ON THE LIFTING PROPERTY (V)[1]

By A. Ionescu Tulcea

University of Illinois

1. Let $(X, \mathfrak{B}, \mu)$ be a measure space (i.e. X is a set, $\mathfrak{B}$ a σ-algebra of subsets of X, μ a positive countably additive measure on $\mathfrak{B}$). Let $\mathfrak{B}_0 = \{B \varepsilon \mathfrak{B} \mid \mu(B) < \infty\}$ and $\mathfrak{N} = \{A \varepsilon \mathfrak{B} \mid \mu(A) = 0\}$. For $A \varepsilon \mathfrak{B}$, $B \varepsilon \mathfrak{B}$ we write $A \equiv B$ if $A \Delta B = (A - B) \cup (B - A) \varepsilon \mathfrak{N}$; this is an equivalence relation in $\mathfrak{B}$. We shall denote by $B \to \tilde{B}$ the canonical mapping of $\mathfrak{B}$ onto the quotient σ-algebra $\mathfrak{B}/\mathfrak{N}$. *Throughout this paper we shall assume that the measure space* $(X, \mathfrak{B}, \mu)$ *satisfies the following conditions*:

(a) The measure space $(X, \mathfrak{B}, \mu)$ is complete (i.e., the relations $A \varepsilon \mathfrak{N}$ and $B \subset A$ imply $B \varepsilon \mathfrak{N}$);

(b) A set $E \subset X$ belongs to $\mathfrak{B}$ if and only if $E \cap B \varepsilon \mathfrak{B}$ for every $B \varepsilon \mathfrak{B}_0$;

(c) For every $E \varepsilon \mathfrak{B}$, $\mu(E) = \sup \{\mu(B) \mid Bc \subset A, B \varepsilon \mathfrak{B}_0\}$;

(d) The quotient σ-algebra $\mathfrak{B}/\mathfrak{N}$ is a complete lattice.

The measure space $(X, \mathfrak{B}, \mu)$ is then a localizable measure space in Segal's sense (see [21] and [13]).

Note that the above setting includes as a particular case $(X, \mathfrak{B}, \mu)$ a complete totally σ-finite measure space. Also, if X is a locally compact space with a given positive Radon measure, the conditions (a)–(d) are satisfied if we take for $\mathfrak{B}$ the σ-algebra of all sets measurable with respect to that Radon measure and for μ the essential measure (see [1]).

In what follows we shall denote by M_R^∞ the algebra of all bounded real-valued measurable functions defined on X. For $f \varepsilon M_R^\infty$, $g \varepsilon M_R^\infty$ we write $f \equiv g$ if f and g coincide almost everywhere; this defines an equivalence relation in M_R^∞. As usual, we denote by L_R^∞ the quotient space of M_R^∞ under this equivalence relation, and by $f \to \tilde{f}$ the canonical mapping of M_R^∞ onto L_R^∞. Endowed with the essential supremum norm, L_R^∞ is a cummutative Banach algebra.

Let now $T: f \to T_f$ be a mapping of M_R^∞ into M_R^∞ and consider the following axioms:

(I) $T_f \equiv f$;

(II) $f \equiv g$ implies $T_f = T_g$;

(III) $T_1 = 1$;

(IV) $f \geqq 0$ implies $T_f \geqq 0$;

(V) $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$;

(VI) $T_{fg} = T_f T_g$.

Let us recall that a mapping $T: f \to T_f$ of M_R^∞ into M_R^∞ satisfying (I)–(VI) is called a *lifting of* M_R^∞; a mapping $T: f \to T_f$ of M_R^∞ into M_R^∞ satisfying (I)-(V) is called a *linear lifting of* M_R^∞ (see [10]).

Received 21 December 1964

[1] Research supported by the U. S. Army Research Office (Durham), under contract DA-31-124-ARO-D-288.

Let now $\theta\colon A \to \theta(A)$ be a mapping of $\mathcal{B}$ into $\mathcal{B}$ and consider the following axioms:

(I′) $\theta(A) \equiv A$;
(II′) $A \equiv B$ implies $\theta(A) = \theta(B)$;
(III′) $\theta(\varnothing) = \varnothing$, $\theta(X) = X$;
(IV′) $\theta(A \cap B) = \theta(A) \cap \theta(B)$;
(V′) $\theta(A \cup B) = \theta(A) \cup \theta(B)$.

Let us recall that if $\theta\colon \mathcal{B} \to \mathcal{B}$ satisfies (I′)–(V′), then θ is called a *lifting of* $\mathcal{B}$ (see [10]). Let us also recall here that in a certain sense there is identity between liftings of M_R^∞ and liftings of $\mathcal{B}$: every lifting of M_R^∞ induces a lifting of $\mathcal{B}$ (consider the restriction to characteristic functions of measurable sets), and conversely every lifting of $\mathcal{B}$ generates in a natural way a lifting of M_R^∞ (see [10]).

It is known (see [10], [14], [18], [19]) that *for every measure space* $(X, \mathcal{B}, \mu)$ *satisfying* (a)–(d) *there is a lifting* $T\colon f \to T_f$ *of* M_R^∞.

For historical reasons (although this does not help to unify our terminology) we shall use the term *lower density of* $\mathcal{B}$ for a mapping $\theta\colon \mathcal{B} \to \mathcal{B}$ satisfying (I′)–(IV′).

