SOME RESULTS ON THE NON-CENTRAL MULTIVARIATE BETA DISTRIBUTION AND MOMENTS OF TRACES OF TWO MATRICES

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1. Introduction and summary. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two symmetric matrices of order of p, $\mathbf{A}_1$, positive definite and having a Wishart distribution [2], [12] with f_1 degrees of freedom, and $\mathbf{A}_2$, at least positive semi-definite and having a (pseudo) non-central (linear) Wishart distribution ([1], [3], [4], [12], [13]) with f_2 degrees of freedom. Now let

$$\mathbf{A}_2 = \mathbf{CYY'C'}$$

where $\mathbf{C}$ is a lower triangular matrix such that $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{CC'}$ and the density function of $\mathbf{Y}: p \times f_2$ is given by

$$(1.1) \qquad k_1 e^{-\lambda^2} \sum_{j=0}^{\infty} (2\lambda y_{11})^j \Gamma[\tfrac{1}{2}(f_1 + f_2 + j)] |\mathbf{I}_p - \mathbf{YY'}|^{\frac{1}{2}(f_1-p-1)}/j!$$

where $\mathbf{I}_p$ is an indentity matrix of order p,

$$k_1 = \textstyle\prod_{i=2}^{p} \Gamma[\tfrac{1}{2}(f_1 + f_2 - i + 1)]/ \pi^{\frac{1}{2}pf_2} \prod_{i=1}^{p} \Gamma[f_1 - i + 1)/2],$$

λ is the only non-centrality parameter in the linear case and y_{11} is the element in the top left corner of the $\mathbf{Y}$ matrix.

Now $V^{(s)}$ criterion suggested by Pillai and $U^{(s)}$ (a constant times Hotelling's T_0^2), [7], [8], [9], [10] are the sums of the non-zero characteristic roots of the matrix $\mathbf{YY'}$ and $(\mathbf{I}_p - \mathbf{YY'})^{-1} - \mathbf{I}_p$ respectively. Here s is minimum (f_2, p). Also we may note that $V^{(s)}$ = trace $\mathbf{YY'}$ = trace $\mathbf{Y'Y}$ and $U^{(s)} = \operatorname{tr}(\mathbf{I}_p - \mathbf{YY'})^{-1} - p = \operatorname{tr}(\mathbf{I}_{f_2} - \mathbf{Y'Y})^{-1} - f_2$. It can be shown that the density function of the characteristic roots of the matrix $\mathbf{Y'Y}$ for $f_2 \leqq p$ can be obtained from that of the characteristic roots of $\mathbf{YY'}$ for $f_2 \geqq p$ if in the latter case the following changes are made: ([12], [5])

$$(1.2) \qquad (f_1, f_2, p) \to (f_1 + f_2 - p, p, f_2).$$

Hence, for the criterion $V^{(s)}$, (and similarly for $U^{(s)}$), we shall only consider the density function of $\mathbf{L} = \mathbf{YY'}$ for $f_2 \geqq p$ which is given by [6]

$$(1.3) \quad f(\mathbf{L}) = ke^{-\lambda^2} {}_1F_1\{\tfrac{1}{2}(f_1 + f_2), \tfrac{1}{2}f_2, \lambda^2 l_{11}\} |\mathbf{L}|^{(f_2-p-1)/2} |\mathbf{I}_p - \mathbf{L}|^{(f_1-p-1)/2},$$

where

$$k = \pi^{-p(p-1)/4} \textstyle\prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(f_1 + f_2 + 1 - i)]/\{\Gamma[\tfrac{1}{2}(f_1 + 1 - i)]\Gamma[\tfrac{1}{2}(f_2 + 1 - i)]\}.$$

Received 29 January 1965.

[1] The work of this author was supported by the Mathematics Division of the Air Force Office of Scientific Research Contract No. AF-AFOSR-84-63.

l_{11} is the element in the top left corner of the matrix $\mathbf{L}$ and ${}_1F_1$ denotes the confluent hypergeometric function. We shall call the distribution of $\mathbf{L}: p \times p$ the non-central (linear) multivariate beta distribution with f_2 and f_1 degrees of freedom.

