ABSTRACTS

(Abstracts of papers presented at the Central Regional meeting, Columbus, Ohio, March 23–25, 1967. Additional abstracts appeared in the February and April issues.)

14. Selecting a subset containing the best hypergeometric population. N. S. Bartlett and Z. Govindarajulu, The Standard Oil Company and Case Institute of Technology.

In many practical situations, such as in acceptance testing, an experimenter is faced with finite populations and finds it necessary or desirable to sample the populations without replacement. The parameter of interest may be p, the proportion of 'good' items, and the experimenter may wish to choose a subset of the k (hypergeometric) populations which will contain the best population (the one with the largest p-value) with specified probability. A selection rule for this situation is specified and its small-sample properties are studied. Tables are provided to facilitate the use of the procedure. A large sample procedure is also studied. (Received 6 February 1967.)

15. Nonparametric tests for 2-factor experiments (preliminary report). C. B. Bell and H. Geller, Case Institute of Technology.

For the case of one observation per cell the hypothesis "no column effect" can be stated $H_0 : F_{ij} = \bar{F}_{i\cdot}$ for $i = 1, 2, \cdots, r$ (where the F_{ij} are continuous). Using the ideas of Pitman [(1937), (1938)], Scheffé (1943), Lehmann and Stein (1949) and Bell and Doksum (1967) one proves: Theorem 1. *Each NP statistic is a function of a permutation statistic* wrt *to the appropriate permutation group* $\mathcal{S}'$. Theorem 2. *Each similar test function has a constant sum over* a.e. *orbit under* $\mathcal{S}'$. Theorem 3. *For non-sequential statistics, a statistic is a rank statistic iff it is* SDF wrt *the appropriate group* $\mathcal{G}$ *of monotone transformations.* Theorem 4. *The most powerful* NP *test against a parametric alternative is based on the permutation statistic of the likelihood function.* (Received 3 February 1967.)

16. Bivariate symmetry tests: parametric and NP (preliminary report). C. B. Bell and H. Smith Haller, Case Institute of Technology.

$H_0 : F(x, y) = F(y, x)$ in the normal case reduces to $H_0' : M_1 = M_2$ and $\sigma_1 = \sigma_2$. (A) A 45° rotation yields: reject H_0' if $|\bar{r}(1 - \bar{r}^2)^{-\frac{1}{2}}(n - 2)^{\frac{1}{2}}| > t(\alpha_1, n - 2)$ or $|\bar{v}n^{\frac{1}{2}}s_v^{-1}| > t(\alpha_2 n - 1)$ where $u = x + y$, $v = x - y$ and $\bar{r} = r(u, v)$. (B) The likelihood ratio test is based on $(1 - \bar{r}^2)(n - 1)[n - 1 - n\bar{v}^2/s_v^2]^{-1}$. For the NP case, (C) the family of all NP statistics is the family of all functions of Pitman statistics and (D) the most powerful NP tests against a specific parametric alternative is based on permutations of the likelihood function. Further, (E) the procedure in (D) can be reversed; and one finds, for example, for normals the Pitman statistics of (i) $\bar{x} - \bar{y}$ is "best" when $\sigma_1 = \sigma_2$, and (ii) $\sum x_r^2 - \sum y_r^2 - 2n\mu(\bar{x} - \bar{y})$ is "best" when $M_1 = M_2 = \mu$. (Received 3 February 1967.)

17. Nonparametric randomness tests. C. B. Bell and E. F. Mednik, Case Institute of Technology.

For the randomness hypothesis $H_0 : F_1 = F_2 = \cdots = F_n$, one finds: Theorem 1. *Each* NP *statistic has a discrete H_0-distribution with probabilities integral multiples of* $(n!)^{-1}$, *and is a function of a permutation statistic. Further, for each discrete distribution F with*

probabilities which are integral multiples of $(n!)^{-1}$, *there exists a* NP *statistic* T *with* H_0-*distribution* F. THEOREM 2. *Each randomized test based on a similar partition is* NP; *and for each preassigned distribution* F, *there exists a randomized statistic with* H_0-*distribution* F. THEOREM 3. *The most powerful* NP *statistic against a specific alternative is based on the permutation statistic of the likelihood function. Conversely, for each* NP *statistic* T *there exists an exponential family of densities against which* T *is most powerful* NP. EXAMPLE:

Alternative	Permutation Statistic	Optimal Class
$\mathfrak{N}(ia, \sigma^2)$	$\sum ix_i$	$\exp\{-iy + k(i) + t(y)\}$
$\mathfrak{N}(0, i\sigma^2)$	$\sum (x_i i^{-1})^2$	$\exp\{-yi^{-1} + k(i) + t(y)\}$
Exponential, λ_i	$\sum \lambda_i x_i$	$\exp\{-\lambda_i y + k(i) + t(y)\}$

(Received 3 February 1967.)

18. Nonparametric independence tests: optimal alternatives (preliminary report). C. B. BELL and P. SMITH, Case Institute of Technology.

It is known [Bell and Doksum (1967)] that the most powerful NP test against a fixed alternative is a permutation test based on the likelihood function. For the converse problem one has the following solution in the continuous case with $H_0 : H(x, y) = F(x)G(y)$ for all x and y. THEOREM. *The permutation test based on* $T = \sum J(x_i, y_i)$ *is* (i) *most powerful* NP *against the class of alternatives with densities* $\exp(aJ(x, y) + b \ln (f(x)g(y)))$; *and* (ii) *locally most powerful* NP *against the families* $\exp(a(\theta)J(x, y) + b(\theta) + \ln(f(x)g(y)) + Q(x, y, \theta))$, *where* $a(\theta)$ *and* $b(\theta)$ *are differentiable,* $a'(\theta)$ *is nonvanishing,* $Q(x, y, \theta) = o(\theta - \theta_0)$ *for all* x *and* y *and* $a(\theta_0) = b(\theta_0) = Q(x, y, \theta_0) \equiv 0$. (Received 3 February 1967.)

19. Degenerate convergence and von Mises-Bernstein theorem for posterior distributions. JAY D. BORWANKER, University of Minnesota.

Let $\{X_n, n = 1, 2, \cdots\}$ be a strictly stationary Markov Chain with a stationary initial probability distribution $P_\theta(\cdot)$ and a transition probability function $P_\theta(x, \cdot)$, the parameter θ belonging to a closed interval Θ of the real line with a non-empty interior. Let $f(x; \theta)$ and $f(x, y; \theta)$ be the derivatives of $P_\theta(\cdot)$ and $P_\theta(x, \cdot)$ respectively, with respect to a σ-finite measure μ. Let $\lambda(\theta)$ be an *a priori* distribution function on Θ and $F_n(\theta)$ be the corresponding posterior distribution given $X_1, \cdots, X_{n+1}$. If θ_0, the true parameter point, is an interior point of Θ and $\lambda(\theta_0 + h) - \lambda(\theta_0 - h) > 0$ for every $h > 0$, then, under some regularity conditions, $P_{\theta_0}[F_n(\theta) \to \delta_{\theta_0}(\theta)] = 1$ where $\delta_{\theta_0}(\theta) = 0$ or 1 according as $\theta <$ or $\geqq \theta_0$. Let $\bar{\theta}_n$ be the maximum likelihood estimator of θ and let $i(\theta) = -\int [(\delta^2/\delta\theta^2) \log f(x_1, x_2; \theta)] f(x_1, x_2; \theta) f(x_1; \theta)\, d\mu(x_1)\, d\mu(x_2)$. If $\lambda'(\theta)$ is a prior density such that it is continuous and positive in a neighbourhood of θ_0, then

$$P_{\theta_0}\left[\lim_n \int_\Theta \left| f_n(\theta) - \left[\frac{ni(\bar{\theta}_n)}{2\pi}\right]^{\frac{1}{2}} \exp\left[-\frac{n}{2}(\theta - \bar{\theta}_n)^2 i(\bar{\theta}_n)\right]\right| d\theta = 0\right] = 1$$

where $f_n(\theta)$ is the posterior density given $X_1, \cdots, X_{n+1}$. These results are an extension of the results of K. Urbanik [Prace Matem. (1959)] and L. LeCam [Univ. of Cal. Pub. (1953)]. Similar results have been obtained for independent identically distributed random variables whose range depends on the parameter θ. (Received 3 February 1967.)

20. C_{λ_i}-minimax estimators. John J. Deely, Martin S. Tierney, and William J. Zimmer, Sandia Corporation.

Let X denote a random variable taking values in $\mathfrak{X}$ and having a density $f(x \mid \theta)$, θ itself being a random variable taking values in Θ according to a distribution G. In this paper we will not assume a specific form for G, but rather only assume its existence and one of the following: (i) the mean of G is known; (ii) the mean and variance of G are known; (iii) the first two moments of G are finite and prior observations on X are available. For any family $\mathcal{G}$ of distributions on Θ an estimator t^* is called *$\mathcal{G}$-minimax* if $\sup_{G \varepsilon \mathcal{G}} R(t^*, G) \leqq \sup_{G \varepsilon \mathcal{G}} R(t, G)$ for every other estimator t, where $R(t, G)$ is the Bayes risk of t. We are mainly interested in estimating parameters from two distributions which are of considerable use in reliability estimation, those being the binomial and gamma distributions. For these distributions, $\mathcal{G}$-minimax estimators are derived when $\mathcal{G}$ is the appropriate family specified by (i) or (ii). When (iii) obtains we define $\mathcal{G}$ as the family of all distributions with the same first two moments (though unknown) and derive "asymptotic" $\mathcal{G}$-minimax estimators using the prior observations to estimate the unknown moments. Comparisons with the usual estimators are also made. (Received 3 February 1967.)