Lower densities have been considered by many authors, for instance J. von Neumann [16], O. Haupt and C. Pauc ([8] and [9]), D. Maharam [14]; see also [6], [7], [15], [20], [22]. We want only to remark here that *a linear lifting* $T\colon f \to T_f$ of M_R^∞ *induces* in a natural way *a lower density* of $\mathcal{B}$; in fact, if for a set $Y \subset X$ we denote by φ_Y the characteristic function of the set Y, and if we define

$$\theta(A) = \{x \mid T_{\varphi_A}(x) = 1\}, \quad \text{for} \quad A \in \mathcal{B},$$

it is easy to verify that $\theta\colon \mathcal{B} \to \mathcal{B}$ satisfies (I′)–(IV′) (this mapping $\theta\colon \mathcal{B} \to \mathcal{B}$ will be called the lower density of $\mathcal{B}$ induced by the linear lifting T.

The idea of defining a topology on X from a lower density of $\mathcal{B}$ was first pointed out to us by John Oxtoby (see [12]); it has been exploited by many authors in the past in one context or another (see [6], [7], [8], [9], [15], [22]). Since we shall make constant use of it, we shall—for the sake of completeness—state and prove in detail the result that we need below:

PROPOSITION 1. *Let* $\theta\colon \mathcal{B} \to \mathcal{B}$ *be a lower density of* $\mathcal{B}$. *Define* $\mathfrak{I}_\theta = \{\theta(A) - N \mid A \in \mathcal{B}, N \in \mathfrak{N}\}$. *Then*:

(1) $\mathfrak{I}_\theta$ *is a topology on* X (= *the topology on* X *induced by* θ); *a set* $A \subset X$ *belongs to* $\mathfrak{N}$ *if and only if* A *is closed and nowhere dense* $\mathfrak{I}_\theta$;

(2) *For a function* $f\colon X \to R$ *the following assertions are equivalent*:

(2a) *There is* $N \in \mathfrak{N}$ *such that* f *is continuous* $\mathfrak{I}_\theta$ *everywhere on* $\complement N$;

(2b) f *is measurable.*

PROOF. (1) We shall show first that $\mathfrak{I}_\theta$ is a topology on X. If $\theta(A_1) - N_1 \in \mathfrak{I}_\theta$, $\theta(A_2) - N_2 \in \mathfrak{I}_\theta$, note that

$$\begin{aligned}(\theta(A_1) - N_1) \cap (\theta(A_2) - N_2) &= (\theta(A_1) \cap \theta(A_2)) - (N_1 \cup N_2) \\ &= \theta(A_1 \cap A_2) - (N_1 \cup N_2)\end{aligned}$$

by (IV′) and that

$$(\theta(A_1) \cup \theta(A_2)) - (N_1 \cup N_2) \subset (\theta(A_1) - N_1) \cup (\theta(A_2) - N_2)$$
$$\subset \theta(A_1) \cup \theta(A_2) \subset \theta(A_1 \cup A_2)$$

since θ is increasing (i.e., the relations $A \varepsilon \mathcal{B}$, $B \varepsilon \mathcal{B}$, and $A \subset B$ imply $\theta(A) \subset \theta(B)$). (The monotonicity of θ is a consequence of axiom (IV′).) (Thus $(\theta(A_1) - N_1) \cap (\theta(A_2) - N_2) \varepsilon \mathfrak{I}_\theta$, $(\theta(A_1) - N_1) \cup (\theta(A_2) - N_2) \varepsilon \mathfrak{I}_\theta$ and $\mathfrak{I}_\theta$ is closed under finite intersections and finite unions. To prove that $\mathfrak{I}_\theta$ is closed under arbitrary unions, it is then enough to verify that for a *directed* (for $\subset$) family $(\theta(A_i) - N_i)_{i \varepsilon I}$ of sets belonging to $\mathfrak{I}_\theta$, the set $\bigcup_{i \varepsilon I} \theta(A_i) - N_i$ belongs to $\mathfrak{I}_\theta$:

Let $\tilde{A}$ be the *supremum* of the family $(\tilde{A}_i)_{i \varepsilon I}$ in $\mathcal{B}/\mathfrak{N}$ (use d)). We shall show that

$$(*) \qquad \bigcup_{i \varepsilon I} \theta(A_i) - N_i \varepsilon \mathcal{B} \quad \text{and} \quad \theta(A) \equiv \bigcup_{i \varepsilon I} \theta(A_i) - N_i .$$

Since $\theta(A) \supset \theta(A_i) \supset \theta(A_i) - N_i$ for each $i \varepsilon I$, the relations $(*)$ will obviously imply that $\bigcup_{i \varepsilon I} \theta(A_i) - N_i \varepsilon \mathfrak{I}_\theta$ and hence will complete the proof of the assertion that $\mathfrak{I}_\theta$ is a topology. In turn, to prove $(*)$, it will be enough by (b) and (c) to show that

$$(**) \quad \bigcup_{i \varepsilon I} (\theta(A_i) - N_i) \cap B \varepsilon \mathcal{B} \quad \text{and} \quad \theta(A) \cap B \equiv \bigcup_{i \varepsilon I} (\theta(A_i) - N_i) \cap B$$

for each $B \varepsilon \mathcal{B}_0$. Let then $B \varepsilon \mathcal{B}_0$; since $\theta(A) \cap B$ is in the class of the "supremum" of the family $(\theta(A_i) \cap B)_{i \varepsilon I}$, there is an increasing sequence $((\theta(A_{i_n}) - N_{i_n}) \cap B)_{1 \leq n < \infty}$ such that

$$\theta(A) \cap B \equiv \bigcup_{n=1}^{\infty} (\theta(A_{i_n}) - N_{i_n}) \cap B.$$

On the other hand

$$\theta(A) \cap B \supset \bigcup_{i \varepsilon I} \theta(A_i) \cap B \supset \bigcup_{i \varepsilon I} (\theta(A_i) - N_i) \cap B \supset \bigcup_{n=1}^{\infty} (\theta(A_{i_n}) - N_{i_n}) \cap B$$

and comparing with the preceding formula we deduce $(**)$.