Pillai [11] had noted that the elements of the matrix $\mathbf{L}$ can be transformed into independent beta variables which he showed for $p = 2, 3, 4$ and 5. In this paper we give a theorem which proves the general case. In addition, when $\lambda = 0$ the first and second order moments of l_{ij} are obtained and used to derive the first two moments of $V^{(s)}$ in the non-central case when $f_2 \geqq p$. The moments of $V^{(s)}$ for $f_2 \leqq p$ can be written down with the help of (1.2). Similar results are obtained for $U^{(s)}$.

2. Independent beta variables. Let

$$\mathbf{L} = \begin{pmatrix} l_{11} & \mathbf{l}' \\ \mathbf{l} & \mathbf{L}_{11} \end{pmatrix} \begin{matrix} 1 \\ p-1, \end{matrix} \qquad \mathbf{L}_{22} = \mathbf{L}_{11} - \mathbf{l}\mathbf{l}'/l_{11},$$
$$\quad 1 \quad p-1$$

and we note that $|\mathbf{L}| = l_{11}\,|\mathbf{L}_{22}|$ and

$$|\mathbf{I}_p - \mathbf{L}| = (1 - l_{11})|\mathbf{I}_{p-1} - \mathbf{L}_{22} - \mathbf{l}\mathbf{l}'/[l_{11}(1 - l_{11})]|.$$

Then it is easy to show that l_{11} and $\{\mathbf{L}_{22}, \mathbf{v} = \mathbf{l}/[l_{11}(1 - l_{11})]^{\frac{1}{2}}\}$ are independently distributed and their respective distributions are

$$(2.1) \quad f_1(l_{11}) = [\beta(\tfrac{1}{2}f_2, \tfrac{1}{2}f_1)]^{-1} \exp(-\lambda^2) l_{11}^{\frac{1}{2}f_2 - 1}(1 - l_{11})^{\frac{1}{2}f_1 - 1} {}_1F_1[\tfrac{1}{2}(f_1 + f_2), \tfrac{1}{2}f_2, \lambda^2 l_{11}]$$

and

$$(2.2) \qquad f_2(\mathbf{L}_{22}, \mathbf{v}) = k_2\, |\mathbf{L}_{22}|^{\frac{1}{2}[(f_2-1)-(p-1)-1]}\, |\mathbf{I}_{p-1} - \mathbf{L}_{22} - \mathbf{v}\mathbf{v}'|^{\frac{1}{2}(f_1-p-1)},$$

where $k_2 = k\beta(\frac{1}{2}f_2, \frac{1}{2}f_1)$.

For further independence, we can use two types of transformations given by

$$(2.3) \qquad \mathbf{u} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{v} \qquad \text{or} \qquad \mathbf{w} = \mathbf{T}^{-1}\mathbf{v},$$

where $\mathbf{I}_{p-1} - \mathbf{L}_{22} = \mathbf{T}\mathbf{T}'$ and $\mathbf{T}: (p - 1) \times (p - 1)$ is a lower triangular matrix. It is easy to show that $\mathbf{u}$ (or $\mathbf{w}$) and $\mathbf{L}_{22}$ are independently distributed and their respective distributions are

$$(2.4) \quad f_3(\mathbf{u}) = \pi^{-\frac{1}{2}(p-1)}\{\Gamma(\tfrac{1}{2}f_1)/\Gamma[(f_1 - p + 1)/2]\} \cdot (1 - \mathbf{u}'\mathbf{u})^{\frac{1}{2}(f_1-p-1)} \qquad [\text{or } f_3(\mathbf{w})]$$

and

$$(2.5) \qquad f_4(\mathbf{L}_{22}) = k_3\, |\mathbf{L}_{22}|^{\frac{1}{2}[(f_2-1)-(p-1)-1]}\, |\mathbf{I} - \mathbf{L}_{22}|^{\frac{1}{2}[f_1-(p-1)-1]},$$

where $k_3 = \pi^{\frac{1}{2}(p-1)}\{\Gamma[(f_1 - p + 1)/2]/\Gamma(\frac{1}{2}f_1)\}k_2$. We may note that the distribution of $\mathbf{L}_{22}: (p - 1) \times (p - 1)$ is central multivariate beta distribution with $(f_2 - 1)$ and f_1 degrees of freedom, and the similar reduction from $\mathbf{L}_{22}$ can be carried successively. We may also note that the transformation

