THE MAXIMUM DEVIATION OF SAMPLE SPECTRAL DENSITIES

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0. Summary. The present paper gives sufficient conditions on a linear process $\{X_t\}$ and its spectral density $f(\cdot)$ for the following limit relation to hold:

$$\|W\|_2^{-1}(N/2m_N \log m_N)^{\frac{1}{2}} \max_{-\pi \leqq \lambda \leqq \pi} [|f_N(\lambda) - f(\lambda)|/f(\lambda)] \to 1 \tag{0.1}$$

in probability as $N \to \infty$ where $f_N(\cdot)$ is the usual windowed sample spectral density, $m_N W(m_N \cdot)$ is the (varying) window, and $m_N \uparrow \infty$ as $N \to \infty$ at a suitable rate. Under the same conditions it is shown that

$$P(a_N^{-1}[N^{\frac{1}{2}} m_N^{-\frac{1}{2}} \|W\|_2^{-1} \max_{|i| \leqq m_N} [|f_N(\lambda_{N,i}^*) - f(\lambda_{N,i}^*)|/f(\lambda_{N,i}^*)] - b_N] \leqq x) \to \exp(-\exp(-x)) \tag{0.2}$$

as $N \to \infty$ for $-\infty < x < \infty$ where $\lambda_{N,i}^*$, a_N, and b_N are defined by (2.1) and (2.2).

Observe that the difference between the maximum deviation and the deviation at a single λ point [5] manifests itself in the factor $(\log m_N)^{-\frac{1}{2}}$. Thus in practice a confidence band for all λ is $O((\log m_N)^{\frac{1}{2}})$ times that for a finite set.

1. Introduction. In this paper we will study spectral estimation in the case of a real-valued, discrete parameter, linear stochastic process $\{X_t: t = 0, \pm 1, \cdots\}$—i.e., a process for which

$$X_t = \sum_{k=-\infty}^{\infty} a_k \xi_{t-k} \tag{1.1}$$

where $\cdots \xi_{-1}, \xi_0, \xi_1, \cdots$ are independent, identically distributed random variables, and $\sum |a_k| < \infty$. In this case $\{X_t\}$ has the spectral density

$$f(\lambda) = (2\pi)^{-1} |\sum_{v=-\infty}^{\infty} a_v e^{-iv\lambda}|^2 \tag{1.2}$$

so that

$$R(v) \equiv E\{X_t X_{t+v}\} = \int_{-\pi}^{\pi} e^{-iv\lambda} f(\lambda)\, d\lambda. \tag{1.3}$$

The most commonly used spectral density estimates, $f_N(\lambda)$, are those obtained by weighting the periodogram,

$$I_N(\lambda) \equiv (2\pi N)^{-1} |\sum_{v=1}^{N} X_v e^{-iv\lambda}|^2, \tag{1.4}$$

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in the following manner [5]:

$$f_N(\lambda) = m_N \int_{-\infty}^{\infty} W(m_N(u - \lambda))I_N(u)\, du \tag{1.5}$$

where $\{m_N\}$ is a sequence of positive integers increasing to ∞ with N at a suitable rate and $W(\cdot)$ is a suitable non-negative, even, weight function. It will be assumed that $W(\cdot)$ has a Fourier representation,

$$W(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iv\lambda} w(v)\, dv \tag{1.6}$$

so that, since $W(\cdot)$ is non-negative,

$$w(v) = \int_{-\infty}^{\infty} e^{iv\lambda} W(\lambda)\, d\lambda. \tag{1.7}$$

We will also require $w(0) = 1$. Then (1.5) can be written

$$f_N(\lambda) = (2\pi)^{-1} \sum_{v=-N+1}^{N-1} e^{-i\lambda v} R_N(v) w(v m_N^{-1}) \tag{1.8}$$

where $R_N(\cdot)$ is the covariance estimate,

$$R_N(v) \equiv N^{-1} \sum_{t=1}^{N-v} X_t X_{t+v} = R_N(-v), \qquad v \geqq 0. \tag{1.9}$$

If $w(\cdot)$ has compact support (i.e., for some V, $w(v) = 0$ if $|v| > V$), $f_N(\cdot)$ will be called a *truncated* spectral estimate.

Our main theorems, Theorems 2.1 and 2.2, describe the asymptotic behavior as $N \to \infty$ of the maximum deviation of $f_N(\cdot)$ from both $f(\cdot)$ and $E(f_N(\cdot))$ where $f_N(\cdot)$ is a truncated estimate of $f(\cdot)$. (0.1) and (0.2) are particular cases. The proof of the theorems is divided into two major parts. First, they are proved for the pure white noise process itself (Section 3). The second part of the proof (Section 4) involves reducing the linear process case to the pure white noise case. Section 2 contains a statement of the theorems together with some preliminary material.