To prove the second assertion in (1), let $A \subset X$ be closed and nowhere dense $\mathfrak{I}_\theta$; then $A \varepsilon \mathcal{B}$ and since $A \equiv \theta(A) \equiv \theta(A) \cap A$ and $\theta(A) \cap A$ is open $\mathfrak{I}_\theta$ and contained in A, we deduce that $\mu(A) = 0$. Conversely, let $A \varepsilon \mathfrak{N}$; then $A = \complement(X - A) = \complement(\theta(X) - A)$ is closed $\mathfrak{I}_\theta$ and A is nowhere dense $\mathfrak{I}_\theta$ (remark that a set which is open $\mathfrak{I}_\theta$ is non-void if and only if it has strictly positive measure).

(2) We shall first prove (2a) $\Rightarrow$ (2b). Suppose that $f: X \to R$ is continuous $\mathfrak{I}_\theta$ on $\complement N$, where $N \varepsilon \mathfrak{N}$. For each $c \varepsilon R$ we have:

$$\{x \mid f(x) > c\} = (\{x \mid f(x) > c\} \cap N) \cup (\{x \mid f(x) > c\} \cap \complement N);$$

since $\{x \mid f(x) > c\} \cap N \varepsilon \mathfrak{N}$ and $\{x \mid f(x) > c\} \cap \complement N \varepsilon \mathfrak{I}_\theta$, we deduce that $\{x \mid f(x) > c\} \varepsilon \mathcal{B}$ and hence that f is measurable.

We shall now prove (2b) $\Rightarrow$ (2*a*). Suppose that $f: X \to R$ is measurable. We may assume without loss of generality that f is bounded (the case f unbounded can be reduced to the bounded case by composing with the mapping $t \to t/(1 + |t|)$

which is a homeomorphism of R onto $(-1, 1)$). In turn, since every bounded measurable function defined on X can be approximated uniformly with simple functions, it is enough to consider the case $f = \varphi_E$, where $E \in \mathcal{B}$. Since φ_E is constant on $\theta(E) \cap E \subset E$ and on $\theta(\complement E) \cap \complement E \subset \complement E$, and since $(\theta(E) \cap E) \cup (\theta(\complement E) \cap \complement E) \equiv X$, the assertion is completely proved. This finishes the proof of Proposition 1.

The next result shows how one can define a lifting of M_R^∞ from a lower density θ of $\mathcal{B}$ by making use of the topology $\mathfrak{I}_\theta$ on X induced by θ; this result was suggested by the technique used by Dixmier in [5], p. 177, to construct a lifting on the Lebesgue space [0, 1].

PROPOSITION 2. *Let $\theta: \mathcal{B} \to \mathcal{B}$ be a lower density of $\mathcal{B}$. For each $y \in X$, let J_y be the set of all $\tilde{f} \in L_R^\infty$ for which there is f in the class $\tilde{f}$ such that f is continuous $\mathfrak{I}_\theta$ at y and $f(y) = 0$. Then*:

(1) *J_y is a closed ideal in L_R^∞ and $J_y \neq L_R^\infty$*;

(2) *If for each $y \in X$ we let χ_y be a character of L_R^∞ vanishing on J_y, then the formula*

$$T_f(y) = \chi_y(\tilde{f}), \qquad \text{for } f \in M_R^\infty,\ y \in X$$

defines a lifting $T: f \to T_f$ of M_R^∞.

PROOF. The proof of (1) is elementary (the fact that $J_y \neq L_R^\infty$ follows from the observation that $\tilde{1} \notin J_y$).

(2) It is clear that the mapping $T: f \to T_f$ defined by the above formula satisfies (II), (III), (IV), (V), (VI). It remains only to show that T satisfies (I). Let $f \in M_R^\infty$; by Proposition 1, there is $N \in \mathfrak{N}$ such that f is continuous $\mathfrak{I}_\theta$ everywhere on $\complement N$. Fix $y \in \complement N$; then $g = f - f(y)$ is continuous $\mathfrak{I}_\theta$ at y and $g(y) = 0$, hence $\tilde{g} \in J_y$. We deduce

$$T_f(y) - f(y) = \chi_y(\tilde{f}) - f(y) = \chi_y(\tilde{g}) = 0$$

and thus $T_f(y) = f(y)$. Since $y \in \complement N$ was arbitrary, it follows that $T_f \in M_R^\infty$ and $T_f \equiv f$. This completes the proof of Proposition 2.

2. By an *automorphism* of $(X, \mathcal{B}, \mu)$ we mean a mapping $s: X \to X$ such that: (i) s is a bijection; (ii) $B \in \mathcal{B}$ implies $s(B) \in \mathcal{B}$, $s^{-1}(B) \in \mathcal{B}$; (iii) $A \in \mathfrak{N}$ implies $s(A) \in \mathfrak{N}$, $s^{-1}(A) \in \mathfrak{N}$.