(2.6) $$x_i = u_i^2/(1 - u_1^2 - \cdots - u_{i-1}^2), \qquad i = 1, 2, \cdots, p - 1, u_0 = 0,$$

in (2.4) gives us the independent beta-variates and their density functions are given by

(2.7) $$g_i(x_i) = \{\beta[\tfrac{1}{2}, \tfrac{1}{2}(f_1 - i)]\}^{-1} x_i^{\frac{1}{2}-1}(1 - x_i)^{\frac{1}{2}(f_1-i)-1}.$$

From the foregone, we have the following theorem:

THEOREM 1. *If the distribution of*

$$\mathbf{L} = \begin{pmatrix} l_{11} & \mathbf{l}' \\ \mathbf{l} & \mathbf{L}_{11} \end{pmatrix}$$

is given by (1.3), *then* l_{11}, $\mathbf{L}_{22} = \mathbf{L}_{11} - \mathbf{l}\mathbf{l}'/l_{11}$ *and* $\mathbf{u} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{l}/[l_{11}(1 - l_{11})]^{\frac{1}{2}}$ [*or* $\mathbf{w} = \mathbf{T}^{-1}\mathbf{l}/[l_{11}(1 - l_{11})]^{\frac{1}{2}}$ *where* $\mathbf{TT}' = \mathbf{I}_{p-1} - \mathbf{L}_{22}$ *and* $\mathbf{T}$ *is a lower triangular matrix*] *are independently distributed and their respective distributions are defined in* (2.1), (2.5) *and* (2.4).

It can be verified for $p = 3$ that from the variates l_{11}, $\mathbf{w}$ and $\mathbf{L}_{22}$, we can obtain the independent β-variates exactly the same as given by Pillai [11], but the use of l_{11}, $\mathbf{u}$ and $\mathbf{L}_{22}$ will give independent β-variates different from those of Pillai [11] in spite of the identical β-distributions.

3. The first and second order moments of l_{ij} when $\lambda = 0$. Let the density function of $\mathbf{L}$ be given by

(3.1) $$k|\mathbf{L}|^{\frac{1}{2}(f_2-p-1)}\,|\mathbf{I}_p - \mathbf{L}|^{\frac{1}{2}(f_1-p-1)},$$

where k is the same as in (1.3). It is easy to see that

(3.2)
$$\begin{aligned} E(l_{ij}) &= E(l_{11}) \quad \text{when} \quad i = j \\ &= E(l_{12}) \quad \text{when} \quad i \neq j \end{aligned}$$

and

(3.3)
$$\begin{aligned} E(l_{ij}l_{i'j'}) &= E(l_{11}^2) && \text{when} \quad i = j = i' = j' \\ &= E(l_{11}l_{12}) && \text{when} \quad i = j = i', i \neq j' \\ &= E(l_{11}l_{22}) && \text{when} \quad i = j, i' \neq i, i' = j' \\ &= E(l_{12}^2) && \text{when} \quad i = i', j = j', i \neq j \\ &= E(l_{11}l_{23}) && \text{when} \quad i = j, i' \neq j' \neq i \\ &= E(l_{12}l_{13}) && \text{when} \quad i = i', j \neq j' \neq i' \\ &= E(l_{12}l_{34}) && \text{when} \quad i \neq j \neq i' \neq j'. \end{aligned}$$

It is easy to see that if $\nu = f_1 + f_2$,

$$E(l_{11}) = f_2/\nu, \qquad E(l_{12}) = 0,$$

and

$$E(l_{11}^2) = f_2(f_2 + 2)/\nu(\nu + 2).$$

For $E(l_{12}^2)$, we integrate over other variates except l_{11}, l_{12} and l_{22}. Then as in Theorem 1, $u_1 = l_{12}/[(1 - l_{11})(1 - z)l_{11}]^{\frac{1}{2}}$, l_{11} and $(l_{22} - l_{12}^2/l_{11}) = z$ are independently distributed. Hence

$$E(l_{12}^2) = E[(1 - l_{11})l_{11}], E(1 - z)E(u_1^2 = x_1)$$
$$= f_1f_2\{\nu(\nu - 1)(\nu + 2)\},$$
$$E(l_{11}l_{12}) = E\{l_{11}[l_{11}(1 - z)(1 - l_{11})]^{\frac{1}{2}}\}E(u_1) = 0,$$

and

$$E(l_{11}l_{22}) = E(l_{11}z) + E\{l_{11}(1 - l_{11})(1 - z)x_1\}$$
$$= \{f_2(f_2 - 1) + f_1f_2/(\nu + 2)\}/\nu(\nu - 1).$$