21. On some percentile estimators of the location parameter of the Weibull and certain other distributions. Satya D. Dubey, Ford Motor Company. (By title)

In this paper an estimator of the location parameter of a Weibull distribution is proposed which is independent of its scale and shape parameters. Several properties of this estimator are established which suggest a proper choice of three ordered sample observations insuring a permissible estimate of the location parameter. This estimator is applicable to every distribution which has the location parameter acting as the origin or threshold parameter. Asymptotic properties of such an estimator of the location parameter of the Weibull distribution is discussed. Finally, the paper contains a brief discussion on a percentile estimator of the location parameter of the Weibull distribution and includes some numerical illustration. (Received 13 February 1967.)

22. On the construction of BIB designs using non-degenerated quadratics in finite Euclidean space. Sakti P. Ghosh, IBM Research Center.

A class of non-degenerated quadrics in finite Euclidean space has been defined. It has been shown that by choosing the points of $EG(N, s)$ as the varieties and these non-degenerated quadrics as the blocks, it is possible to construct three new different series of BIB designs. (Received 30 January 1967.)

23. Distribution-free confidence bounds for $P(X < Y)$. Zakkula Govindarajulu, Case Institute of Technology.

Let $X_1, \cdots, X_m (Y_1, \cdots, Y_n)$ denote a random sample from a continuous population having $F(x)(G(y))$ for its distribution function. Also, let $X(Y)$ be the underlying random variable. We wish to estimate $p = \int_{-\infty}^{\infty} F\, dG$. The best unbiased estimate of p is $\hat{p} = \int F_m\, dG_n)$ where $F_m(G_n)$ denotes the empirical distribution function based on the X's$(Y$'s$)$. The problem is, for given γ $(0 < \gamma < 1)$, to determine ϵ free of F and G such that $P(p \leqq \hat{p} + \epsilon) = P(p \geqq \hat{p} - \epsilon) \geqq \gamma$ or $P(|\hat{p} - p| \leqq \epsilon) \geqq \gamma$. In this paper, asymptotic normality of $\hat{p}$ is established under somewhat milder assumptions than assumed in the literature. A distribution-free upper bound for the asymptotic variance of $\hat{p}$ is derived, and using this large-sample one-sided and two-sided distribution-free confidence bounds for $\hat{p} - p$ are explicitly derived. The one-sided ones are one-half of the corresponding bounds due to Birnbaum and McCarty [*Ann. Math. Statist.* **29** (1958) 558–562]. These distribution-free bounds are also about 80% as efficient as those based on normal samples of X and Y. An unbiased, consistent and distribution-free estimate of the variance of $\hat{p}$ is obtained which could be used instead of the upper bound, and thus shorten the confidence bounds. These results are extended to the situations of random sample sizes and of discontinuous F and G. (Received 6 February 1967.)

24. On the Glivenko-Cantelli theorem for infinite invariant measures. EUGENE M. KLIMKO, Ohio State University.

Let τ be an ergodic measure preserving point transformation on the sigma-finite measure space $(\Omega, \mathfrak{A}, \mu)$. In probability theory, null-recurrent Markov chains and Markov processes satisfying the Harris condition give rise to such transformations. Let X and Y be $\mathfrak{A}$-measurable functions and $X_n = X \circ \tau^n$, $Y_n = Y \circ \tau^n$, for $n = 0, 1, \cdots$. For extended real numbers s, x, t, y, let $F_n^s(x) = 1$ if $X_n \varepsilon (s, x)$, 0 otherwise; $G_n^t(y) = 1$ if $Y_n \varepsilon (t, y)$, 0 otherwise; $F^s(x) = \mu[X \varepsilon (s, x)]$ and $G^t(y) = \mu[Y \varepsilon (t, y)]$.

THEOREM. *Let $s, t \varepsilon \bar{R}$ (extended real line). Let C and D be sets in $\bar{R}$ such that for some positive constants c, d, $C = [x: F^s(x) \leqq c]$ and $D = [y: G^t(y) \geqq d]$. Let $B = C \times D$ and*

$$\Delta_n = \sup_{(x,y)\varepsilon B} |\textstyle\sum_{i=0}^{n-1} F_i^s(x)/\sum_{i=0}^{m-1} G_i^t(y) - F^s(x)/G^t(y)| .$$

Then for almost all $\omega \varepsilon \Omega$ $\lim_{n\to\infty} \Delta_n = 0$. (Received 1 February 1967.)

25. On the probabilistic limit theorems for a class of positive matrices. L. A. KLIMKO and L. SUCHESTON, Ohio State University.

A new approach to the ergodic theory of positive matrices is developed by the method of Banach limits. Let $T = (t_{ij})$ be irreducible and recurrent. One considers $L(t_{ij}^n)$, t_{ij}^n bounded, and $L(\sum_1^N t_{ij}^n / \sum_1^N t_{11}^n)$, obtaining quite simply the Doeblin ratio limit theorem, the existence and uniqueness of invariant measures and the unaveraged convergence of t_{ij}^n in the aperiodic case. Assume that there is a positive right invariant function; (this holds, as proved by Vere-Jones in 1962, if 1 is the radius of convergence of the series $\sum_n t_{ij}^n s^n$). Let f be summable, g positive and summable. A necessary and sufficient condition for convergence of $\sum_1^N fT^n / \sum_1^N dT^n$ at each coordinate is that v be bounded. (Received 1 February 1967.)

26. A selection problem (preliminary report). DESU M. MAHAMUNULU, State University of New York at Buffalo.

We have k populations $\Pi_1, \cdots, \Pi_k$ at our disposal, where the quality of each one of them is characterized by a scalar parameter θ. The population with the largest θ-value is defined to be the best. Population Π_i is said to be good if $d(\theta_i, \theta_{\max}) \leqq \Delta$ and bad if $d(\theta_i \, \theta_{\max}) > \Delta$, where d is a suitable distance function and Δ is a specified constant. The problem is to devise a procedure for selecting the largest subset such that the probability is at least P^* (a specified number) that all the bad populations are excluded from the selected set. The suggested procedure is the following: Select Π_i whenever $d(T_i, T_{\max}) \leqq c$ where T_i is a real-valued statistic from Π_i and $T_{\max} = {}_{\max}(T_1, \cdots, T_k)$. The constant c is determined so as to satisfy the above probability requirement. Two cases—T_i has the density $f(x - \theta_i)$ for $-\infty < x < \infty$ and T_i has the density $f(x/\theta_i)$ for $0 < x < \infty$—are considered. The supremum of the expected number of bad populations that are included in the selected subset is obtained. Other related formulations of the problem are also considered. (Received 2 February 1967.)

27. Approximations to the generalized negative binomial distribution. HAJIME MAKABE and ZAKKULA GOVINDARAJULU, Tokyo Institute of Technology and Cast Institute of Technology.

$S_N = X_1 + X_2 + \cdots + X_N$, where the X's are independent (not necessarily identical) geometric random variables, is called a generalized negative binomial random variable. Govindarajulu (*Proc. Internat. Symp. Classical and Contagious Discrete Distributions*, Montreal (1963) p. 99) has obtained a sufficient condition for the asymptotic normality of S_N, when suitably standardized. However, it is of interest to improve the normal approximation and to explore other approximations. In this investigation, a correction term to the normal approximation and a bound on the error term are dèrived. Also, approximations by a suitable Poisson distribution with a correction term and a bound on the error are obtained. Further, approximation by an infinitely divisible law is considered. The main tools are expansion of the characteristic function and application of Lévy's inversion formula. We employ techniques similar to those used by Makabe and Morimura [*Kodai Math. Sem. Rep.* **8** (1956) 31–40] and Makabe [*Kodai Math. Sem. Rep.* **14** (1962) 123–133]. (Received 6 February 1967.)

28. Selecting a subset containing the best one of several IFRA populations (preliminary report). JAGDISH K. PATEL, University of Minnesota. (Introduced by Milton Sobel.)