The set $\mathfrak{A}$ of all automorphisms of $(X, \mathcal{B}, \mu)$ is a group for the usual composition $(s, t) \to s \circ t$; we shall denote by e the unit element ($=$ the identity automorphism) of $\mathfrak{A}$. Remark that for each $s \in \mathfrak{A}$, the mapping $\tilde{f} \to \tilde{f} \circ s = \widetilde{f \circ s}$ is an isomorphism of the algebra L_R^∞ onto itself.

Let $\mathcal{G}$ be a subgroup of $\mathfrak{A}$. We shall give the following definitions:

We say that *a lower density θ of $\mathcal{B}$ commutes with $\mathcal{G}$* if

$$s(\theta(B)) = \theta(s(B)) \qquad \text{for every } B \in \mathcal{B},\ s \in \mathcal{G}.$$

Remark that if θ commutes with $\mathcal{G}$, then every $s \in \mathcal{G}$ is a *homeomorphism of X when endowed with the topology $\mathfrak{I}_\theta$* (in fact, if $U = \theta(A) - N \in \mathfrak{I}_\theta$, then $s(U) = \theta(s(A)) - s(N)$, $s^{-1}(U) = \theta(s^{-1}(A)) - s^{-1}(N)$ belong again to $\mathfrak{I}_\theta$).

We say that *a linear lifting* $T: f \to T_f$ *of* M_R^∞ *commutes with* $\mathcal{G}$ if

$$T_{f\circ s} = T_f \circ s \qquad \text{for every } f \in M_R^\infty,\ s \in \mathcal{G}.$$

THEOREM 1. *Let $\mathcal{G}$ be a subgroup of $\mathcal{A}$ and suppose that $\mathcal{G}$ has the following property*:

$(*)$ *For every $x \in X$, the mapping $s \to s(x)$ of $\mathcal{G}$ into X is injective.*

Then the following assertions are equivalent:

(i) *There is a lower density of $\mathcal{B}$ commuting with $\mathcal{G}$;*

(ii) *There is a lifting of M_R^∞ commuting with $\mathcal{G}$;*

(iii) *There is a linear lifting of M_R^∞ commuting with $\mathcal{G}$.*

PROOF: (ii) $\Rightarrow$ (iii) obviously.

(iii) $\Rightarrow$ (i) is immediate. In fact, let $T: f \to T_f$ be a linear lifting of M_R^∞ commuting with $\mathcal{G}$ and let $\theta: \mathcal{B} \to \mathcal{B}$ be the lower density of $\mathcal{B}$ induced by T. We have for each $B \in \mathcal{B}$ and $s \in \mathcal{G}$,

$$\begin{aligned}\theta(s(B)) &= \{x \mid T_{\varphi_{s(B)}}(x) = 1\} = \{x \mid T_{\varphi_B \circ s^{-1}}(x) = 1\} \\ &= \{x \mid T_{\varphi_B}(s^{-1}(x)) = 1\} = s(\theta(B));\end{aligned}$$

hence θ commutes with $\mathcal{G}$.

(i) $\Rightarrow$ (ii). Let $\theta: \mathcal{B} \to \mathcal{B}$ be a lower density of $\mathcal{B}$ commuting with $\mathcal{G}$. Consider in X the equivalence relation: $x \sim y$ if and only if there is $s \in \mathcal{G}$ such that $s(x) = y$. Let $(X_i)_{i\in I}$ be the corresponding partition of X into equivalence classes. For each $i \in I$ *choose* $x_i \in X_i$.

With the notations of Proposition 2 it is now easily seen that for each $i \in I$ and $s \in G$,

$$J_{s(x_i)} = J_{x_i} \circ s^{-1} (= \{\tilde{f} \circ s^{-1} \mid \tilde{f} \in J_{x_i}\}) \tag{1}$$

(use the fact that each $s \in \mathcal{G}$ is a homeomorphism of X when endowed with the topology $\mathfrak{I}_\theta$). It follows that if χ_{x_i} is a character of L_R^∞ vanishing on J_{x_i}, then $\chi_{x_i}^s$ defined by

$$\chi_{x_i}^s(\tilde{g}) = \chi_{x_i}(\tilde{g} \circ s), \qquad \text{for } \tilde{g} \in L_R^\infty \tag{2}$$

is a character of L_R^∞ vanishing on $J_{s(x_i)}$.

We remark now that given $y \in X$, there are: *a unique $i \in I$ such that $y \in X_i$* and (by condition $(*)$) *a unique $s \in \mathcal{G}$ such that $y = s(x_i)$*. Now for $f \in M_R^\infty$ define T_f by the equations

$$T_f(y) = \chi_{x_i}^s(\tilde{f}) \quad \text{if} \quad y \in X_i \quad \text{and} \quad y = s(x_i). \tag{3}$$

By Proposition 2 (make use of (2) above), the mapping $T: f \to T_f$ is a lifting of M_R^∞. It remains to verify only that the lifting T commutes with $\mathcal{G}$:

Let $f \in M_R^\infty$, $t \in \mathcal{G}$ and $y \in X$. Then $y \in X_i$ for some unique $i \in I$ and $y = s(x_i)$ for a unique $s \in \mathcal{G}$; it follows that $t(y) = (t \circ s)(x_i)$. We have

$$\begin{aligned}T_{f\circ t}(y) &= \chi_{x_i}^s(\tilde{f} \circ t) = \chi_{x_i}((\tilde{f} \circ t) \circ s) = \chi_{x_i}(\tilde{f} \circ (t \circ s)) \\ &= \chi_{x_i}^{t\circ s}(\tilde{f}) = T_f(t(y)).\end{aligned}$$

Since f, t, y were arbitrary, T commutes with $\mathcal{G}$ and hence (i) $\Rightarrow$ (ii) is proved. This completes the proof of Theorem 1.