Similarly for obtaining $E(l_{11}l_{23})$ and $E(l_{12}l_{13})$, we consider (3.1) with $p = 3$ only. Using the successive reduction of Theorem 1, it can be shown that $E(l_{11}l_{23}) = E(l_{12}l_{13}) = 0$. The same type of reduction gives us after some algebra $E(l_{12}l_{34}) = 0$. Hence, we have the following theorem:

THEOREM 2. *Let the distribution of* $\mathbf{L}: p \times p$ *be given by* (3.1). *Then*

(3.4) $$E(l_{ij}) = f_2/\nu \quad \text{if} \quad i = j$$
$$= 0 \qquad \text{otherwise},$$

and

(3.5)
$$\begin{aligned} E(l_{ij}l_{i'j'}) &= f_2(f_2 + 2)/\{\nu(\nu + 2)\} && \text{if } i = j = i' = j' \\ &= f_1f_2/\{\nu(\nu - 1)(\nu + 2)\} && \text{if } i = i', j = j', i \neq j \text{ and } i = j', i' = j, i \neq j \\ &= f_2\{(f_2 - 1) + f_1/(\nu + 2)\}/\{\nu(\nu - 1)\} && \text{if } i = j, i' = j', i \neq i' \\ &= 0 && \text{otherwise.} \end{aligned}$$

4. First two moments of $V^{(s)}$ criterion. We note that

(4.1) $$V^{(p)} = \text{tr}\,\mathbf{L} = l_{11} + \text{tr}\,\mathbf{L}_{22} + (1 - l_{11})\mathbf{u}'(\mathbf{I}_{p-1} - \mathbf{L}_{22})\mathbf{u},$$

where l_{11}, $\mathbf{u}$ and $\mathbf{L}_{22}$ are independently distributed and their respective distributions are given by (2.1), (2.4) and (2.5). With the help of Theorem 2, we find that

(4.2) $$E(\mathbf{I}_{p-1} - \mathbf{L}_{22}) = \mathbf{I}_{p-1}\{f_1/(\nu - 1)\},$$

(4.3) $$E[(\text{tr}\,\mathbf{L}_{22})(\mathbf{I}_{p-1} - \mathbf{L}_{22})] = \delta_1\mathbf{I}_{p-1},$$

and

$$E(\mathrm{tr}\,\mathbf{L}_{22})^2 = [(p-1)(f_2-1)/(\nu-1)]\{(f_2+1)/(\nu+1) + (p-2)(f_2-2)/(\nu-2) + f_1(p-2)/(\nu+1)(\nu-2)\}, \tag{4.4}$$

where

$$\delta_1 = [(f_2-1)/(\nu-1)]\{(p-1) - (f_2+1)/(\nu+1) - (f_2-2)(p-2)/(\nu-2) - f_1(p-2)/(\nu+1)(\nu-2)\}. \tag{4.5}$$

Moreover,

$$E[\mathbf{u}'(\mathbf{I}_{p-1} - \mathbf{L}_{22})\mathbf{u}] = \{f_1/(\nu-1)\}E(\mathbf{u}'\mathbf{u}) = (p-1)/(\nu-1), \tag{4.6}$$

$$E[(\mathrm{tr}\,\mathbf{L}_{22})\mathbf{u}'(\mathbf{I}_{p-1} - \mathbf{L}_{22})\mathbf{u}] = \delta_1 E(\mathbf{u}'\mathbf{u}) = \delta_1(p-1)/f_1, \tag{4.7}$$