There are some related results in the literature. Similar problems in the case of the periodogram itself are discussed in [7], and the maximum of trigonometric polynomials with random coefficients is studied in [9]. [3], Chapter 6, contains a version of our Theorem 2.2 in which first $N \to \infty$ with m fixed and then $m \to \infty$. See also [4].

2. The main theorems. Since our theorems hold under varying combinations of assumptions, it is convenient to label the more common ones:

(A_1) $\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots$ are independent and identically distributed with $E(\xi_t) = 0$, $E|\xi_t|^2 = 1$, and $E|\xi_t|^8 < \infty$.

(A_2) X_t has the representation (1.1), and $|a_k| = O(k^{-(1+\beta)})$, as $k \to \infty$, where $\beta > \frac{1}{5}$.

(A_3) $f(\cdot)$ is everywhere positive and satisfies a uniform Lipschitz condition.

(A_3') $f(\cdot)$ is everywhere positive and has a bounded second derivative.

(A_4) $f_N(\cdot)$ is truncated, and $m_N = O(N^\alpha)$ as $N \to \infty$, $\alpha < \frac{2}{5}$.

(A_5) $W(\cdot)$ is a non-negative, even, bounded, integrable function satisfying (1.6) and (1.7). $w(0) = 1$, and $w''(0)$ exists.

In addition, we will need the following notation: for N so large that $m_N \geqq 2$, say $N \geqq N_0$, let

$$\lambda^*_{N,i} = \pi \cdot |i|/m_N, \qquad i = -m_N, \cdots, m_N, \tag{2.1}$$

$$a_N = (2 \log (2m_N))^{-\frac{1}{2}}, \tag{2.2}$$

$$b_N = (2 \log (2m_N))^{\frac{1}{2}} - \tfrac{1}{2}(2 \log (2m_N))^{-\frac{1}{2}}\{\log \log (2m_N) + \log 2\pi\}.$$

THEOREM 2.1. *Assume* (A_1)–(A_5); *if* $N \log N = o(m_N^{\gamma})$ *as* $N \to \infty$, $\gamma \leqq 4$, *then*

$$\max_{|\lambda| \leqq \pi} (N/2m_N \log m_N)^{\frac{1}{2}} |f_N(\lambda) - E(f_N(\lambda))|/\|W\|_2 f(\lambda) \to 1 \tag{2.3}$$

in probability as $N \to \infty$, *and* (0.1) *holds if* $\gamma \leqq 3$. *If, in addition,* (A_3') *is satisfied, then* (2.3) *is true provided* $\gamma \leqq 8$, *and* (0.1) *is true provided* $\gamma \leqq 5$.

THEOREM 2.2. *Assume* (A_1)–(A_5); *if* $N \log N = o(m_N^{\gamma}), \gamma \leqq 4$, *as* $N \to \infty$, *then*

$$\begin{aligned} &\lim_{N \to \infty} P(a_N^{-1}[N^{\frac{1}{2}} m_N^{-\frac{1}{2}} \|W\|_2^{-1} \\ &\cdot \max_{|i| \leqq m_N} [|f_N(\lambda^*_{N,i}) - E(f(\lambda^*_{N,i}))|/f(\lambda^*_{N,i})] - b_N] \leqq x) \\ &\qquad = \exp(-\exp(-x)) \end{aligned} \tag{2.4}$$

for $-\infty < x < \infty$; *and* (0.2) *holds if* $\gamma \leqq 3$. *If, in addition,* (A_3') *holds then* (2.4) *is true provided* $\gamma \leqq 8$, *and* (0.2) *is true provided* $\gamma \leqq 5$.

We conclude this section with a lemma which will be used in the next.

LEMMA 2.1. *Let* $p(\lambda) = \sum_{v=-k}^{k} \alpha_v \exp(iv\lambda)$ *be a trigonometric polynomial. Then*

$$\max_\lambda |p'(\lambda)| \leqq (2k + 1) \max_\lambda |p(\lambda)|$$

where $p'(\cdot)$ *denotes the derivative of* $p(\cdot)$.