REMARKS. (1) The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) in the above theorem do not remain true if the group $\mathcal{G}$ does not satisfy the condition $(*)$. If $(X, \mathcal{B}, \mu)$ is the Lebesgue space of the real line and $\mathcal{G} = \{e, \gamma\}$ where $\gamma: x \to -x$, then it is known that: there is a lower density of $\mathcal{B}$ commuting with $\mathcal{G}$ and there is a linear lifting of M_R^∞ commuting with $\mathcal{G}$; however, as it was remarked in [13], p. 4, there is *no* lifting of M_R^∞ commuting with $\mathcal{G}$.

(2) Let $\mathcal{G}$ be a subgroup of $\mathcal{A}$ satisfying the condition $(*)$ of Theorem 1. Suppose that $T: f \to T_f$ is a linear lifting of M_R^∞ commuting with $\mathcal{G}$ and define $\theta'(A) = \{x \mid T_{\varphi_A}(x) = 1\}$, $\theta''(A) = \{x \mid T_{\varphi_A}(x) > 0\}$ for every $A \varepsilon \mathcal{B}$. Let $\mathfrak{D}$ be the set of all linear liftings $S: f \to S_f$ of M_R^∞ satisfying the inequalities $\varphi_{\theta'(A)} \leqq S_{\varphi_A} \leqq \varphi_{\theta''(A)}$ for all $A \varepsilon \mathcal{B}$. Below we identify $\mathfrak{D}$ with a subset of the Cartesian product R^H, where $H = M_R^\infty \times X$ (by identifying every $S \varepsilon \mathfrak{D}$ with the element $(S_f(x))_{(f,x)\varepsilon H}$ of R^H). Let $\mathfrak{D}_1$ be the set of all $S \varepsilon \mathfrak{D}$ which commute with $\mathcal{G}$. *Then $\mathfrak{D}_1$ is convex and an element $S \varepsilon \mathfrak{D}_1$ is extremal in $\mathfrak{D}_1$ if and only if S is extremal in $\mathfrak{D}$.* In fact, assume that $S \varepsilon \mathfrak{D}_1$ is extremal in $\mathfrak{D}_1$, but is not extremal in $\mathfrak{D}$. There are then $S^{(1)} \varepsilon \mathfrak{D}$, $S^{(2)} \varepsilon \mathfrak{D}$, $S^{(1)} \neq S^{(2)}$ and $0 < \lambda < 1$ such that $S = \lambda S^{(1)} + (1 - \lambda)S^{(2)}$. Since $S^{(1)} \neq S^{(2)}$, there are $g \varepsilon M_R^\infty$ and $z \varepsilon X$ such that $S_g^{(1)}(z) \neq S_g^{(2)}(z)$. As in the proof of Theorem 1, let $(X_i)_{i\varepsilon I}$ be the partition of X into the classes of intransitivity of $\mathcal{G}$ and for each $i \varepsilon I$ choose a point $x_i \varepsilon X_i$; moreover, if $z \varepsilon X_{i_0}$, choose $x_{i_0} = z$. For each $i \varepsilon I$ and $j = 1, 2$, define $x_i'^{(j)}: L_R^\infty \to R$ by

$$x_i'^{(j)}(\tilde{f}) = S_f^{(j)}(x_i), \qquad \text{for } \tilde{f} \varepsilon L_R^\infty;$$

it is clear that $x_i'^{(j)} \varepsilon (L_R^\infty)'$. Now for $j = 1, 2$ and $f \varepsilon M_R^\infty$ define $T_f^{(j)}$ by the formula

$$T_f^{(j)}(y) = x_i'^{(j)}(\tilde{f} \circ s) \quad \text{if} \quad y \varepsilon X_i \quad \text{and} \quad y = s(x_i).$$

It is easily seen that $T^{(1)} \neq T^{(2)}$, $S = \lambda T^{(1)} + (1 - \lambda)T^{(2)}$ and $T^{(1)} \varepsilon \mathfrak{D}_1$, $T^{(2)} \varepsilon \mathfrak{D}_1$ (see Propositions 3 and 4 in [10]). This contradicts the assumption that S is extremal in $\mathfrak{D}_1$ and thus proves the assertion made above.

(3) Remark (2) permits to give a different proof of the implication (iii) $\Rightarrow$ (ii) in Theorem 1. With the notations of Remark (2), it is enough to note that the set $\mathfrak{D}_1$ is non-void, convex and compact in R^H and hence contains an extremal point S; but S is then extremal in $\mathfrak{D}$ and hence S has the "multiplicative" property (VI) (see Proposition 4 in [10]).