and

$$\begin{aligned} E\{\mathbf{u}'(\mathbf{I}_{p-1} - \mathbf{L}_{22})\mathbf{u}\}^2 &= E\{\mathbf{u}'\mathbf{S}\mathbf{u}\}^2 \quad \text{if} \quad \mathbf{S} = \mathbf{I}_{p-1} - \mathbf{L}_{22} \\ &= E(s_{11}^2) \textstyle\sum_{i=1}^{p-1} E(u_i^4) + \{E(s_{11}s_{22}) + 2E(s_{12}^2)\} \\ &\quad \cdot \textstyle\sum_{i\neq j=1}^{p-1} E(u_i u_j) \\ &= 3(p-1)/(\nu-1)(\nu+1) + [(p-1)(p-2)/ \\ &\quad (\nu-1)(\nu-2)(f_1+2)]\{(f_1-1) \\ &\quad + 3(f_2-1)/(\nu+1)\}. \end{aligned} \tag{4.8}$$

Hence, we get

$$E(V^{(p)}) = 1 + (p-1)(f_2-1)/(\nu-1) + f_1\{(p-1)/(\nu-1) - 1\}a_1 \tag{4.9}$$

and

$$\begin{aligned} E(V^{(p)})^2 = 1 &+ [(p-1)(f_2-1)/(\nu-1)]\{2 + (f_2+1)/(\nu+1) \\ &+ (p-2)(f_2-2)/(\nu+2) + f_1(p-2)/(\nu+1)(\nu-2)\} \\ &- 2[f_1\{1 - (p-1)/(\nu-1) + (p-1)(f_2-1)/(\nu-1)\} \\ &+ [(p-1)(f_2-1)/(\nu-1)]\{1 - p + (f_2+1)/(\nu+1) \\ &+ (f_2-2)(p-2)/(\nu-2) + f_1(p-2)/(\nu+1) \\ &\quad \cdot (\nu-2)\}]a_1 \\ &+ f_1(f_1+2)[1 - 2(p-1)/(\nu-1) + 3(p-1)/(\nu-1) \\ &\quad \cdot (\nu+1) \\ &+ [(p-1)(p-2)/(\nu-1)(\nu-2)(f_1+2)] \\ &\quad \cdot \{f_1 - 1 + 3(f_2-1)/(\nu+1)\}]a_2, \end{aligned} \tag{4.10}$$

where

(4.11) $$a_1 = \{\textstyle\sum_{i=0}^{\infty} (\lambda^2)^i/[i!(\nu + 2i)]\} \exp(-\lambda^2),$$

and

(4.12) $$a_2 = \{\textstyle\sum_{i=0}^{\infty} (\lambda^2)^i/[i!(\nu + 2i)(\nu + 2i + 2)]\} \exp(-\lambda^2).$$

The expressions for the moments of $V^{(p)}$ given by (4.9) and (4.10) reduce to the results for $s = 2$ given by Pillai [11] when $p = 2$. However, Pillai has provided the first four moments of $V^{(2)}$ in that paper [11]. For obtaining the moments of $V^{(s)}$ when $f_2 \leqq p$ replace in the expression of the moments in (4.9) and (4.10) f_1 by $f_1 + f_2 - p$, f_2 by p and p by f_2 as in (1.2).

5. The first two moments of U^s. We prove first the following theorem for obtaining the moments of $U^{(p)}$ [7], [8], [9], [10].

THEOREM 3. *Let $M: p \times p = (m_{ij})$ be distributed as*

(5.1) $$k\,|\mathbf{M}|^{\frac{1}{2}(f_2-p-1)}\,|\mathbf{I}_p + \mathbf{M}|^{-\frac{1}{2}(f_1+f_2)}\,d\mathbf{M}.$$