Consider $k \geqq 2$ IFRA (increasing failure rate average) populations π_i $(i = 1, 2, \cdots, k)$. The IFRA class of populations with non-decreasing average failure rate, $\gamma(T)$, was defined by Birnbaum et al. [*Ann. Math. Statist.* **37** (1966) 816–825]. Let $\gamma_{[1]} \leqq \gamma_{[2]} \leqq \cdots \leqq \gamma_{[k]}$ be the unknown ordered values of $\gamma = \gamma(T)$. Any selection of a subset which contains at least one population with $\gamma_{[1]}$ is regarded as a correct selection (CS). Then, for a preassigned probability P^*, a procedure R is studied which satisfies the condition $P\{CS \mid R\} \geqq P^*$ regardless of true unknown γ-values. This procedure R is: Retain π_i in the selected subset if and only if $N_i + c \leqq (N_{\min} + c)/d$, where $c > 0$ and $0 < d \leqq 1$ are determined subject to R satisfying the probability condition and $N_i = N_i(T)$ is the number of failures observed from π_i by some fixed common time T. A condition to determine uniquely a pair (c, d) is obtained in terms of the expected size $E\{S \mid R\}$ of the selected subset S whenever $\gamma_{[i]} - \gamma_{[1]} \geqq \delta^* > 0$, where δ^* is preassigned. The basic method of approach is to show that the lower bound of $P\{CS \mid R\}$ and the upper bound of $E\{S \mid R\}$ can be expressed in terms of the corresponding results for Poisson processes. Another result is that the same lower bound of $P\{CS \mid R\}$ is also the correct asymptotic expression for an analogous problem dealing with binomial populations when the common sample size is large. (Received 1 February 1967.)

29. Maximizing the expectation of a step function of a sum of independent random variables. S. M. SAMUELS, University of California, Berkeley.

Let $\mathcal{S}$ be the class of all random variables $S_n = X_1 + \cdots + X_n$ where the X_i's are independent and non-negative, with prescribed means $\nu_1, \cdots, \nu_n$ and let g be a bounded, increasing step function on $[0, \infty)$. We consider the problem of finding $\sup_{S_n \varepsilon \mathcal{S}} Eg(S_n)$. The problem for $g =$ the indicator function of $[\lambda, \infty)$ was studied in my paper "On a Chebyshev-type inequality for sums of independent random variables." *Ann. Math. Statist.* **37** (1966)

248–259. As in that paper, solutions are obtained for small values of n, and for subclasses of $\mathcal{S}$ for arbitrary n. (Received 3 February 1967.)

30. Relations and bigraphs (preliminary report). JAGBIR SINGH and W. A. THOMPSON, JR., The Florida State University.

This work extends the results of Thompson and Remage [*Ann. Math. Statist.* **35** (1964) 739–747] to cover the treatment of ties in paired comparisons. A bigraph consists of a set $X = \{x_1, x_2, \cdots, x_m\}$ of points, a set of directed lines of the type $x \to y$ and a disjoint set of undirected lines of the type $x—y$. The lines are relations defined on X, i.e., they are disjoint subsets of $X \times X$. Bigraphs having no undirected lines are digraphs studied by Thompson and Remage and Harary and Moser [*Amer. Math. Monthly* **73** (1966) 231–245]. We define various relations on X by means of a bigraph and study their properties. An arrangement $(p_1, p_2, \cdots, p_m)$ of X is a partial rank order (pro) determined by a relation R on X if $(p_i, p_j) \notin R$ whenever $i > j$. Conditions are found under which one can determine a pro and a unique pro. In general, given a bigraph, these conditions will not be satisfied. This problem is resolved by deleting or changing the orientation of a minimum number of lines. It is immaterial whether we delete a line or change its orientation. There is a possibility of determining a unique pro even if the bigraph has undirected lines to start with. (Received 29 January 1967.)

31. Exponential ergodicity in Markov renewal processes (preliminary report). J. L. TEUGELS, Purdue University.

Consider a regular, irreducible Markov Renewal Process (MRP) as defined by R. Pyke [*Ann. Math. Statist.* **32** (1961) 1231–1259] with renewal function $\mathfrak{M}(t) = (M_{ij}(t))$ and matrix of transition probabilities $Q(t) = (Q_{ij}(t))$, where i and j run over the nonnegative integers. In the case of a positive-recurrent MRP it is known that the functions $M_{ij}(t)$ tend asymptotically to $t \cdot \mu_j^{-1}$ where μ_j is the mean recurrence time of the process to the state j. In this paper it is shown that if $\eta_i = E[\sum_j Q_{ij}(t)]$ is finite for all i, then the following "solidarity theorem" holds. If for some fixed pair of states i and j, there exist constants $0 < K_{ij} < \infty$ and $\lambda_j > 0$ such that $|M_{ij}(t) - t \cdot \mu_j^{-1}| \leqq K_{ij} e^{-\lambda_j t}$ for large t, then for every pair of states k and r, there exist constants K_{kr} and λ_r, where $0 < K_{kr} < \infty$, $\lambda_r > 0$, such that for large t $|M_{kr}(t) - t \cdot \mu_r^{-1} - L_{kr}| \leqq K_{kr} e^{-\lambda_r t}$, where $L_{kr} = [\mu_{kr}'' - 2\mu_{kr}\mu_r][2\mu_r^2]^{-1}$ and μ_{kr}'' is the second moment of the recurrence time distribution. A similar result holds in the transient case, even without the condition on η_j. Moreover in this case there exists a common "best-possible" positive constant λ such that λ_r in the above relation may be replaced by λ. These theorems extend results obtained by D. G. Kendall [U. Grenander (ed.) *Probability and Statistics* (1959)] and D. Vere-Jones [*Quart. J. Math.* **13** (1962) 7–28] in the case of a discrete time Markov Process, and of J. F. C. Kingman [*Proc. London Math. Soc.* (1963) 593–604] in the case of a continuous time Markov Process. (Received 2 February 1967.)

32. On partitioning a set of normal populations into those worse and those better than a standard by a single stage, two stage and sequential procedure. YUNG LIANG TONG, University of Minnesota.

Let $\pi_0, \pi_1, \cdots, \pi_k$ be $(k+1)$ normal populations with means $\mu_0, \mu_1, \cdots, \mu_k$ and a common variance σ^2; and let π_0 denote the standard or control population with μ_0 either known or unknown. For arbitrary but fixed δ_1^* and δ_2^* such that $\delta_1^* < \delta_2^*$, we define three subsets of the set $E = \{\pi_1, \pi_2, \cdots, \pi_k\}$ by

$$A_W = \{\pi_i : \mu_i \leqq \mu_0 + \delta_1^*\}, \quad A_I = \{\pi_i : \mu_0 + \delta_1^* < \mu_i < \mu_0 + \delta_2^*\} \quad \text{and}$$
$$A_B = \{\pi_i : \mu_i \geqq \mu_0 + \delta_2^*\}.$$

After observations have been taken, the set E is partitioned into two disjoint subsets S_W and S_B. A correct decision (CD) is defined by $A_W \subset S_W$ and $A_B \subset S_B$, or equivalently, $S_W \subset (A_W \cup A_I)$ and $S_B \subset (A_B \cup A_I)$. A population π_i is said to be misclassified if $\pi_i \varepsilon A_W \cap S_B$ or $\pi_i \varepsilon A_B \cap S_W$. Let P^* be an arbitrary but preassigned constant such that $2^{-k} < P^* < 1$. The statistical problem is to find a rule R (including a sampling procedure and a terminal decision rule) such that: (1) when $\sigma^2 = \sigma_0^2$ is known, $P[\text{CD} \mid \mathbf{u}, \sigma_0^2; R] \geqq P^*$ for every mean vector $\mathbf{u}$; (2) when σ^2 is unknown, $P[\text{CD} \mid \mathbf{u}, \sigma^2; R] \geqq P^*$ for every $\mathbf{u}$ and every $\sigma^2 > 0$. A similar decision rule has been proposed for known and unknown σ^2. When σ^2 is known, a single stage procedure is used and some optimal properties of the rule have been proved. When σ^2 is unknown, there is no single stage procedure that can solve this problem; a two stage procedure and a sequential procedure based on the idea of stage estimation of the unknown σ^2 have been considered. Tables of a multivariate normal distribution and a multivariate t distribution for determining the sample sizes required under the above procedures have been constructed. The expected sample size functions and the expected misclassification size functions and their asymptotic behavior have been investigated. (Received 31 January 1967.)

33. Some theorems on stochastically larger random vectors. Yung Liang Tong, University of Minnesota. (By title)

Lehmann [*Testing Statistical Hypotheses*, (1959) Wiley] has given the definition and a necessary and sufficient condition for one random variable to be stochastically larger (sl) than another. In the present paper an analogous definition is given to random vectors and two theorems have been obtained in terms of Lehmann's definition. Let $\mathbf{X} = (X_1, X_2, \cdots, X_k)$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_k)$ be any random vectors. $\mathbf{X}$ is said to be stochastically larger (sl) than $\mathbf{Y}$ if for every constant vector $\mathbf{c} = (c_1, c_2, \cdots, c_k)$ the inequality $P[X_1 \leqq c_1, X_2 \leqq c_2, \cdots, X_k \leqq c_k] \leqq P[Y_1 \leqq c_1, Y_2 \leqq c_2, \cdots, Y_k \leqq c_k]$ holds. If there exist k nondecreasing functions $f_1, f_2, \cdots, f_k$ such that $(f_1(X_1), f_2(X_2), \cdots, f_k(X_k))$ has the same joint distribution as $(Y_1, Y_2, \cdots, Y_k)$, then it can be shown that: (1) $\mathbf{X}$ is sl than $\mathbf{Y}$ iff for $i = 1, 2, \cdots, k$, X_i is sl than Y_i. (2) Let $g: R^k \to R^1$ be any real function such that $g(\mathbf{x}) = g(x_1, x_2, \cdots, x_k) \uparrow x_i$ for $i = 1, 2, \cdots, k$. If $\mathbf{X}$ is sl than $\mathbf{Y}$, then $g(\mathbf{X})$ is sl than $g(\mathbf{Y})$. In particular, this applies to linear combinations of the x_i's with non-negative coefficients and to the maximum of the x_i's. It should be noted that there is no assumption made here on independence. (Received 31 January 1967.)