(4) If X is a Lie group, with corresponding left invariant Haar measure, then using a "derivation theorem" and the "ultrafilter device" of J. Dieudonné (see [4], p. 80; see also [17]) one can construct a linear lifting of M_R^∞ commuting with the group of all left-translations (see also [2] for the existence of a sequence of "quasi-spheres" in a Lie group and for the corresponding version of the Vitali covering theorem). By Theorem 1, we deduce from this the existence of a lifting of M_R^∞ commuting with the group of left-translations. Although there are many examples of locally compact groups which are not Lie groups and for

which the existence of a lifting of M_R^∞ commuting with the group of left-translations can be established (these will be discussed elsewhere), it is not known whether or not this is true for an arbitrary locally compact group, with corresponding left invariant Haar measure.

Consider again, as in Remark (2) above, the locally convex space (= Cartesian product)

$$R^H = \prod_{(f,x)\varepsilon H} R_{(f,x)}$$

where $H = M_R^\infty \times X$ and $R_{(f,x)} = R$ for every $(f, x) \varepsilon H$. We denote a general element of R^H by $z = (z(f, x))_{(f,x)\varepsilon H}$. In what follows we shall *identify* a linear lifting $T: f \to T_f$ of M_R^∞ with the element $(T_f(x))_{(f,x)\varepsilon H}$ of R^H.

For each $s \varepsilon \mathfrak{A}$, define the mapping $U_s : R^H \to R^H$ by

$$U_s z = (z(f \circ s, s^{-1}(x)))_{(f,x)\varepsilon H} \quad \text{if} \quad z = (z(f, x))_{(f,x)\varepsilon H}.$$

It is obvious that $U_s : R^H \to R^H$ is *linear* and *continuous*, hence $U_s \varepsilon \mathfrak{L}(R^H, R^H)$.

Note that if $\mathfrak{G}$ is a subgroup of $\mathfrak{A}$, the mapping $s \to U_s$ is an *isomorphism* of $\mathfrak{G}$ into $\mathfrak{L}(R^H, R^H)$. We remark also that, with the above notations, *a linear lifting $T: f \to T_f$ of M_R^∞ commutes with $\mathfrak{G}$ if and only if $U_s T = T$ for every $s \varepsilon \mathfrak{G}$.*

The following is an instance when the existence of a linear lifting of M_R^∞ commuting with a group $\mathfrak{G} \subset \mathfrak{A}$ can be established by an application of a fixed-point theorem:

THEOREM 2. *Let $\mathfrak{G} \subset \mathfrak{A}$ be a countable, amenable group [in the sense of Day (see [3])]. There is then a linear lifting $T: f \to T_f$ of M_R^∞ commuting with $\mathfrak{G}$.*

(The author is indebted to C. Ionescu Tulcea for many valuable discussions and in particular for suggestions that led to the formulation of Theorem 2.)

PROOF. Let $\rho: \mathfrak{B} \to \mathfrak{B}$ be a *lifting* of $\mathfrak{B}$; for each $A \varepsilon \mathfrak{B}$, define correspondingly the "lower orbit" and the "upper orbit" under G:

$$\theta^{(1)}(A) = \bigcap_{s\varepsilon\mathfrak{G}} s(\rho(s^{-1}(A))) \quad \text{and} \quad \theta^{(2)}(A) = \bigcup_{s\varepsilon\mathfrak{G}} s(\rho(s^{-1}(A))).$$

Since ρ is a lifting of $\mathfrak{B}$ and $\mathfrak{G}$ is countable, it is clear that $\theta^{(1)}(A) \varepsilon \mathfrak{B}$, $\theta^{(2)}(A) \varepsilon \mathfrak{B}$ and $\theta^{(1)}(A) \equiv \theta^{(2)}(A) \equiv A$. Note also that for each $A \varepsilon \mathfrak{B}$ and each $t \varepsilon \mathfrak{G}$ we have

$$(1) \qquad t(\theta^{(j)}(t^{-1}(A))) = \theta^{(j)}(A) \qquad \text{for } j = 1, 2$$

(since the mapping $s \to t \circ s$ is a bijection of $\mathfrak{G}$ onto $\mathfrak{G}$). Define now the set $\mathfrak{D}$ of all linear liftings $S: f \to S_f$ of M_R^∞ satisfying the inequalities

$$(2) \qquad \varphi_{\theta^{(1)}(A)} \leqq S_{\varphi_A} \leqq \varphi_{\theta^{(2)}(A)} \qquad \text{for all } A \varepsilon \mathfrak{B}.$$

Note that, since $\varphi_{\theta^{(1)}(A)} \leqq \varphi_{\rho(A)} \leqq \varphi_{\theta^{(2)}(A)}$ for all $A \varepsilon \mathfrak{B}$, the lifting of M_R^∞ generated by ρ, belongs to $\mathfrak{D}$. Hence the set $\mathfrak{D}$ is *non-void*. Note also that $\mathfrak{D}$ as a subset of R^H is *convex* and *compact* (see the proof of Proposition 4 in [10]). Finally, note that

$$(3) \qquad U_t : \mathfrak{D} \to \mathfrak{D} \qquad \text{for each } t \varepsilon \mathfrak{G}.$$

In fact, let $S \varepsilon \mathfrak{D}$ and $t \varepsilon \mathfrak{G}$; for $A \varepsilon \mathfrak{B}$ and $x \varepsilon X$ we have

$$(U_tS)_{\varphi_A}(x) = S_{\varphi_A \circ t}(t^{-1}(x)) = S_{\varphi_{t^{-1}(A)}}(t^{-1}(x));$$

but by (2) and (1)

$$S_{\varphi_{t^{-1}(A)}}(t^{-1}(x)) \geqq \varphi_{\theta(1)(t^{-1}(A))}(t^{-1}(x)) = \varphi_{t(\theta(1)(t^{-1}(A)))}(x) = \varphi_{\theta(1)(A)}(x)$$

and

$$S_{\varphi_{t^{-1}(A)}}(t^{-1}(x)) \leqq \varphi_{\theta(2)(t^{-1}(A))}(t^{-1}(x)) = \varphi_{t(\theta(2)(t^{-1}(A)))}(x) = \varphi_{\theta(2)(A)}(x).$$

Thus $\varphi_{\theta(1)(A)}(x) \leqq (U_tS)\varphi_A(x) \leqq \varphi_{\theta(2)(A)}(x)$; since $A \in \mathfrak{B}$ and $x \in X$ were arbitrary, we deduce that $U_tS \in \mathfrak{D}$ and (3) is proved.