Then for $f_1 > (p + 1)$,

(5.2) $$\begin{aligned} E(m_{ij}) &= f_2/(f_1 - p - 1) \quad \textit{if} \quad i = j \\ &= 0 \quad \textit{otherwise} \end{aligned}$$

and for $f_1 > (p + 3)$,

(5.3) $$\begin{aligned} E(m_{ij}m_{i'j'}) &= f_2(f_2 + 2)/\{(f_1 - p - 1)(f_1 - p - 3)\} \quad \text{if} \quad i = j = i' = j' \\ &= f_2(f_2 + f_1 - p - 1)/\{(f_1 - p)(f_1 - p - 1)(f_1 - p - 3)\} \quad \text{if} \quad i = i', j = j', i \neq j \\ &= f_2\{(f_1 - p)(f_1 - p - 1)\}^{-1}[(f_2 - 1) + (f_2 + f_1 - p - 1) \\ &\quad \cdot (f_1 - p - 3)^{-1}] \quad \text{if} \quad i = j, i' = j', i \neq i' \\ &= 0 \quad \textit{otherwise}. \end{aligned}$$

PROOF. $\mathbf{M}$ is symmetric and positive definite and for evaluating $E(m_{ij})$ and $E(m_{ij}m_{i'j'})$ it is easy to see from (3.2) and (3.3) the various cases which should be considered separately.

Moreover, we may note that

$$m_{11}, \ \mathbf{w}: (p - 1) \times 1 = \{m_{11}(1 + m_{11})\}^{-\frac{1}{2}} T_1^{-1}\mathbf{m} \quad \text{and} \quad \mathbf{M}_{22.1} = \mathbf{M}_{11} - \mathbf{m}\mathbf{m}'/m_{11}$$

are independently distributed and their respective density functions are

(5.4) $$\{\beta[\tfrac{1}{2}f_2, \tfrac{1}{2}(f_1 - p + 1)]\}^{-1} m_{11}^{\frac{1}{2}f_2-1}(1 + m_{11})^{-\frac{1}{2}(f_1+f_2-p+1)},$$

(5.5) $$\pi^{-\frac{1}{2}(p-1)}\{\Gamma[(f_1 + f_2 - p + 1)/2]\}^{-1}\{\Gamma[(f_1 + f_2)/2]\}(1 + \mathbf{w}'\mathbf{w})^{-\frac{1}{2}(f_1+f_2)},$$

and

$$k_3 \, |\mathbf{M}_{22.1}|^{\frac{1}{2}(f_2-p-1)} \, |\mathbf{I}_{p-1} + \mathbf{M}_{22.1}|^{-\frac{1}{2}(f_1+f_2-1)}, \tag{5.6}$$

where

$$\mathbf{M}_{22.1} = (m_{ij.1}\,,\, i, j = 2, 3, \cdots, p),\ \mathbf{I}_{p-1} + \mathbf{M}_{22.1} = \mathbf{T}_1\mathbf{T}_1',$$

$\mathbf{T}_1 : (p - 1) \times (p - 1)$ is a lower-triangular matrix and $\mathbf{M}_{11}$ is obtained from $\mathbf{M}$ by deleting the first row and column.

From the above results, it is easy to verify the following,

$$\begin{aligned}
E(m_{11}) &= f_2/(f_1 - p - 1);\\
E(m_{12}) &= (Ew_1)E[m_{11}(1 + m_{11})(1 + m_{22.1})]^{\frac{1}{2}} = 0,\\
E(m_{12}^2) &= E(w_1^2)[Em_{11}(1 + m_{11})][E(1 + m_{22.1})]\\
&= f_2(f_1 + f_2 - p - 1)/\{f_1 - p)(f_1 - p - 1)(f_1 - p - 3)\};\\
E(m_{11}m_{22}) &= E(m_{11}m_{22.1}) + E(m_{12}^2) = f_2(f_2 - 1)\{(f_1 - p)(f_1 - p - 1)\}^{-1}\\
&\qquad + E(m_{12}^2);\\
E(m_{11}m_{12}) &= E(w_1)[Em_{11}^{3/2}(1 + m_{11})^{\frac{1}{2}}m_{22.1}^{\frac{1}{2}}] = 0;\\
E(m_{12}m_{13}) &= E\{m_{11}(1 + m_{11})w_1^2m_{23.1}\} + E\{m_{11}(1 + m_{11})[(1 + m_{33.1})\\
&\qquad - m_{23.1}^2/(1 + m_{22.1})]^{\frac{1}{2}}w_1w_2\}\\
&= 0,
\end{aligned}$$

where w_1 and w_2 are the first two elements in w. Again $E(m_{12}m_{34}) = 0$.

This proves the theorem 3.

