey's result to obtain (4.6) above.

Before turning to the limit theorems, the following result which will be needed below, is stated. Its proof is omitted since it may be proved either by paralleling the proof of Theorem I.14.4 of [5] or by applying Theorem 6.2 of [12].

LEMMA 4.4. *For $r > 0$, set $\mu_k = E|Y_n^{(k)}|^r$. Then for two distinct states i, j in the same recurrence class, μ_i and μ_j are either both finite or both infinite.*

The main consequence of this lemma is the fact that the mean recurrence times μ_{jj} (and their variances) are either all finite or all infinite for all states j in the same irreducible recurrence class.

5. Strong Law of Large Numbers. Consider an irreducible recurrent MRP and a real-valued function f defined on $I^+ \times R_1$ for which the series (2.9) converges for all t, a.s. As in Section 4, f and the state $j = 0$ are considered fixed. Throughout this section, assume $E|Y_n| < \infty$ and set $m = E(Y_n)$, $(n \geqq 1)$. Set $G(u) = f(Z_u, X_u)/X_u$ for all $u \geqq 0$ (as in (4.8)). Set $a_t = T_{N_0(t)}$, $b_t = t - U_t$, and

$$R_1(t) = I_{[T_1 \leqq t]} \int_0^{T_1} G(u)\,du, \qquad R_2(t) = \int_{a_t}^{b_t} G(u)\,du, \tag{5.1}$$
$$V(t) = I_{[T_1 \leqq t]} \int_{T_1}^{a_t} G(u)\,du = \sum_{n=1}^{N_0(t)-1} Y_n$$

which yields the decomposition $W(t) = R_1(t) + V(t) + R_2(t)$.

LEMMA 5.1. *Under the assumption that $E|Y_n| < +\infty$,*

$$R_1(t)/t \to 0 \quad \text{(a.s.)} \tag{5.2}$$

and

$$V(t)/t \to m/\mu_{00} \quad \text{(a.s.)}, \tag{5.3}$$

the latter limit being zero if $\mu_{00} = \infty$.

PROOF. Clearly $|R_1(t)| \leqq |\int_0^{T_1} G(u)\,du|$. Since the right hand side of this inequality is a well defined finite r.v. which does not depend on t, (5.2) is immediate. To prove (5.3), observe that by the S.L.L.N. for independent and identically distributed r.v.'s, the assumption of the lemma implies that $n^{-1}\sum_{s=1}^{n} Y_s \to m$ (a.s.). Moreover, it is known in Renewal theory that $N_0(t)/t \to \mu_{00}^{-1}$ (a.s.), even if $\mu_{00} = +\infty$. These two results together imply (5.4).

Because of the factorization of $W(t)$ given in (5.1), it follows that under the assumptions of Lemma 5.1, $W(t)/t$ will converge almost surely, if and only if $R_2(t)/t$ does. Set

$$Y_n^+ = \int_{T_n}^{T_{n+1}} |G(u)|\, du, \qquad (n \geqq 1), \tag{5.4}$$

which is to say that Y_n^+ is the analogue of Y_n with f replaced by $|f|$.

THEOREM 5.1. (S.L.L.N.). *If* $E(Y_1^+) < \infty$, *then* $W(t)/t \to m/\mu_{00}$ (a.s.).

PROOF. Notice first that $E(Y_1^+) < \infty$ implies $E|Y_1| < \infty$. Moreover $|R_2(t)| \leqq Y^+_{N_0(t)}$. Since $E(Y_1^+) < \infty$, one has $Y_n^+/n \to 0$ (a.s.) and since $N_0(t)/t \to \mu_{00}^{-1}$ (a.s.) it follows that

$$t^{-1}Y^+_{N_0(t)} = [Y^+_{N_0(t)}/N_0(t)]\cdot[N_0(t)/t] \to 0 \quad \text{(a.s.)}$$

as required. This theorem together with Lemma 4.4 leads to

COROLLARY 5.1. *If* $E(Y_1^+) < \infty$, $m/\mu_{00} \equiv A$ *does not depend on the choice* $j = 0$.