(Abstracts of papers presented at the Eastern Regional meeting, Atlanta, Georgia, April 3–5, 1967. Additional abstracts appeared in earlier issues.)

12. Exact Bahadur efficiency for the Kolmogorov-Smirnov and Kuiper one- and two-sample statistics. Innis G. Abrahamson, Columbia University.

The exact slopes of the weighted Kolmogorov-Smirnov statistic,

$$K_{\psi,n} = n^{\frac{1}{2}} \sup [|F_n - F| \psi(F))]$$

(under mild restrictions on ψ), and the Kuiper statistic

$$V_n = n^{\frac{1}{2}} [\sup (F_n - F) + \sup (F - F_n)]$$

are found. V_n is always more efficient than $K_{\psi,n}$ when $\psi \equiv 1$. Limiting efficiencies are computed for several classes of alternatives. The limiting efficiencies of $K_{\psi,n}$ for $\psi \equiv 1$ relative to $\psi(F) = [F(1 - F)]^{-\frac{1}{2}}$ are computed for various alternatives and the weighted statistic appears to be superior. In the two-sample case, a parallel investigation is made of the corresponding unweighted Kolmogorov-Smirnov and Kuiper statistics. (Received 6 February 1967.)

13. Tests for a specified correlation matrix. Murray A. Aitkin, Karen H. Reinfurt, and W. C. Nelson, University of North Carolina and Virginia Polytechnic Institute.

A test for a specified correlation matrix in a p-variate normal distribution has been proposed by Bartlett and Rajalakshman (*J. Roy. Statist. Soc. Ser. B* **15** 107–24). The asymptotic χ^2 distribution of the test statistic is inadequate in small samples, and an appropriate correction is derived. The power of the test based on this corrected statistic is compared with the power of the test based on the likelihood ratio statistic. (Received 13 February 1967.)

14. Some integral transforms of characteristic functions. Gerald R. Anderson and Tatsuo Kawata, Bellcomm, Inc. and The Catholic University of America.

This paper considers the Poisson and Fejér integrals of the characteristic functions (cf's) $f(t)$ and $f(t)e^{-itx}$, respectively. These transforms are applied to the convergence problem of sequences of distribution functions (df's) to obtain nasc's for a sequence of df's to converge to a non-decreasing function. An additional condition is imposed on the transforms to obtain nasc's for the limiting function to be a df. The usual Lévy continuity theorem follows. The Poisson integral of a cf is characterized in terms of functions harmonic in the upper half plane. It is shown that $g(x, y)$, $-\infty < x < +\infty$, $y > 0$, is the Poisson integral of a positive definite function if, and only if, (i) $g(x, y)$ is harmonic on $-\infty < x < +\infty$, $y > 0$, (ii) the function $x \rightarrow g(x, y)$ is bounded on $(-\infty, +\infty)$ for each $y > 0$ and (iii) for all $\phi(x) \varepsilon L_1(-\infty, +\infty)$ with non-negative Fourier transforms $\hat{\phi}(t)$, $0 \leqq \int_{-\infty}^{+\infty} \phi(x)g(x, y)\, dx \leqq M \max \hat{\phi}(t)$, where M is a constant independent of ϕ and y $(y > 0)$. This theorem is extended to Poisson integrals of Fourier Stieltjes transforms of functions of bounded variation on $(-\infty, +\infty)$. (Received 6 February 1967.)

15. Testing for uniformity of a distribution on a compact topological group. R. J. W. Beran, The Johns Hopkins University.

Ajne (1966) has derived simple tests for uniformity of a distribution on a circle. His methods are extended as follows: Let the sample space X be a compact topological group with Haar measure μ. Let $E \subset X$ be measurable and $E \cup \bar{E} = X$. Define a class of densities $F = \{f_g(\cdot) \mid g \varepsilon X\}$ by $f_g(x) = p/\mu(E)$ if $x \varepsilon gE$ and $f_g(x) = (1 - p)/\mu(\bar{E})$ if $x \varepsilon g\bar{E}$. For $p = \mu(E)/\mu(X)$ these densities become uniform. Given n independent observations on a random variable with density in F, let $N(x)$ be the number of these observations belonging to xE. To test $H_0 : p = \mu(E)/\mu(X)$ versus $H_i : p \neq \mu(E)/\mu(X)$ the tests obtained by rejecting for large values of $\int_X [N(x) - n\mu(E)/\mu(X)]^2\, d\mu(X)$ and $\max_x N(x)$ are, respectively, locally most powerful invariant and most powerful invariant against distant alternatives in F. Asymptotic distributions are found on the sphere and torus [i.e. a rectangle with opposite edges identified]. The methods used apply, in principal, to any compact topological group. (Received 6 March 1967)

16. Tables for the moments of gamma order statistics. M. C. Breiter and P. R. Krishnaiah, Aerospace Research Laboratories, Wright-Patterson AFB. (By title)

Let $x_1, \cdots, x_n$ be n independent and identically distributed gamma variables and let the density function of each be given by $g_r(x) = e^{-x}x^{r-1}/\Gamma(r)$, $0 \leqq x < \infty$. Gupta (*Technometrics* **2** (1960) 243–262) gave tables for the first four moments of different gamma order statistics when $n = 1(1)15$ and $r = 1(1)5$. In this paper, we tabulated the first four moments of the gamma order statistics when $n = 1(1)16$ and $r = 1.5(1)10.5$. (Received 27 February 1967.)

17. On the minimum number of assemblies required to construct some partially balanced arrays with 2 symbols.

DHARAM VIR CHOPRA, Southern Colorado State College.

A partially balanced array of strength d, m constraints, N assemblies, 2 symbols [see Chakravarty "Fractional replications in asymmetrical factorial designs and partially balanced arrays," *Sankhyā* **17** (1956)] is an $(m \times N)$ matrix T with elements 0 and 1, with the following property: Let T^* be any d-rowed submatrix of T and let $\mathbf{V}_1 = (j_{11}, j_{12}, \cdots, j_{1d})'$, $\mathbf{V}_2 = (j_{21}, \cdots, j_{2d})'$ be any two column vectors of T^* where the j's $= 0$ or I and $\mathbf{V}_1$ is obtainable from $\mathbf{V}_2$ by permuting its elements. Then $\lambda(\mathbf{V}_1) = \lambda(\mathbf{V}_2)$ where $\lambda(\mathbf{V}_i)$ is the number of times $\mathbf{V}_i$ occurs as a column in T^*. Let μ_i be the number of times each distinct column vector of weight i appears in T^* and put $\boldsymbol{\mu}' = (\mu_0, \mu_1, \mu_2, \cdots, \mu_d)$. In this paper it has been shown, by the use of diophantine equations, that for $m = 8$, $d = 4$, $\mu_2 = I, 4, 6$, N must be >27, 55 and 69 respectively. As a consequence of this the existence of partially balanced arrays with $N = 28, 56, 70$ for $\mu_2 = I, 4, 6$ respectively and $m = 8$, $d = 4$ has been established and furthermore it has been shown that each of these arrays is unique. The values of $\boldsymbol{\mu}'$ for each of these arrays has been shown to be (6, 4, 1, 0, 0), (4, 6, 4, 1, 0) and (1, 4, 6, 4, 1). (Received 15 February 1967.)

18. Some extensions of Somerville's procedure for ranking means of normal populations.

WILLIAM R. FAIRWEATHER, Cornell University and the MITRE Corporation. (By title).

Somerville [*Biometrika* **41** (1954)] proposed a one-stage and a two-stage procedure (which eliminates one population after the first stage) for selecting the population with the largest mean from a set of normal populations with unknown means and a common, known variance. He assumed that a certain loss was incurred on making an incorrect selection and also assumed a sampling cost. He showed numerically, for the special case of three populations, that the two-stage procedure, when using appropriate allocations of observations between stages, has a smaller maximum expected loss (maximum over the possible configuration of the true population means) than does the one-stage procedure. The formulation of Sommerville's two-stage procedure is extended to arbitrary numbers of populations, and the procedures are studied numerically for the case of four populations. For this case, two 2-stage procedures are possible (eliminating either one or two populations after the first stage) and it is found that both two-stage procedures, using appropriate allocations, have smaller maximum expected losses than does the corresponding one-stage procedure, but which two-stage procedure is the better depends on the allocation. Methods of extending the formulation to multi-stage procedures are also considered. Numerical studies which are undertaken require the evaluation of 5-variate normal integrals with arbitrary correlation coefficients and limits of integration. This is accomplished using a method suggested by Plackett [*Biometrika* **41** (1954)]. (Received 10 February 1967.)

