TIMID PLAY IS OPTIMAL, II

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An animal population reproduces under the following conditions. Within each generation, given the past, animals reproduce independently of one another. Each animal products 0, 2, 3, $\cdots$ descendants; the mean number of descendants is positive but at most $M < 2$. How is the survival probability of the population maximized? The answer is, each animal should produce 0 or 2 descendants, with respective probabilities $1 - \frac{1}{2}M$ and $\frac{1}{2}M$. This theorem will now be stated formally and proved.

Let $Z_0, Z_1, Z_2, \cdots$ be a sequence of integer-valued random variables. For nonnegative integer j and $0 < M < 2$, say $(Z_0, Z_1, \cdots)$ is a (j, M)*-population* iff $Z_0 = j$ and for all $n > 0$, given $Z_0, \cdots, Z_{n-1}$, the conditional distribution of Z_n is that of a sum of Z_{n-1} independent random variables, each taking only values 0, 2, 3, $\cdots$ and having mean at most M.

The (j, M)-population $(Z_0, Z_1, \cdots)$ is *timid* iff, for all $n > 0$, given $Z_0, \cdots, Z_{n-1}$, the conditional distribution of Z_n is that of a sum of Z_{n-1} independent random variables, each being 0 with probability $(1 - \frac{1}{2}M)$ and 2 with probability $\frac{1}{2}M$. For the timid (j, M)-population, $P(Z_k > 0) = f(j, M, k)$ is given by the formula

$$f(j, M, k) = 1 - g(M, k)^j$$

where

$$g(M, k) = (1 - \tfrac{1}{2}M) + (\tfrac{1}{2}M)g(M, k - 1)^2$$

and

$$g(M, 0) = 0.$$

For $k > 0$, $0 < g(M, k) < 1$.

Theorem. *For any (j, M)-population $(Z_0, Z_1, \cdots)$, and for any nonnegative integer k, the probability that $Z_k > 0$ is at most $f(j, M, k)$.*

A slightly more careful argument shows the inequality to be strict unless $Z_0, \cdots, Z_k$ are distributed like the first $k + 1$ terms in the timid (j, M)-population.

Proof. The theorem is trivial for $k = 0$. Suppose it holds for some $k > 0$, and let $(Z_0, Z_1, \cdots)$ be a (j, M)-population. Of course, $P(Z_{k+1} > 0) = E\{P[Z_{k+1} > 0 \mid Z_1]\}$. Now Z_1 is distributed like $X_1 + \cdots + X_j$, the X_i's being

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independent, taking values 0, 2, 3, $\cdots$, and having mean at most M. Given Z_1, the process $(Z_1, Z_2, Z_3, \cdots)$ is conditionally a (Z_1, M)-population. Thus,

$$P[Z_{k+1} > 0] \leqq E\{f(Z_1, M, k)\} = 1 - \prod_{i=1}^{j} E[g(M, k)^{X_i}].$$

Now use the lemma below.

LEMMA. *Let* $0 < g < 1$ *and* $M < 2$. *Among all random variables* X *taking values* 0, 2, 3, $\cdots$ *and having mean at most* M, $E(g^X)$ *is minimized when* X *takes the values* 0 *and* 2 *with rbspective probabilities* $(1 - \frac{1}{2}M)$ *and* $\frac{1}{2}M$.

PROOF. Let $h(x) = (1 - \frac{1}{2}x) + (\frac{1}{2}x)g^2$. Then $h(n) \leqq g^n$ for $n = 0, 2, 3, \cdots$, and h is decreasing, so $E(g^X) \geqq E[h(X)] = h[E(X)] \geqq h(M)$.

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