Since $\mathfrak{D}$ is non-void, convex and compact, since $U_t(\mathfrak{D}) \subset \mathfrak{D}$ for each $t \in \mathcal{G}$ and since the group $\mathcal{G}$ is amenable, we can apply the Kakutani-Markov-Day fixed-point theorem (see [3]) and we deduce the existence of a $T \in \mathfrak{D}$ such that $U_tT = T$ for every $t \in \mathcal{G}$; T is the required linear lifting of M_R^∞ commuting with $\mathcal{G}$. This completes the proof of Theorem 2.

COROLLARY 1. *If τ is an automorphism of $(X, \mathfrak{B}, \mu)$ (in particular, if τ is a measure-preserving automorphism), there is a linear lifting $T: f \to T_f$ of M_R^∞ such that $T_{f\circ\tau} = T_f \circ \tau$ for every $f \in M_R^\infty$.*

REMARK. Theorems 1 and 2 above bring contributions to Problem 4 raised in [13].

3. In what follows we shall denote by $\mathfrak{M}_R$ the algebra of all real-valued measurable functions defined on X. Let $\mathcal{I}$ be the set of all $f \in \mathfrak{M}_R$ which vanish almost everywhere. For $f \in \mathfrak{M}_R$, $g \in \mathfrak{M}_R$ we write $f \equiv g$ if $f - g \in \mathcal{I}$. We denote by M_R the quotient space $\mathfrak{M}_R/\mathcal{I}$ and by $f \to \tilde{f}$ the canonical mapping of $\mathfrak{M}_R$ onto M_R. As usual, for $1 \leqq p < \infty$, we denote by $\mathfrak{L}_R^p$ the vector space of all $f \in \mathfrak{M}_R$ for which $|f|^p$ is integrable.

THEOREM 3. *Let $\theta: \mathfrak{B} \to \mathfrak{B}$ be a lower density of $\mathfrak{B}$ and $\mathcal{G}$ a subgroup of $\mathfrak{A}$ having the property $(*)$ of Theorem 1. Assume that θ commutes with $\mathcal{G}$. There is then a mapping $S: f \to S_f$ of $\mathfrak{M}_R$ into $\mathfrak{M}_R$ having the following properties:*

(j) $S_f \equiv f$;

(jj) *$f \equiv g$ implies $S_f = S_g$;*

(jjj) $S_1 = 1$;

(jv) $S_{\alpha f + \beta g} = \alpha S_f + \beta S_g$;

(v) *$S_f = f$ if f is continuous $\mathfrak{I}_\theta$;*

(vj) *$S_{f\circ s} = S_f \circ s$ for $f \in \mathfrak{M}_R$ and $s \in \mathcal{G}$;*

(vjj) *The restriction of S to M_R^∞ is a lifting of M_R^∞.*

PROOF. Here is a sketch of the proof:

Let $y \in X$. Let J_y and χ_y be as in Proposition 2 and $H_y = \{\tilde{f} \in L_R^\infty \mid \chi_y(\tilde{f}) = 0\}$. Let K_y be the set of all $\tilde{g} \in M_R$ for which there is g in the class $\tilde{g}$ such that g is continuous $\mathfrak{I}_\theta$ at y and $g(y) = 0$. Let V_y be the vector subspace of M_R spanned by H_y and K_y; clearly $\tilde{1} \notin V_y$. Finally let x_y' be a linear functional on M_R such that

$$(1) \qquad x_y'(\tilde{1}) = 1 \quad \text{and } x_y' \text{ vanishes on } V_y.$$

It is obvious that $x_y'(\tilde{f}) = \chi_y(\tilde{f})$ for each $\tilde{f} \in L_R^\infty$.

Let now $(X_i)_{i \varepsilon I}$ be the partition of X into the classes of intransitivity of $\mathcal{G}$ and for each $i \varepsilon I$ choose a point $x_i \varepsilon X_i$. For $f \varepsilon \mathfrak{M}_R$ define S_f by:

$$(2) \qquad S_f(y) = x'_{x_i}(\tilde{f} \circ s) \quad \text{if} \quad y \varepsilon X_i \quad \text{and} \quad y = s(x_i).$$

It can be verified that (see the proof of Proposition 2 and of the implication (i) $\Rightarrow$ (ii) in Theorem 1) S satisfies (j)–(vjj).