19. A test of goodness of fit based on sample-spacings.

JAMES R. GEBERT and B. K. KALE, Iowa State University.

Let $u_1, u_2, \cdots, u_n$ be the order statistics of a random sample of size n from a continuous cdf F. Let F_n be the empirical df. Define

$$I(F_n, F) = \sum_{i=1}^{n+1} [F(u_i) - F(u_{i-1})] \log \{(n+1)[F(u_i) - F(u_{i-1})]\}$$

with $u_0 = -\infty$, $u_{n+1} = +\infty$. Then $I(F_n, F)$ is the discriminatory information (in the sense of Kullback) provided by F against F_n. The limiting distribution of $I(F_n, F)$ is shown to be normal with mean $1 - C - [2(n+1)]^{-1}$ and variance $(n+1)^{-1}[\pi^2/3 - 3]$ where C is Euler's constant. For testing $H_0 : F = F_0$ a test is proposed on the basis of the statistic $I(F_n, F_0)$. The paper studies the power of this statistic, Greenwood's statistic $\sum_{i=1}^{n+1} V_i^r$ and the information statistic $I(F_0, F_n) = (n+1)^{-1}\sum_{i=1}^{n+1} \log \{(n+1)[F_0(u_i) - F_0(u_{i-1})]\}$ against Weiss' alternatives, $G_n(X) = F_0(X) + n^{-\delta} \int_0^{F_0(X)} r(x)\,dx$, where $\int_0^1 r(x)\,dx = 0$, and $\delta > 0$. (Received 13 February 1967.)

20. Small sample properties of percentile modified rank tests. JEAN D. GIBBONS and JOSEPH L. GASTWIRTH, University of Pennsylvania and The Johns Hopkins University.

This paper is a study of small sample properties of the two sample rank tests proposed recently by Gastwirth (*J. Amer. Statist. Assoc.* (1965) 1127–1141). The tests are percentile modifications of the well-known Wilcoxon test for location and Barton-David test for scale in that the same scoring method is used but only for the extremes of the rank order of the combined samples. Gastwirth's calculations of asymptotic relative efficiency implied that the tests would be effective against normal alternatives in large samples. Power results obtained here indicate that the same general conclusions apply to small and moderate sized samples, and that the percentile modified tests are almost as good as the asymptotically optimum tests for small samples from normal populations but are much easier to use. The tests are shown to have higher power than the Rosenbaum and Siegel-Tukey tests. Tables of the exact null probability distribution of the test statistics are given for equal sized small samples, and the normal approximation to the null distribution is shown to be accurate enough to use in practice even for small sample sizes. (Received 31 January 1967.)

21. Structural analysis for the multivariate regression model. M. SAFIUL HAQ, University of British Columbia.

The multivariate regression model has been analysed, treating it as a structural model [Fraser, *Biometrika* **53** (1966) 1–10]. The conditional probability element of the position statistic on the orbit, the structural probability elements for the parameters and, finally, the prediction distribution of the future response variable have been obtained. The results thus obtained have been applied to multivariate normal regression model as a special case of rotationally symetric error variable. The structural probability for the regression matrix β and the covariance matrix Σ are the same as obtained by Tia and Zellner (*J. Roy. Statist. Soc.* **26** 277–285) as posterior distribution of β and Σ which they arrived as assuming independence of β and Σ in the prior distribution with $\rho(\beta) =$ constant and $\rho(\Sigma)$ proportional to $|\Sigma|^{-\frac{1}{2}(k+1)}$, where k represents the rank of Σ. (Received 6 February 1967.)

22. The dominance of the mean for binary choice. MELVIN HINICH, Hudson Laboratories of Columbia University.

Let $x' = (x_1, \cdots, x_n)$ represent the preference values for each of n issues for an individual chosen at random from a population $N(\mu, \Sigma)$. Suppose $X = x$ has to choose between two positions, $\theta' = (\theta_1, \cdots, \theta_n)$ and $\phi' = (\phi_1, \cdots, \phi_n)$, and x chooses θ over ϕ if $\|x - \theta\|_A < \|x - \phi\|_A$ where $\|x - \theta\|_A{}^2 = (x - \theta)'A(x - \theta)$ and A is a positive definite. Let $r = P(\|X - \theta\|_A < \|X - \phi\|_A)$. Then $r > \frac{1}{2}$ if and only if $\|\theta - \mu\|_A < \|\phi - \mu\|_A$. This result can be extended to a stochastic A. Let $A = \bar{A} + E$ where E is a $n \times n$ matrix with elements ϵ_{ij} which are independent normal variates with mean zero and variance σ^2. Assume that $\sigma^2 \ll \lambda_{\min}$ where $\lambda_{\min}$ is the minimum ev of $\bar{A}$, and assume X and the ϵ_{ij}'s are independent. Then for large n, $r > \frac{1}{2}$ if and only if $\|\theta - \mu\|_{\bar{A}} < \|\phi - \mu\|_{\bar{A}}$. (Received 3 February 1967.)

23. Some properties of the sample coefficient of variation. BORIS IGLEWICZ and RAYMOND H. MYERS, Western Reserve University and Virginia Polytechnic Institute.

The sample coefficient of variation (scv) has many applications in statistics. Despite this there seems to be a lack of appropriate probability tables to facilitate its use. Such a table of percentage points of the scv for a normal parent distribution has been developed by the authors. Included in this presentation are several illustrations of the use of this table to find the percentage points of the scv, obtain the power of certain tests, find confidence intervals for the population coefficient of variation and calculate the percentage points of certain other statistics related to the non-central t distribution.

24. Monotone convergence of binomial probabilities with an application to ML estimator. KUMAR JOGDEO, Courant Institute of Mathematical Sciences, New York University.

Suppose for every n one performs a set of $(2n + 1)$ Bernoulli trials where the probability of success in an individual trial is $(n + 1)/(2n + 1)$. If S_{2n+1} denotes the number of successes then it is proved that the probability of S_{2n+1} exceeding its expected value $(n + 1)$, decreases monotonically to $\frac{1}{2}$, as n increases. This has been used to construct an example which underlines the known misbehaviour of the maximum likelihood (ML) estimator in the presence of *a priori* information. It has been shown that for a given sample size n, there exists an interval (length depending on n) around $\frac{1}{2}$ such that if *a priori*, p is known to be in it then the ML estimator is uniformly worse than a trivial estimator which does not utilize any knowledge from the sample. This example is amplification of the one given in Lehmann's notes on the theory estimation. (Received 6 February 1967.)

25. Remarks on some classes of characteristic functions. TATSUO KAWATA, The Catholic University of America.

The following two theorems are given: THEOREM 1. *Suppose that $F(x)$ is a distribution function satisfying*: (1) $F(x) = O(\exp - \theta(|x|))$ *as* $x \to -\infty$ *where* $\theta(u)$ *is a non-negative, non-decreasing function for* $u > 0$ *such that* (2) $\int_1^\infty \theta(u)/u^2\, du = \infty$. *If* $g(z)$, $z = t + i\tau$ *is a function analytic in* $a < t < b$, $0 < \tau \leqq R$ *and continuous in* $a < t < b$, $0 \leqq \tau \leqq R$ *and if the characteristic function* $f(t)$ *of* $F(x)$ *is equal to* $g(t)$ *in an interval* $(\alpha, \beta)\,(a < \alpha < \beta < b)$, *then* $f(t) = g(t)$ *holds in* (a, b). *Conversely if* $\theta(u)$ *is a non-negative, non-decreasing and is such that* $\int_1^\infty \theta(u)/u^2\, du < \infty$, *then for any* $L > 0$ *there is a distribution function* $F(x)$ *for which* (1) *holds, as well as* $1 - F(x) = O(\exp(-\theta(x)))$ *as* $x \to \infty$, *and the corresponding characteristic function vanishes over* $|t| > L$. THEOREM 2. *Suppose that the distribution function is constant except in* $I_n = (-\mu_n - \delta_n, -\mu_n + \delta_n)$ *for* $x < 0$, $\delta_n > 0$, $\mu_n > 0$, $n = 1, 2, \cdots$, *where* $\{I_n\}$ *is a sequence of non-overlapping intervals such that* $\mu_{n+1} - \mu_n \to \infty$, $\delta_n \to 0$, *as* $n \to \infty$. *Then we have the same conclusion as in the first part of Theorem* 1. We also give some theorems which assert the convergence of distribution functions from the convergence in a small interval of the corresponding characteristic functions. (Received 6 February 1967.)

26. Tests based on a symmetrical optimality criterion. OLAF KRAFFT, University of North Carolina.

A test φ^* for two hypotheses H and K on the parameter θ of a class of distribution functions is called optimal if it minimizes a weighted sum of the maximum error probabilities for all tests φ. After discussing simple hypotheses, a notion of least favorable *a priori* distribution is introduced which is based on the usual metric in L_1-space. With the help of that notion and using methods of infinite linear programming the problem of composite hypotheses is studied. The sufficient conditions, given by Lehmann (*Testing Statistical Hypotheses*, 327–328), for a test to be a maximin test turn out to be—in a modified version—necessary and sufficient for the above defined optimality of a test. (Received 1 February 1967.)