The condition $E(Y_1^+) < \infty$ is quite strong, as is indicated by Example (i) below. It is desirable to try to obtain weaker conditions, and, if possible, necessary and sufficient conditions for the a.s. convergence of $R_2(t)/t$. The next theorem gives a much weaker sufficient condition for the S.L.L.N. than that given in Theorem 5.1, as well as a necessary and sufficient condition in the positive recurrent case. For each $v \geqq 0$ define

$$Y_n(v) = \left\{\int_{T_n}^{T_n+v} G(u)\, du\right\} I_{[T_{n+1}-T_n \geqq v]} \tag{5.5}$$

and

$$M_n = \sup\nolimits_{0<v<\infty} |Y_n(v)|, \qquad (n \geqq 1). \tag{5.6}$$

THEOREM 5.2. (a) *If* $E(M_n) < \infty$, *then*

$$t^{-1}W(t) \to m/\mu_{00} \quad \text{(a.s.)}. \tag{5.7}$$

(b) *If* $\mu_{00} < \infty$, *then* (5.7) *holds if and only if* $E(M_n) < \infty$.

PROOF. Observe first of all that $E(M_n) < \infty$ implies $E|Y_n| < \infty$, so that Lemma 5.1 applies. Also, on the event $[N_0(t) > 0]$, one has that

$$R_2(t) = Y_{N_0(t)}(t - U_t - a_t). \tag{5.8}$$

It therefore follows that on this event

$$|Y_{N_0(t)}(t - a_t - U_t)|/T_{N_0(t)+1} \leqq t^{-1}|R_2(t)| \leqq |Y_{N_0(t)}(t - a_t - U_t)|/a_t \tag{5.9}$$

so that

$$\begin{aligned} \limsup_{n \geqq 1} (M_n/n)(n/T_{n+1}) &\leqq \limsup_{t\to\infty} t^{-1}|R_2(t)| \\ &\leqq \limsup_{n\geqq 1} (M_n/n)(n/T_n). \end{aligned} \tag{5.10}$$

To obtain the first inequality in (5.10) observe that its left member may be written as

$$\lim_{t\to\infty} \sup_{n \geqq N_0(t)} \sup_{0 \leqq u < T_{n+1} - T_n} \{|Y_n(u)|/T_{n+1}\}.$$

The right hand inequality in (5.10) suffices to prove (a) because if $E(M_n) < \infty$, and hence $M_n/n \to 0$ a.s., then $R_2(t)/t \to 0$ a.s., since, in any case, $T_n/n \to \mu_{00}$ a.s. If $\mu_{00} < \infty$ then from the left hand inequality in (5.10) it follows that $R_2(t)/t \to 0$ a.s. implies that $M_n/n \to 0$ a.s. and hence $E(M_n) < \infty$, thereby proving (b).

In order to illustrate the importance of the various assumptions used in this and the remaining sections, the following examples of strongly-regular MRP's have been constructed.

Examples. Let the MRP under consideration actually be a Markov Chain with transition probabilities of the form $p_{i,i+1} = 1 - p_{i0}$, $p_{2i,2i+1} = 1$, $(i \geqq 0)$. Therefore, the mean recurrence time μ_{00} is equal to $2\sum_{i=1}^{\infty} d_i$ where $d_1 = 1$ and $d_i = \prod_{k=1}^{i-1} p_{2k-1,2k} = P[T_1 \geqq 2i]$, for $i > 1$.

(i) This first example will illustrate that neither the condition $E(Y_1^+) < \infty$, of Theorem 5.1 nor the condition $\mu_{00} < \infty$, is a necessary one. Define the function f to be $f(0) = 0$, $f(2i) = -f(2i - 1) = 1 (i \geqq 1)$, and choose the d_i's so that $\mu_{00} = \infty$. Then

$$\begin{aligned} R_2(t)/t &= 0 && \text{if } [t] - a_t \text{ is odd} \\ &= -1/t && \text{if } [t] - a_t \text{ is even} \end{aligned}$$

(where $[t]$ is the greatest integer $\leqq t$), is obviously convergent to zero a.s. Moreover $Y_1 = -1$ and $E(Y_1^+) = \mu_{00} - 1 = \infty$.