Lemma 1. *If* $\mathbf{L}: p \times p$ *is a symmetric and positive definite matrix and* $U^{(p)} = \operatorname{tr}(\mathbf{I}_p - \mathbf{L})^{-1} - p$, *then*

$$1 + U^{(p)} = \{(1 - l_{11})(1 - \mathbf{u}'\mathbf{u})\}^{-1} + (1 - \mathbf{u}'\mathbf{u})^{-1}(\mathbf{u}'\mathbf{M}\mathbf{u}) + \operatorname{tr}\mathbf{M}, \tag{5.7}$$

where

$$\mathbf{L} = \begin{pmatrix} l_{11} & \mathbf{l}' \\ \mathbf{l} & \mathbf{L}_{11} \end{pmatrix},$$

$$\mathbf{l}: (p - 1) \times 1 = \{l_{11}(1 - l_{11})\}^{\frac{1}{2}}(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{\frac{1}{2}}\mathbf{u},$$

$\mathbf{L}_{22} : (p - 1) \times (p - 1) = \mathbf{L}_{11} - \mathbf{l}\mathbf{l}'/l_{11}$ *and* $\mathbf{M}: (p - 1) \times (p - 1) = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} - \mathbf{I}_{p-1}$.

Proof. We may note that

$$(\mathbf{I}_p - \mathbf{L})^{-1} = \begin{pmatrix} (1 - l_{11})^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -(l_{11})^{\frac{1}{2}}\mathbf{u}' \\ -(l_{11})^{\frac{1}{2}}\mathbf{u} & \mathbf{I}_{p-1} - (1 - l_{11})\mathbf{u}\mathbf{u}' \end{pmatrix}^{-1}$$
$$\cdot \begin{pmatrix} (1 - l_{11})^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -(l_{11})^{\frac{1}{2}}\mathbf{u}' \\ -(l_{11})^{\frac{1}{2}}\mathbf{u} & \mathbf{I}_{p-1} - (1 - l_{11})\mathbf{u}\mathbf{u}' \end{pmatrix}^{-1} = \begin{pmatrix} 1 + l_{11}\,\mathbf{u}'\mathbf{u}/(1 - \mathbf{u}'\mathbf{u}) & (l_{11})^{\frac{1}{2}}\mathbf{u}'/(1 - \mathbf{u}'\mathbf{u}) \\ (l_{11})^{\frac{1}{2}}\mathbf{u}/(1 - \mathbf{u}'\mathbf{u}) & \mathbf{I}_{p-1} + \mathbf{u}\mathbf{u}'/(1 - \mathbf{u}'\mathbf{u}) \end{pmatrix}.$$

Hence

$$\operatorname{tr}(\mathbf{I}_p - \mathbf{L})^{-1} = 1 - (1 - \mathbf{u}'\mathbf{u})^{-1} + \{(1 - l_{11})(1 - \mathbf{u}'\mathbf{u})\}^{-1} + \operatorname{tr}(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} + \mathbf{u}'(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1}\mathbf{u}/(1 - \mathbf{u}'\mathbf{u}).$$

From this, the lemma is obvious.

THEOREM 4. *If the distribution of* $\mathbf{L}$ *is non-central (linear) multivariate beta distribution* (1.3) *and* $U^{(p)} = \operatorname{tr}(\mathbf{I}_p - \mathbf{L})^{-1} - p$, *then for* $f_1 > (p + 1)$,

$$E(U^{(p)}) = (pf_2 + 2\lambda^2)/(f_1 - p - 1), \tag{5.8}$$

and for $f_1 > (p + 3)$,

$$\operatorname{Var}(U^{(p)}) = 2[4\lambda^4(f_1 - p) + (4\lambda^2 + pf_2)(f_1 - 1)(f_1 + f_2 - p - 1)]/\{(f_1 - p)(f_1 - p - 1)^2(f_1 - p - 3)\}. \tag{5.9}$$

PROOF. By Theorem 1, we may note that l_{11}, $\mathbf{u}$ and $\mathbf{M} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} - \mathbf{I}_{p-1}$ are independently distributed and their respective density functions are given by (2.1), (2.4) and

$$k_3|\mathbf{M}|^{\frac{1}{2}(f_2-p-1)}|\mathbf{I}_{p-1} + \mathbf{M}|^{-\frac{1}{2}(f_1+f_2-1)}.$$