REMARKS. (1) If $\mathcal{G}$ reduces to $\{e\}$ the existence of the mapping S given by Theorem 3 can be obtained by a "convenient splitting." In fact, let T be the lifting of M_R^{∞} given by Proposition 2; let $\mathcal{V}$ be the vector subspace of $\mathfrak{M}_R$ spanned by $\{T_f \mid f \varepsilon M_R^{\infty}\}$ and $\{g \varepsilon \mathfrak{M}_R \mid g \text{ is continuous } \mathfrak{I}_\theta\}$. Then $\mathcal{V} \cap \mathcal{I} = \{0\}$, hence $\mathcal{V}$ can be "completed" to a direct summand $\mathcal{W}$ of $\mathcal{I}$ in $\mathfrak{M}_R$ and the projection of $\mathfrak{M}_R$ onto $\mathcal{W}$ gives the required mapping in Theorem 3.

(2) It may be of interest to compare Theorem 3 above with the observation made by von Neumann in [16]. He showed there, that in the case when $(X, \mathcal{B}, \mu)$ is the Lebesgue space of the real line, there is *no* mapping $S: f \to S_f$ of $\mathfrak{M}_R$ into $\mathfrak{M}_R$ satisfying (j)–(jv) and the additional condition of *multiplicativity*: $S_{fg} = S_f S_g$ for $f \varepsilon \mathfrak{M}_R$, $g \varepsilon \mathfrak{M}_R$.

(3) Theorem 3 above remains true if in its formulation we strike out condition (jjj) and we replace everywhere $\mathfrak{M}_R$ by $\mathcal{L}_R^p$ (here $1 \leqq p < \infty$). It is known that (see [11], p. 791) under the assumption that $(X, \mathcal{B}, \mu)$ is *non-atomic*, there is *no* mapping $S: f \to S_f$ of $\mathcal{L}_R^p$ into $\mathcal{L}_R^p$ satisfying (j), (jj), (jv) above and the additional condition of *positivity*: (P) $f \geqq 0$ implies $S_f \geqq 0$, or the weaker condition of *continuity*: (Co) There is a set $A \subset X$ with $0 < \mu^*(A) < \infty$ such that $f \to S_f(y)$ is a continuous linear functional on $\mathcal{L}_R^p$ for each $y \varepsilon A$ (see also [13], Problem 5).

REFERENCES

[1] BOURBAKI, N. (1956). *Integration*, (Chapter V). Hermann, Paris.

[2] COMFORT, W. and GORDON, H. (1961). Vitali's theorem for invariant measures. *Trans. Amer. Math. Soc.* **99** 83–90.

[3] DAY, M. M. (1961). Fixed-point theorems for compact convex sets. *Illinois J. Math.* **5** 585–590.

[4] DIEUDONNÉ, J. (1951). Sur le théorème de Lebesgue-Nikodym (IV). *J. Indian Math. Soc.* **15** 77–86.

[5] DIXMIER, JACQUES (1957). *Les algèbres d'opérateurs dans l'espace Hilbertien.* Gauthier-Villars, Paris.

[6] GOFFMAN, C., NEUGEBAUER, C., and NISHIURA, T. (1961). Density topology and approximate continuity. *Duke Math. J.* **28** 497–506.

[7] GOFFMAN, C. and WATERMAN, D. (1961). Approximately continuous transformations. *Proc. Amer. Math. Soc.* **12** 116–121.

[8] HAUPT, O. and PAUC, C. (1952). La topologie approximative de Denjoy envisagée comme vraie topologie. *C. R. Acad. Sci. Paris* **234** 390–392.

[9] HAUPT, O. and PAUC, C. (1954). Über die durch allgemeine Ableitungsbasen bestimmten Topologien. *Ann. Mat. Pura Appl. Ser. IV* **36** 247–271.

[10] IONESCU TULCEA, A. and IONESCU TULCEA, C. (1961). On the lifting property (I). *J. Math. Anal. Appl.* **3** 537–546.

[11] IONESCU TULCEA, A. and IONESCU TULCEA, C. (1962). On the lifting property (II),

Representation of linear operators on spaces $L_E{}^r$, $1 \leqq r \leqq \infty$. *J. Math. Mech.* **11** 773–796.

[12] IONESCU TULCEA, A. and IONESCU TULCEA, C. (1964). On the lifting property (III). *Bull. Amer. Math. Soc.* **70** 193–197.

[13] IONESCU TULCEA, A. and IONESCU TULCEA, C. (1963). Problems and remarks concerning the lifting property. Technical Report, U. S. Army Research Office (Durham).

[14] MAHARAM, D. (1958). On a theorem of von Neumann. *Proc. Amer. Math. Soc.* **9** 987–994.

[15] MARTIN, N. F. G. (1964). A topology for certain measure spaces. *Trans. Amer. Math. Soc.* **112** 1–18.

[16] VON NEUMANN, J. (1931). Algebraische Repräsentanten der Funktionen bis auf eine Menge vom Masse Null. *J. Crelle* **165** 109–115.

[17] RINGROSE, J. R. (1963). On well-bounded operators II. *Proc. London Math. Soc.* **13** 613–638.

[18] RYAN, R. (1962). The lifting property and direct sums. MRC, Technical Report No. 328, U. S. Army, University of Wisconsin.

[19] RYAN, R. (1964). Representative sets and direct sums. *Proc. Amer. Math. Soc.* **15** 387–390.

[20] SAKS, S. (1937). *Theory of the Integral.* Hafner, New York.

[21] SEGAL, I. (1951). Equivalence of measure spaces. *Amer. Math. J.* **73** 275–313.

[22] TROYER, R. and ZIEMER, W. (1963). Topologies generated by outer measures. *J. Math. Mech.* **12** 485–494.