(ii) One can modify the above example slightly to show that even when $\mu_{00} < \infty$, the condition $E(Y_1^+) < \infty$, is still not a necessary one. To do this, choose the d_i's so that $\mu_{00} = \sum_{i=1}^{\infty} d_i < \infty$, and yet $\sum_{i=1}^{\infty} i^2(d_i - d_{i+1}) = \infty$. Define the function f by $f(2i) = -f(2i + 1) = i \ (i \geqq 0)$. Then

$$\begin{aligned} R_2(t)/t &= 0 && \text{if } [t] - a_t \text{ is even} \\ &= ([t] - a_t - 1)/2t && \text{otherwise} \end{aligned}$$

which again converges to zero a.s. However, in this case $Y_1 = 0$ and $E(Y_1^+) = E[(T_2 - T_1)^2/4] - E[(T_2 - T_1)/2] = \infty$, by construction.

(iii) If the above example is modified to make $\mu_{00} = \infty$, then it becomes one for which $E|Y_1| < \infty$, and yet $R_2(t)/[t]$, and hence $W(t)/[t]$, can only converge in distribution if it converges at all. This follows from results of Dynkin and Lamperti (cf. Theorem 3.2 of [8]).

(iv) Consider now an example in which the d_i's are so chosen as to make

$\sum_{i=1}^{\infty} i^{\frac{1}{2}}(d_i - d_{i+1}) = \infty$ and hence, a fortiori, $\mu_{00} = \infty$. Define the function f by $f(2i) = -f(2i+1) = i^{\frac{1}{2}}$ $(i \geqq 0)$. Then

$$\begin{aligned} R_2(t)/t &= 0 \qquad\qquad\qquad\qquad \text{if } [t] - a_t \text{ is even} \\ &= \{([t] - a_t - 1)/2\}^{\frac{1}{2}}/t \qquad \text{otherwise} \end{aligned}$$

and, since $[t] - a_t - 1 \leqq t$, this converges to zero a.s. However, in this example

$$M_1 = [(T_2 - T_1 - 1)/2]^{\frac{1}{2}}$$

has an infinite expectation, thus showing that the condition, $\mu_{00} < \infty$, cannot be dropped from (b) of Theorem 5.2.

It may be that one is interested in the convergence of normed sums of the form $W(t)/N(t)$. In this case an analogue to Theorem 5.2 may be proved in which m/μ_{00} is replace by m/μ_{00}^* in (5.7) and in which the condition, $\mu_{00} < \infty$, in (b) is replaced by $\mu_{00}^* < \infty$.

6. Weak Law of Large Numbers. Assume throughout this section that Y_1 is a r.v. which satisfies the Weak Law of Large Numbers (W.L.L.N.). That is, its characteristic function is differentiable at zero with derivative equal to m, say, or, equivalently,

(i) $E(Y_1 I_{[|Y_1| \leqq x]}) \to m$ as $x \to \infty$ and

(ii) $xP[|Y_1| > x] \to 0$ as $x \to \infty$ (cf. Pitman [13]). Using the same notation as in Section 5, one obtains

Lemma 6.1. *If Y_1 satisfies the* W.L.L.N., *then*

$$R_1(t)/t \to 0 \quad \text{(a.s.)}, \tag{6.1}$$

$$V(t)/t \to_p m/\mu_{00}, \tag{6.2}$$

the latter limit being zero if $\mu_{00} = \infty$.

The proof is analogous to that of Lemma 5.1 and is omitted. The important implication of this result is that under the assumption that Y_1 satisfies the W.L.L.N., $W(t)/t$ will converge in probability if and only if $R_2(t)/t$ does. Thus in the proof of the following theorems attention will be directed to the quantity $R_2(t)$ only.

Theorem 6.1. (W.L.L.N.). *If Y_1 satisfies the* W.L.L.N. *and if $\mu_{00}^* < \infty$, then $R_2(t)$ converges in distribution and hence*

$$W(t)/t \to_p m/\mu_{00}. \tag{6.3}$$

Proof. Using (5.8), one may obtain, by direct computation, that for each state i,

$$\begin{aligned} P[R_2(t) \leqq x \mid Z_0 = i, U_0 = 0] &= P[R_2(t) \leqq x, N_0(t) \\ &= 0 \mid Z_0 = i, U_0 = 0](1 - \delta_{i0}) \\ &+ \int_0^t P[Y_0(t - u - U_{t-u}) \leqq x, N_0(t-u) = 0 \mid Z_0 = 0, U_0 = 0]\, dM_{i0}(u), \end{aligned} \tag{6.4}$$