COROLLARY 2.1. *Let* $\lambda_i = \pi \cdot (i/rk)$, $|i| \leqq rk$. *Then*

$$\max_{|\lambda| \leqq \pi} |p(\lambda)| \leqq \max_{|i| \leqq rk} |p(\lambda_i)/(1 - 3\pi r^{-1})|.$$

Lemma 2.1 is proved in [12]; the corollary then follows from

$$\max_{|\lambda| \leqq \pi} |p(\lambda)| \leqq \max_{|i| \leqq rk} |p(\lambda_i)| + (\pi/rk) \max_{|\lambda| < \pi} |p'(\lambda)|.$$

3. The pure white noise case. In this section we consider the special case in which $X_t = \xi_t$ for all t. In this case we will denote the spectral density estimate by $g_N(\cdot)$. Theorem 3.1 (below) says that $(N/m_N)^{\frac{1}{2}}(2\pi g_N(\lambda) - 1)$ is equal for fixed λ to a sum of independent, identically distributed random variables plus an error term which is uniformly negligible as $N \to \infty$. We remind the reader that $Y_N = o_p(a_N)$ as $N \to \infty$ iff $a_N^{-1} Y_N \to 0$ in probability as $N \to \infty$. A sufficient condition for this is $E|Y_N|^2 = o(a_N^2)$ as $N \to \infty$. Throughout this section and the next B will denote a positive real number which is independent of N and λ and may change from one usage to the next.

THEOREM 3.1. *Assume* (A_1), (A_4), *and* (A_5), *and let* $U_N(\cdot)$ *be defined by* (3.6).

Then as $N \to \infty$

$$\max_{0 \leqq \lambda \leqq \pi} |(N/m_N)^{\frac{1}{2}}(2\pi g_N(\lambda) - 1) - U_N(\lambda)| = o_p([\log m_N]^{-1}).$$

PROOF. Since $m = m_N = o(N^\alpha)$ as $N \to \infty$ iff cm does for every constant c, we may assume that $w(u) = 0$ for $|u| \geqq 1$. Thus from (1.8) and (1.9) we find that

$$(N/m)^{\frac{1}{2}}(2\pi g_N(\lambda) - 1) = Z_N(\lambda) + r_N(\lambda) + r_N \tag{3.1}$$

where for $0 \leqq \lambda \leqq \pi$, $1 \leqq t \leqq N$, and $N = 1, 2, \cdots$

$$Z_N(\lambda) = N^{-\frac{1}{2}} \textstyle\sum_{t=1}^N Z_{N,t}(\lambda), \tag{3.2a}$$

$$Z_{N,t}(\lambda) = 2m^{-\frac{1}{2}} \textstyle\sum_{v=1}^{m-1} \xi_t \xi_{t+v} w(vm^{-1}) \cos(v\lambda),$$

$$r_N(\lambda) = 2(Nm)^{-\frac{1}{2}} \textstyle\sum_{t=N-m+2}^{m} \sum_{v=N-t+1}^{m-1} \xi_t \xi_{t+v} w(vm^{-1}) \cos(v\lambda), \tag{3.2b}$$

$$r_N = (Nm)^{-\frac{1}{2}} \textstyle\sum_{t=1}^N (\xi_t^2 - 1).$$

Since for $N = 1, 2, \cdots$

$$E\{\max_\lambda |r_N(\lambda)|\} \leqq 2(Nm)^{-\frac{1}{2}} \textstyle\sum_{v=1}^m E\,|\sum_{t=N-v}^N \xi_t \xi_{t+v}|,$$

$$E\,|\textstyle\sum_{t=N-v}^N \xi_t \xi_{t+v}|^2 \leqq v \leqq m, \qquad v = 1, \cdots, m, \quad \text{and} \tag{3.3}$$

$$E\,|r_N|^2 \leqq m^{-1} E\,|\xi_1^2 - 1|^2,$$

we have $\max_\lambda |r_N(\lambda) + r_N| = o_p([\log m]^{-1})$ as $N \to \infty$, and it suffices to consider the stochastic processes $Z_N(\lambda)$, $0 \leqq \lambda \leqq 1$, $N = 1, 2, \cdots$ defined by (3.2a) and (3.2b).

LEMMA 3.1. *Assume* (A_1) *and* (A_5); *then the random variables* $Z_{N,1}(\lambda), \cdots, Z_{N,N}(\lambda)$ *have zero means and covariances*

$$\operatorname{Cov}(Z_{N,1}(\lambda_1), Z_{N,1}(\lambda_2)) = (4/m) \textstyle\sum_{v=1}^{m-1} w(vm^{-1})^2 (\cos v\lambda_1)(\cos v\lambda_2)$$

for $0 \leqq \lambda_i \leqq \pi$, $i = 1, 2$, *and* $N = 1, 2, \cdots$. *If* $t_1 < t_2 < t_3 < t_4$ *and* $0 \leqq \lambda_i \leqq \pi$, $i = 1, \cdots, 4$, *then*

$$E(Z_{N,t_1}(\lambda_1) Z_{N,t_2}(\lambda_2)) = 0 = E(\textstyle\prod_{i=1}^4 Z_{N,t_i}(\lambda_i)).$$