27. Simultaneous tests for multiple comparisons of polynomial growth curves when errors are autocorrelated. P. R. KRISHNAIAH, Aerospace Research Laboratories, Wright-Patterson AFB. (By title)

For $i = 1, 2, \cdots, k$, let $\{X_{it}\}$ be a discrete stochastic process of normal variates and suppose $X_{it} = \beta_{i0} + \beta_{i1}t + \beta_{i2}t^2 + \cdots + \beta_{ip}t^p + \epsilon_{it}$ where $\{\epsilon_{it}\}$ is a Gaussian stationary Markov process of order one with zero mean value. Also let $H = \bigcap_{j=0}^{p} H_j$ and $H_j = \bigcap_{u=1}^{m} H_{ju}$ where $H_{ju} : c_{u1}\beta_{1j} + \cdots + c_{uk}\beta_{kj} = 0$ and $H_j : \beta_{1j} = \cdots = \beta_{kj}$. In this paper we propose exact procedures to test the hypotheses H_{ju} simultaneously against two-sided and one-sided alternatives. These results are then extended to the case where $\{\epsilon_{it}\}$ is a Gaussian stationary Markov process of order h. (Received 10 February 1967.)

28. On the convergence of discrimination information. SOLOMON KULLBACK, The George Washington University.

A sufficient condition involving an hypothesis of mean square convergence is given for the convergence of discrimination information. A stronger condition requiring uniform convergence has already been given. An example is given which satisfies the mean square convergence hypothesis but not the uniform convergence hypothesis. A weaker sufficient condition requiring an hypothesis of convergence in mean of order one and the existence of sufficient partitions of the space is also given for the convergence of discrimination information. An example is given which satisfies the latter conditions but not the former. (Received 27 February 1967.)

29. A distribution arising from random points on the circumference of a circle (preliminary report). NICO F. LAUBSCHER and G. J. RUDOLPH, CSIR, Pretoria.

Suppose n points are observed on the circumference of a circle. If the points cluster closely together then the probability is small that the placing of points occurred by means of a "random" process. We derive the distribution of the smallest angle, α, (at the center of the circle) which subtends all n points, under the assumption that the points are independently and uniformly distributed on the circumference of the circle. The distribution is arrived at by two different approaches. A table is provided from which critical values of α can be obtained, for rejecting the null-hypothesis of a uniform distribution of points on the circumference of a circle. (Received 30 January 1967.)

30. Some limit theorems for dependent 'rare' events. RICHARD M. MEYER, University of North Carolina.

The object of this paper is to establish and apply some limit theorems for 'rare' events which are not necessarily independent. Specific results are given for certain sequences of m-dependent, $f(n)$-dependent, and strongly mixing 'rare' events. In particular, these results serve to establish the robustness of the Poisson limit for Bernoulli and Poisson-Bernoulli trials with respect to the assumption of independence. For example, under mild assumptions on the probabilities of $f(n)$-dependent 'rare' events, it is shown that the probability that exactly k among n of them occur is asymptotically Poisson as $n \to \infty$. As an application, the limit distribution of the number of upcrossings of a high level by the class of processes considered by Watson [*Ann. Math. Statist.* **25** (1954) 798–800] is determined. A related result for certain sequences of strongly mixing events enables one to determine the same limit distribution for a class of processes similar to that considered by Loynes [*Ann. Math. Statist.* (1965) **36** 993–999]. (Received 6 February 1967.)

31. The extended negative hypergeometric distribution (ENHD) (preliminary report). PAUL R. MILCH, Naval Postgraduate School.

This distribution is defined by the probability mass function $f(x; t; N, n, A) = S^{-1}(t)\binom{N}{n}^{-1}\binom{A+x-1}{x}\binom{N-A-x}{n-x}t^x$ for $x = 0, 1, \cdots, n$, where $S(t) = \binom{N}{n}^{-1}\sum_{x=0}^{n}\binom{A+x-1}{x}\binom{N-A-x}{n-x}t^x$ is the probability generating function of the negative hypergeometric distribution as defined by Bolshev (*Theor. Prob. Appl.* **9** 619–24, English trans.); Patil and Seshadri (*J. Roy. Statist. Soc. Ser. B* **26** 286–92), and others. The ENHD appears as the conditional distribution of X given $X + Y = n$ when X and Y are independent negative binomial random variables with parameters r_1, p_1, and r_2, p_2, respectively. Then $A = r_1$, $N = r_1 + r_2 + n - 1$, and $t = p_1/p_2$. Therefore, the ENHD may be used to build confidence intervals for the ratio p_1/p_2 of two Bernoulli parameters. For this purpose, tables for the percentiles of the ENHD were obtained for various values of r_1, r_2, and n. The general problem of estimation of $t = p_1/p_2$ is also discussed. The probability generating function of the ENHD may be expressed as the ratio of two Jacobi polynomials, a result that leads to several formulae among the moments of this distribution. Various limit distributions are also available. (Received 1 February 1967.)

32. On a k-sample model of Conover. P. V. RAO, University of Florida.

In a recent paper Conover (*Ann. Math. Statist.* **36** 1223–1225) presented a k-sample model in order statistics, in which k random samples of equal size are first ordered within themselves in the usual manner, and are ordered among themselves by considering the size of the maximum value in each sample. As an application of this model, Conover suggests that one may pick the k best items from nk items by first dividing them randomly into k groups of n items each and then picking the best in each group. In the present paper, the probability $p_{n,k}(0, m)$ of correctly picking the m best items $(0 < m \leqq k)$ from nk items is calculated for Conover's model. In general, this probability is very small when $m = k$, so that Conover's model is not very efficient for picking the k best items. However, given n and m, it is always possible to choose a k such that $p_{n,k}(0, m)$ is as large as we please. This implies that Conover's model may be used efficiently for picking m best items from nk items if k is sufficiently large. (Received 6 February 1967.)

33. Testing for clusters in a Poisson process. E. ROTHMAN, The Johns Hopkins University.

Let t_i and X_i denote the time of the ith event and the time between the $(i - 1)$st and ith event in a Poisson process. The testing problem H_0 : Poisson (λ) against H_1 : Poisson (λ) for $0 \leqq t \leqq t_k$ or, $t_{k+r} \leqq t \leqq t_n$ and Poisson $(\lambda + \mu)$ for $t_k \leqq t \leqq t_{k+r}$ where k is arbitrary is considered. The statistic $S(r) = \min_j[t_{j+r-1} - t_{j-1}]$, $1 \leqq j \leqq n - r + 1$, is shown to be UMP and its distribution is given in the case $n/r = 1, 2, 3, \cdots$. An analogous statistic, $\max_j \sum_{i=j}^{j+r-1} X_i$, is suggested for testing the significance of a band of r frequencies when the null hypothesis is a flat spectrum. The distribution of this statistic is found and an approximation is given. (Received 6 March 1967.)

34. Contributions to central limit theory for dependent variables. R. J. SERFLING, Research Triangle Institute and University of North Carolina.

The investigation examines the class of sequences $\{X_i\}$ satisfying $EX_i = 0$, $E|X_i|^{2+\delta} < M < \infty$ $(\delta > 0)$ and (A): $E(\sum_1^n X_i)^2 \sim n$. Under various alternative further restrictions upon the dependence in the sequence, several central limit theorems are obtained, using the characteristic function approach. The further restrictions do not curtail the dependence a great deal more than what is implied by condition (A). Also, though conditional expectations are involved, the restrictions are well adapted to interpretation in terms of properties of $\sum_1^n X_i$, the quantity of interest. These general results yield interesting theorems for weakly stationary sequences, for sequences of martingale differences, for bounded sequences, and for some other special classes; they generalize the Hoeffding-Robbins theorem for m-dependent sequences. Included in the investigation are some results on moments of sums and some results relating the dependence restrictions to others in the literature. (Received 30 January 1967.)

35. An Iterative solution for the simple stochastic epidemic. NORMAN C. SEVERO, State University of New York at Buffalo.

The system of differential-difference equations describing the simple stochastic epidemic with infection rate equal to one are $p_r'(t) = (r + 1)(N - r - 1)p_{r+1}(t) - r(N - r)p_r(t)$ for $r = 0, 1, \cdots, N$, where N is the total population size, $p_r(t)$ is the probability of r infectives at time t, and $p_r(t) \equiv 0$ if $r < 0$ or $r > N$. Subject to the arbitrary initial conditions $p_r(0) = a_r$, $(a_r \geqq 0, \sum_{r=0}^N a_r = 1)$, the state probabilities may be shown to be $p_{N+1-k}(t) = \sum_{j=1}^{N+1} c(k, j) \exp(b_jt)$, $k = 1, \cdots, N + 1$, where $c(i, j)$ is defined as 0 for $i < j$, a_N for $i = j = 1$, $(N - i + 2)(i - 2)\epsilon(i - 2)[c_1(i - 1, j)\delta(b_j - b_i) - c_2(i - 1, j)\delta^2(b_j - b_i) + c_2(i - 1, j)\delta(b_j - b_i)t]$ for $i > j$, and $a_{N+1-i} - \sum_{u=1}^{i-1} c_1(i, u)$ for $i = j > 1$. In these expressions $b_i = -(N + 1 - i)(i - 1)$ for $i = 1, \cdots, N + 1$; $\delta(x)$ is equal to t when $x = 0$ and x^{-1} when $x \neq 0$; $\epsilon(x)$ is equal to 1 for $x \geqq 0$ and 0 for $x < 0$; and c_1 and c_2 are defined recursively (in i) as the coefficients in the linear function of t, $c(i, j) = c_1(i, j) + c_2(i, j)t$. (Received 13 February 1967.)