where $Y_0(v) = \{\int_0^v G(u)\,du\} I_{[T_1 \geqq v]}$. Now under the condition $Z_0 = 0$, $U_0 = 0$, $Y_0 = Y_0(T_1)$ has the same distribution as Y_1. Also one obtains that as u increases, the events $[Y_0(u - U_u) \leqq x, N_0(u) = 0]$ will alternatively occur and not occur until finally $N_0(u) > 0$ at $u = T_1$. Let $t_1, t_2, \cdots$ be the successive time points at which such a change occurs. Set $t_{-1} = t_0 = 0$ and define for $k \geqq 0$, $V_u^+ = k$ on $[t_{2k-1} < u \leqq t_{2k+1}]$, $V_u^- = k$ on $[t_{2k} < u < t_{2k+2}]$. Note that V_u^+ and V_u^- are finite (a.s.) because $\mu_{00}^* < +\infty$, so that upon setting $V_u = V_u^+ - V_u^-$, one has $[V_u = 0, u < T_1] = [Y_0(u) \leqq x, N_0(u) = 0, u < T_1]$. Therefore,

$$P[Y_0(u) \leqq x, N_0(u) = 0 \mid Z_0 = 0, U_0 = 0] = E(1 - V_u \mid Z_0 = 0, U_0 = 0),$$

is a function of bounded variation in u, and hence is continuous a.e. The version of the Key Renewal theorem due to Beneš [3] is then applicable to each term of (6.4) to give

$$\begin{aligned} &\lim_{t\to\infty} P[R_2(t) \leqq x \mid Z_0 = i, U_0 = 0] \\ &\quad = \mu_{00}^{-1} \int_0^\infty P[Y_0(u - U_u) \leqq x, N_0(u) = 0 \mid Z_0 = 0, U_0 = 0]\,du. \end{aligned} \tag{6.5}$$

Since $P[R_2(t) \leqq x] = \sum_i a_i P[R_2(t) \leqq x \mid Z_0 = i, U_0 = 0]$, the theorem follows by the dominated convergence theorem. (In taking the limit in (6.5), it should be understood that if G_{00} is a lattice distribution function, then t approaches infinity over multiples of its span.) The integral on the right hand side of (6.5) is the (conditional) expected amount of time before the first visit to state 0 during which W_f does not exceed x.

It cannot, unfortunately, be concluded from this theorem that m/μ_{00} does not depend upon the choice of $j = 0$, nor that the assumption, "$Y_1^{(j)}$ satisfies the W.L.L.N." is true for all or no j. This is true, but it requires a separate argument which is left for future work.

The finiteness condition imposed on the mean recurrence time μ_{00}^* in the above theorem is somewhat unnatural. It implies in particular that the MRP is strongly-regular. It is really the family of r.v.'s $Y_1(u)$, and not the mean recurrence times, that should be considered as playing the major role in determining the convergence of $R_2(t)$. For example, if the r.v.'s $Y_1(u)$ are uniformly bounded, and hence $R_2(t)$ is uniformly bounded (and vice versa), then clearly $R_2(t)/t$ converges to zero in probability regardless of the value of μ_{00}^*. On the other hand, one cannot expect $R_2(t)/t$ to converge in probability to zero without some conditions on the joint behavior of the r.v.'s $Y_1(u)$ and the recurrence times. [For example, consider Example (iii) of Section 5.] Unfortunately, there do not seem to be any practical alternatives for the condition, "$\mu_{00}^* < \infty$", that merit mentioning.

7. Central Limit theorem. Assume in this section that $\mu_{00}^* < \infty$ and Var $(Y_1) \equiv \sigma^2 < \infty$. In particular, this assumption implies through Lemma 6.1 and Theorem 6.1 that in the expression $W(t) = V(t) + R_1(t) + R_2(t)$, both $t^{-\frac{1}{2}}R_1(t)$ and $t^{-\frac{1}{2}}R_2(t)$ converge to zero in probability as $t \to \infty$. Therefore, when studying the limiting behavior of $t^{-\frac{1}{2}}W(t)$, it suffices to consider that of $t^{-\frac{1}{2}}V(t)$. To avoid the need for an additional assumption, the d.f. which places unit mass at zero will be considered as a $N(0, 0)$ d.f.