Let $l_{11,0}$ be the variate whose distribution is the same as that of l_{11} when $\lambda^2 = 0$. Then

$$E(1 - l_{11})^{-1} = E(1 - l_{11,0})^{-1} + 2\lambda^2/(f_1 - 2),$$

$$E(1 - l_{11})^{-2} = E(1 - l_{11,0})^{-2} + 4\lambda^2\{(f_1 + f_2 - 2) + \lambda^2\}/\{(f_1 - 2)(f_1 - 4)\}.$$

If $U_0^{(p)}$ be the $U^{(p)}$ statistic when l_{11} is replaced by $l_{11,0}$, then

$$E(U^{(p)}) = E(U_0^{(p)}) + [2\lambda^2/(f_1 - 2)]E(1 - \mathbf{u}'\mathbf{u})^{-1} \tag{5.10}$$

and

$$E[1 + U^{(p)}]^2 = E[1 + U_0^{(p)}]^2 + \{4\lambda^2/(f_1 - 2)\} \cdot E\{(1 - \mathbf{u}'\mathbf{u})^{-1}[\operatorname{tr}\mathbf{M} + (1 - \mathbf{u}'\mathbf{u})^{-1}(\mathbf{u}'\mathbf{M}\mathbf{u})]\} + [4\lambda^2(f_1 + f_2 - 2 + \lambda^2)/\{(f_1 - 2)(f_1 - 4)\}]E(1 - \mathbf{u}'\mathbf{u})^{-2}. \tag{5.11}$$

That is,

$$\operatorname{Var}(U^{(p)}) = \operatorname{Var}(U_0^{(p)}) + \alpha, \tag{5.11a}$$

where

$$\alpha = \{4\lambda^2/(f_1 - 2)\}E\{(1 - \mathbf{u}'\mathbf{u})^{-1}[\text{tr } \mathbf{M} + (1 - \mathbf{u}'\mathbf{u})^{-1}(\mathbf{u}'\mathbf{M}\mathbf{u})]\}$$
$$+ [4\lambda^2(f_1 + f_2 - 2 + \lambda^2)/\{(f_1 - 2)(f_1 - 4)\}]E(1 - \mathbf{u}'\mathbf{u})^{-2}$$
$$- [4\lambda^4/(f_1 - 2)^2][E(1 - \mathbf{u}'\mathbf{u})^{-1}]^2 - 2[2\lambda^2/(f_1 - 2)]E(1 + U_0^{(p)})$$
$$\cdot E(1 - \mathbf{u}'\mathbf{u})^{-1}.$$

We note that

$$E(U_0^{(p)}) = pf_2/(f_1 - p - 1), \quad E(\mathbf{M}) = (f_2 - 1)\mathbf{I}_{p-1}/(f_1 - p),$$
$$E(\text{tr } \mathbf{M}) = (p - 1)(f_2 - 1)/(f_1 - p),$$
$$E(1 - \mathbf{u}'\mathbf{u})^{-1} = (f_1 - 2)/(f_1 - p - 1),$$

and

$$E(1 - \mathbf{u}'\mathbf{u})^{-2} = (f_1 - 4)(f_1 - 2)/\{(f_1 - p - 1)(f_1 - p - 3)\}.$$

Putting these values in α, we get

$$(5.12) \quad \alpha = 8\lambda^4/(f_1 - p - 1)^2(f_1 - p - 3) + 8\lambda^2(f_1 - 1)(f_1 + f_2 - p - 1)/ (f_1 - p)(f_1 - p - 1)^2(f_1 - p - 3).$$

From theorem 3, it is easy to find Var $(U_0^{(p)})$. However the first four (central) moments of $U_0^{(p)}$ are available in [7], [9], [10] and substituting the value of Var $(U_0^{(p)})$ in (5.11a), we get Theorem 4.

The expressions for moments of $U^{(p)}$ given above check with those obtained by Pillai [11] for $p = 2$.

The third and fourth moments of $V^{(s)}$ and $U^{(s)}$ and some approximations to their distributions will be presented in a later report.

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