36. Partial diallel cross designs. K. R. SHAH, Michigan State University.

Connection between partially balanced incomplete block (PBIB) designs and partial diallel cross (PDC) was noted by Hinklemann and Kempthorne (*Biometrika* (1963)). It is noted here that to obtain a PDC design one needs *only* an association scheme. One may start with an association scheme as defined by Bose and Mesner in *Ann. Math. Statist.* (1959). For a choice of $\lambda_1, \cdots, \lambda_m$ where each $\lambda_i = 0$ or 1 (except that at least one $\lambda_i \neq 0$) there exists a PDC design which can be analyzed using a method given by Hinkelmann and Kempthorne. Thus for *any* association scheme there are $2^m - 1$ PDC designs. Some PDC designs obtained in this manner are examined here for their efficiency in estimating general combining ability parameters. (Received 15 February 1967.)

37. Maximum likelihood rankings from paired comparisons when ties are allowed (preliminary report). JAGBIR SINGH and W. A. THOMPSON, JR., The Florida State University.

This work extends the scheme of Thompson and Remage [*Ann. Math. Statist.* (1964) **35** 739–747] to establish the maximum likelihood order from paired comparison data when ties are also allowed. The members of a set X of m objects are compared independently in pairs by a "subject" who states his preferences and indifferences. x_i and x_j are compared on independent trials; each trial having three possible outcomes denoted respectively by $x_i \rightarrow$

x_j, $x_j \to x_i$ and $x_i - x_j$. It is natural to think of such direct comparisons as constituting a sample and to consider the population parameters $\pi_{ij} = P(x_i \to x_j)$ and $\gamma_{ij} = P(x_i - x_j)$. If every pair is compared at least once, then there are $2\binom{m}{2}$ functionally independent parameters. The parameter space is a $2\binom{m}{2}$ dimensional unit cube with typical point denoted by π. For an arbitrary relation $R(\pi)$ defined in terms of probabilities let ω be that portion of the parameter space where $R(\pi)$ is a preference relation. One can define $R(\pi)$ in various possible ways but we consider only three alternative definitions. For each case, given paired comparison data, we estimate the preference relation on X by maximizing the likelihood function with respect to π over ω. In practice the maximum may not be unique and in that case we determine all those points of ω where the likelihood is maximized. (Received 10 February 1967.)

38. The use of OLUMV estimators in inference robustness study of the location parameter of a class of symmetric distributions. G. C. Tiao and D. Lund, University of Wisconsin and State University of Wisconsin, Eau Claire.

Box and Tiao [A note on criterion robustness and inference robustness. *Biometrika* **51** (1964) 169–73] made the distinction between *criterion* robustness and *inference* robustness. Criterion robustness refers to the sensitivity of the distribution of a criterion, which is optimal under a certain set of assumptions about the model, to departures from such assumptions. Inference robustness, on the other hand, refers to the sensitivity of the conclusions which we are entitled to draw from the data to changes in the underlying assumptions, when the criterion is appropriately adjusted. In this paper, we present an inference robustness study of the location parameter θ of the class of distributions $p(y \mid \theta, \sigma, \beta) = K \exp\{-\frac{1}{2}\cdot |(y - \theta)/\sigma|^{2/(1+\beta)}\}$. Suppose that N independent observations are drawn from a member of the class. For a fixed value of β, the OLUMV estimators $\hat{\theta}_\beta$ and $\hat{\sigma}_\beta$ of θ and σ can be obtained, and (except for $\beta = 0$), we adopt the quantity $(\hat{\theta}_\beta - \theta)/s(\hat{\theta}_\beta)$, where $s(\hat{\theta}_\beta)$ is proportional to $\hat{\sigma}_\beta$ and is the estimated standard deviation of $\hat{\theta}_\beta$, as the corresponding optimal criterion for making inferences about θ. Tables of the OLUMV coefficients of $(\hat{\theta}_\beta, \hat{\sigma}_\beta)$ are given for $\beta = 0.75(0.25)1$ and $N = 2(1)20$. A numerical example is given showing how conclusions about θ in terms of significance levels might be affected by changes in β. Extension to the problem of linear contrasts in the one way analysis of variance model is also discussed.

39. Inference with tested priors. V. B. Waikar and S. K. Katti, Florida State University.

In this paper, the authors have extended the ideas introduced first by Katti (*Biometrics* (1962) **18** 139–147) to multivariate populations. A two stage sampling procedure has been proposed to estimate the unknown mean vector μ of a multivariate population when the covariance matrix Σ is known and prior information about the mean vector is available in the form of a guessed value μ_0. The first stage sample is used to test the accuracy of the prior knowledge. The proposed estimate $\hat{\mu}$ depends upon choosing a region R in the sample space. It is shown that the optimal region R^* which minimizes the generalized expected mean square of $\hat{\mu}$ given μ_0 is the true mean vector, is an ellipsoid. (GEMS $(\hat{\mu} \mid \mu_0)$ = determinant of $E[(\hat{\mu} - \mu_0)(\hat{\mu} - \mu_0)']$.) Some properties of R^* are derived. The expressions for the 'expected total sample size' and the 'expected percentage saving in the sample size' are derived. For the case of multivariate normal distribution the efficiency of $\hat{\mu}$ with respect to sample mean vector based on a fixed sample is obtained. The behaviour of the generalized expected mean square of $\hat{\mu}$ is studied when the guessed value μ_0 departs from the true mean μ. (Received 13 February 1967.)

(Abstracts of papers presented at the Western Regional meeting, Missoula, Montana, June 15–17, 1967. Additional abstracts will appear in future issues.)

2. Necessary equations for the existence of some partially balanced arrays with 2 symbols. DHARAM VIR CHOPRA, Southern Colorado State College.

A partially balanced array of strength d, m constraints, N assemblies, 2 symbols [See Chakravarty "Fractional replications in asymmetrical factorial designs and partially balanced arrays," *Sankhyā* **17** (1956)] is an $(m \times N)$ matrix T with elements 0 and 1, with the following property: let T^* be any d-rowed submartrix of T and let $\mathbf{V}_1 = (J_{11}, J_{12}, \cdots, J_1 d)'$, $\mathbf{V}_2 = (J_{21}, \cdots J_{2d})'$ be any two column vectors of T^* where J's $= 0$ or 1 and $\mathbf{V}_1$ is obtainable from $\mathbf{V}_2$ by permuting its elements. Then $\lambda(\mathbf{V}_1) = \lambda(V_2)$ where $\lambda(\mathbf{V}_i)$ is the number of times $\mathbf{V}_i$ occurs as a column in T^*. Let μ_i be the number of times each distinct column vector of weight i appears in T^* and put $\mathbf{\mu}' = (\mu_0, \mu_1, \cdots, \mu_d)$. Let $m = 8$, $d = 4$ and X_i $(i = 0, 1, 2, \cdots, 8)$ be the number of vectors of length 8 and weight i which constitute the partially balanced array T. In this paper a set of equations in X_i's have been obtained by calculating the total number of vectors of length 4 and weight i $(i = 0, 1, \cdots, 4)$ in two different ways. The method of forming these equations is easily extendable to any m, d. Under the conditions that each X_i $(i = 0, 1, 2, \cdots 8)$ be a non-negative integer, these equations have been solved for $m = 8$ and as a consequence the least upper bound for N has been obtained which is shown to be attainable when $\mu_2 = 6$. (Received 24 February 1967.)

3. Characterizations of independence in certain families of bivariate and multivariate distributions. KUMAR JOGDEO, Courant Institute of Mathematical Sciences, New York University.

Lehmann [*Ann. Math. Statist.* **37** (1966) 1137–1153] showed that in the family of bivariate distributions with quadrant positive (or negative) dependence uncorrelatedness implies independence. This characterization is extended to multivariate families of distributions. It is shown that the conditions for independence can be expressed through the product moments and the result for the bivariate distribution is a special case. An earlier result of Jogdeo and Patil [Abstract, *Ann. Math. Statist.* **38** (1967) 642] gives a class of bivariate distributions where independence of *any* two events of the type $[X_1 \leqq a]$, $[X_2 \leqq b]$ characterizes independence of X_1 and X_2. This class has been broadened and generalized to a class of multivariate distributions where the total independence of *any* n events of the type $[X_1 \leqq a_1]$, $[X_2 \leqq a_2]$, $\cdots$, $[X_n \leqq a_n]$ implies that the same holds for the random variables $X_1, \cdots, X_n$. Some implications of these results to the tests of independence are pointed out. (Received 6 February 1967.)