From Renewal theory, $N_j(t)/t \rightarrow_{\text{a.s.}} 1/\mu_{jj}$. Therefore, one may apply the theorem of Anscombe [2] on sums of random numbers of random variables (see also the recent, more general paper by Billingsley [4] or the Central Limit theorem of Smith [17] for cumulative processes) to obtain

LEMMA 7.1. *If* $\mu_{00}^* < \infty$ *and* $\text{Var}(Y_1) \equiv \sigma^2 < \infty$, *then*

$$t^{-\frac{1}{2}}[W(t) - N_0(t)m] \rightarrow_L \text{a } N(0, \sigma^2/\mu_{00}) \text{ r.v.} \tag{7.1}$$

Consider now the limiting behavior of $t^{-\frac{1}{2}}[W(t) - tA]$, where $A = m/\mu_{00}$ is independent of j, as given by Corollary 5.1. Note first that if $\mu_{00} < \infty$, if one replaces the function f with the function $g = f - m/\mu_{00}^*$, and if $\text{Var } Y_1(g) < \infty$, then a consequence of Lemma 7.1 is that

$$t^{-\frac{1}{2}}[W(t) - N(t)Ap] \rightarrow_L \text{a } N(0, B_g) \text{ r.v.} \tag{7.2}$$

where $p = \mu_{00}/\mu_{00}^*$ does not depend upon the choice $j = 0$ by (4.7) and $B_g \equiv \mu_{00}^{-1} \text{ Var } Y_1(g)$. Since neither the left hand side of (7.2) nor the assumptions used, depend on the choice $j = 0$, then neither does B_g. In a similar manner, upon replacing f in Lemma 7.1 with the function h defined by $h(i, x) = f(i, s) - xA$ one obtains that if $\text{Var } Y_1(h) < \infty$, then

$$t^{-\frac{1}{2}}[W(t) - S_{N(t)}A] \rightarrow_L \text{a } N(0, B_h) \text{ r.v.} \tag{7.3}$$

where $B_h \equiv \mu_{00}^{-1} \text{ Var } Y_1(h)$ does not depend on $j = 0$. Moreover since $t - S_{N(t)} \leqq t - S_{N_0(t)}$, which by Renewal Theory converges in law when $\mu_{00} < \infty$, one has $t^{-\frac{1}{2}}[t - S_{N(t)}] \rightarrow_p 0$. This completes the proof of

THEOREM 7.1. *Let* $\mu_{00}^* < \infty$, $\mu_{00} < \infty$, *and let* g *and* h *be as defined in the preceding paragraph. Then*

(a) *if* $\text{Var } Y_1(g) < \infty$, $t^{-\frac{1}{2}}[W(t) - N(t)Ap] \rightarrow_L$ a $N(0, B_g)$ r.v.,

(b) *if* $\text{Var } Y_1(h) < \infty$, $t^{-\frac{1}{2}}[W(t) - tA] \rightarrow_L$ a $N(0, B_h)$ r.v.

with B_g *and* B_h *as defined by* (7.3) *and* (7.4) *respectively.*

Perhaps the most important case of an MRP with more than one state (thereby excluding the basic case of a Renewal process) is the two state MRP. For in any MRP all problems concerning the duration time (sojourn time) of a particular state may be reduced to this case. In [20], Takács studies the limiting distribution of the duration time spent in one of two states. This is the special case of the above with $f(j, k, x) = x$ whenever j is the specified state. It should also be pointed out that the joint asymptotic normality of the $N_k(t)$'s follows from the above theorems upon setting $f(k, j, x) = w_k$, a constant, for the set of indices k being considered, and equal to zero otherwise.

When studying the limiting behavior of the maximum likelihood estimators of the parameters in a Birth and Death process, Albert [1] obtains special cases of the above Central Limit theorems. These limit theorems have also been applied in [11] to study the limiting behavior of estimators of the transition functions $\{Q_{ij}\}$ in an arbitrary finite state MRP.

Many other limit theorems for MRP's are possible. For example, the law of the iterated logarithm, as well as the limiting distribution of the maximum partial sum or of the maximum f_n may be derived, (cf. [5], [9] and [12]). Moreover, the a.s. convergence of the ratios in Section 3 can be studied. Of greater interest, however, would be a study of the Central Limit problem for the case in which $Y_1(f)$ is in the domain of attraction of a (non-Normal) stable law. Several limit theorems for sojourn times in the non-Normal case have been outlined by Kesten [7]. A recent paper by Taga [19] studies limit theorems for MRP's with a finite number of states.

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