4. Confidence intervals for difference of frequencies in matched samples. JOHN E. WALSH, System Development Corporation.

A. Stuart has developed a test for the comparison of frequencies in matched samples [The comparison of frequencies in matched samples. *British J. Statist. Psych.* **10** (1957) 29–32]. He assumes that the data consist of independent bivariate observations such that the first coordinates, considered by themselves, constitute the trials of one binomial population and, likewise, the second coordinates are the trials of a second binomial population. The null hypothesis asserts that these binomial populations are equal (have the same probability for "success"). This paper extends these results to obtain approximate confidence intervals for the difference between the probabilities of success for the two binomial populations. Not only is correlation permitted to occur between the entries of a bivariate observation but different observations can have different values for this correlation. (Received 20 February 1967.)

(Abstracts of papers to be presented at the Annual meeting, Washington, D. C., December 27–30, 1967. Additional abstracts will appear in future issues.)

1. On the trace comparison of some partially balanced arrays with 2 symbols. DHARAM VIR CHOPRA, Southern Colorado State College. (By title)

A partially balanced array of strength d, m constraints, N assemblies and 2 symbols [See Chakravarty "Fractional replications in asymmetrical factorial designs and partially balanced arrays," *Sankhyā* **17** (1956)] is an $(m \times N)$ matrix T with elements 0 and 1, with the following property: Let T^* by any d-rowed submatrix of T and let $\mathbf{V}_1 = (J_{11}, J_{12}, \cdots, J_{1d})'$, $\mathbf{V}_2 = (J_{21}, J_{22}, \cdots, J_{1d})'$ be any two column vectors of T^* where the J's $= 0$ or 1 and $\mathbf{V}_1$ is obtainable from $\mathbf{V}_2$ by permuting its elements. Then $\lambda(\mathbf{V}_1) = \lambda(\mathbf{V}_2)$ where $\lambda(\mathbf{V}_i)$ is the number of times $\mathbf{V}_i$ occurs as a column in T^*. Let μ_i be the number of times each distinct column vector of weight i appears in T^* and put $\boldsymbol{\mu}' = (\mu_0, \mu_1, \cdots, \mu_d)$. An array T of strength 4 is called (1, 0) symmetric [See Bose and Srivastava "Analysis of irregular factorial fractions," *Sankhyā Ser. A* **26** (1964)] if $\mu_0 = \mu_4$ and $\mu_1 = \mu_3$. In the paper "Optimal balanced 2^m fractional factorial designs" (to be published in S. N. Roy memorial volume, University of North Carolina) J. N. Srivastava gives the trace criterion to compare two arrays T_1 and T_2 according to which T_1 is 'better than' T_2 if tr $(M^{-1})_{T_1} <$ tr $(M^{-1})_{T_2}$ where M is obtained from the normal equations. In this paper, it has been shown that of all arrays T, with $m = 8$, $d = 4$ and μ_2 fixed, the array with the least trace is the one which is (1, 0) symmetric. (Received 27 February 1967.)

2. A new approach to the classification problem. H. O. HARTLEY and J. N. K. RAO, Texas A & M University.

In this paper the problem of classification of individuals into one of several groups is reduced to an estimation problem. It is shown how the problem may be solved by well-known estimation techniques. The new approach leads to a more general method of classification than the classical approach. Certain comparisons (such as the probabilities of misclassification) are made between the new and classical approaches for some simple cases. The following problems are solved using the new approach: (1) classification when the groups form a one-way classification; (2) classification when the groups form a two-way classification and certain groups are not represented in the initial sample; (3) classification when some individuals in the initial sample are misclassified. The univariable and multivariable cases are both considered, however, the detailed investigations are confined to the univariable case because of its simplicity. (Received 30 January 1967.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. Convergence of likelihood ratios of probabilities based on rank orders. ROBERT H. BERK and I. RICHARD SAVAGE, University of California, Berkeley, and The Florida State University.

Assume $X_1, \cdots, X_n$ (resp. $Y_1, \cdots, Y_n$) are mutually independent and identically distributed real valued random variables having continuous distribution functions F^* (resp. G^*). Define $H^* = (F^* + G^*)/2$, $U^* = F^*H^{*-1}$ and $V^* = G^*H^{*-1}$. Define $Z = (Z_1, \cdots, Z_{2n})$ by $Z_i = O(1)$ if the ith smallest of the $2n$ random variables is an $X_i(Y_i)$. Also define

$$\exp [nI_n(Z)] = (2n)' \cdot \int \cdots \int_{0<x_1<\cdots<x_{2n}<1} \{\textstyle\prod_{i=1}^{2n} [f^{1-Z_i}(x_i) g^{Z_i}(x_i)\, dx_i]\}$$

where $f(\cdot)$ and $g(\cdot)$ are piece-wise continuous densities on (0, 1). Then it is shown

$$\text{limit}_{n\to\infty} I_n(Z) =_{\text{wp1}} \sup_p \int_0^1 [(\ln f(x))\, dU^*P(x) + (\ln g(x))\, dV^*P(x) - 2(\ln p(x))\, dP(x)]$$

where $P(x)$ is an absolutely continuous distribution on (0, 1) with density $p(x)$. This result includes Lemma 2 and solves the general problem (pp. 1158–1159) of Savage and Sethuraman (*Ann. Math. Statist.* **37** (1966) 1154–1160). (Received 9 March 1967.)

2. Continuous time sequential Markovian control processes (preliminary report). S. S. CHITGOPEKAR, The Florida State University.

We consider a stochastic system with a finite state space and a finite action space. We assume that between actions, the waiting time to transition is a random variable with a continuous distribution depending only on the previous state and the action taken. The system earns at a rate depending upon the state of the system and the action taken. In contrast to R. A. Howard [*Proc. Internat. Statist. Inst.* (1963) 625–652], we allow actions to be taken between transitions. If a policy is such that at some stage there is a positive probability of an action being taken between transitions, we say that the policy involves "hesitation." For any policy S the criterion of interest is $I(S)$, given by

$$I(S) = \liminf_{N\to\infty} \textstyle\sum_{n=1}^{N} E(i_n(S))/\sum_{n=1}^{N} E(T_n(S))$$

where $i_n(S)$ is the income earned under the nth action and $T_n(S)$ is the time spent by the system under the nth action. It is shown that when the waiting-time distributions are exponential, the optimal policy is a stationary policy not involving "hesitation." However, when the waiting times are not exponentially distributed, we cannot always eliminate "hesitation." The "hesitation" involved in the optimal policy in the class of stationary policies is such that after the action in any state has been selected, the expected time to transition is either maximized or minimized. The case where action is costly has also been considered. (Received 15 February 1967.)

3. A theorem about infinitely divisible distributions. MEYER DWASS, Northwestern University.

Suppose that $p_0, p_1, \cdots$ are probability masses on $0, 1, \cdots$, with generating function $P(t) = \sum p_n t^n$. Suppose the distribution is not defective and has first moment equal to 1. That is $P(1) = 1$ and $P'(1) = 1$. Then $u = u(t) = t/P(t)$ defines a one-to-one mapping of $[0, 1]$ onto itself with inverse $t = h(u)$. Suppose that $p_1 \neq 1$. THEOREM. *$h(u)$ is a probability generating function of an infinitely divisible distribution which assigns its mass to $1, 2, \cdots$ and which has infinite first moment. This is equivalent to saying that*

$$h(u) = u \exp \textstyle\sum_{n=1}^{\infty} a_n(u^n - 1), \qquad 0 \leqq u \leqq 1,$$

with $a_n \geqq 0$ and $\sum a_n < \infty \cdot h'(1) = \infty$. (Received 24 February 1967.)

4. Invariance in the multivariate normal and Wishart distributions. MORRIS L. EATON, University of Chicago.

Let V have a Wishart distribution with expectation $n\Sigma$ and n degrees of freedom. If g is a 1–1 bimeasurable mapping of the sample space of V onto itself, and if g preserves the Wishart family for Σ varying in an open set, it is shown that $gV = AVA'$ (a.e.) for some nonsingular matrix A. A similar characterization of transformations preserving the multivariate normal family (when the natural parameters vary over an open set) is obtained. Both of the above results use results given in an abstract by Lehmann and Stein [*Ann. Math. Statist.* **25** (1953) 142]. The above results are used to characterize all transformations preserving the multivariate analysis of variance hypothesis testing problem. Similar results are obtained for other invariant hypothesis testing problems. (Received 6 March 1967.)

5. On the asymptotic theory of estimating the mean by sequential confidence intervals of prescribed accuracy. ARTHUR J. NADAS, Columbia University.

Asymptotically efficient [in the sense of Chow, Y. S. and Robbins, H. (1965), On the asymptotic theory of fixed-width confidence intervals for the mean. *Ann. Math. Statist.* **36** 457–462] random sample sizes are obtained for the problem of estimating the mean (of a population whose variance is finite, but whose distribution is otherwise arbitrary) within prescribed accuracy. Results are obtained under various definitions of accuracy, such as fixed-width confidence intervals (as in Chow and Robbins), proportional closeness and combinations of the two. The cases where variance or squared coefficient of variation is known are also treated. Bounds are obtained for the expected sample size in (Chow and Robbins). (Received 24 February 1967.)