Moreover, there exists a constant B for which

$$\begin{aligned} |E\{\textstyle\prod_{i=1}^4 Z_{N,t_i}(\lambda_1)\}| &\leqq B \qquad \text{if } t_1 = t_2 \text{ and } t_3 = t_4 \\ &\leqq Bm^{-1} \quad \text{if } t_1 = t_2 \neq t_3 \neq t_4 \end{aligned}$$

for $0 \leqq \lambda_i \leqq \pi$, $i = 1, \cdots, 4$ *and* $N = 1, 2, \cdots$.

PROOF. The first assertion is obvious. The second follows from the fact that if $t_1 < t_i$ for $i \neq 1$, then $E\{\prod_i \xi_{t_i} \xi_{t_i+v_i}\} = E\{\xi_{t_1}\} \cdot E\{\xi_{t_1+v_1} \prod_{i \neq 1} \xi_{t_i} \xi_{t_i+v_i}\}$ in each of the multiple sums which compose its left and right-hand sides. The final assertion involves a rather tedious consideration of cases the details of which will be omitted.

LEMMA 3.2. *Assume* (A_5); *if* $h(N) = m\lambda_N \to \infty$ *and* $0 \leqq \lambda_N < \pi$ *as* $N \to \infty$, *then*

(i) $(2/m) \sum_{v=0}^{m-1} w(vm^{-1})^2 \cos v\lambda_N = O(h(N)^{-1})$

as $N \to \infty$; *and if* $\liminf_{N\to\infty} h(N) \geqq 1$, *then*

(ii) $\limsup_{N\to\infty} (2/m)|\sum_{v=1}^{m-1} w(vm^{-1})^2 \cos v\lambda_N| < \|W\|_2^2$.

PROOF. Routine Fourier analysis yields

$$m^{-1} \sum_{v=0}^{m-1} w(vm^{-1})^2 \cos v\lambda_N = \int_{-\pi}^{\pi} [\sin (m - \tfrac{1}{2})(y + \lambda_N) W_N(y)/\sin (\tfrac{1}{2})(y + \lambda_N)]dy$$

where $W_N(y) = \sum_{-\infty}^{\infty} W * W(my + 2km\pi)$, $*$ denotes convolution in $L_1(-\infty, \infty)$, and the sum converges in $L_1(-\pi, \pi)$. Since for $\lambda_N \leqq \pi$,

$$\int_{2|y| \leqq \lambda_N} |\sin (m - \tfrac{1}{2})(y + \lambda_N)/\sin (\tfrac{1}{2})(y + \lambda_N)| W_N(y)\, dy \leqq |\sin \tfrac{1}{4}\lambda_N|^{-1} \int_{-\pi}^{\pi} W_N(y)\, dy \leqq 2\pi/h(N),$$

and

$$\int_{2|y| \geqq \lambda_N} |\sin (m - \tfrac{1}{2})(y + \lambda_N)/\sin \tfrac{1}{2}(y + \lambda_N)| W_N(y)\, dy \leqq 2m \int_{2|y| \geqq m^{-1}h(N)} W_N(y)\, dy = o(h(N)^{-1}),$$

the first assertion of the lemma follows. To establish the second, consider a subsequence $\{N_j\}$ for which the left side of (ii) is approached and $h(N_j) \to h$, $1 \leqq h \leqq \infty$ as $j \to \infty$. If $h = \infty$, then (ii) follows from (i). If $h < \infty$, then as $j \to \infty$

$$(2/m)|\sum_{v=1}^{m-1} w(vm^{-1})^2 \cos v\lambda_{N_j}| \to |2 \int_0^1 w(u)^2 \cos (hu)\, du| < 2 \int_0^1 w(u)^2\, du = \|W\|_2^2.$$

COROLLARY 3.1. *Let* $\sigma_N^2(\lambda) = \mathrm{Var}\,(Z_{N,1}(\lambda))$, $0 \leqq \lambda \leqq \pi$, $N = 1, 2, \cdots$; *then* $\sigma_N^2(\lambda)$ *is uniformly bounded and*

$$\sigma_N^2(\lambda) \to \|W\|_2^2 \quad as \quad N \to \infty$$

uniformly on $[m^{-1} \log m, \pi]$.

COROLLARY 3.2. *Let* $r_N(\lambda_1, \lambda_2)$ *be the correlation coefficient of* $Z_{N,1}(\lambda_1)$ *and* $Z_{N,1}(\lambda_2)$, $0 \leqq \lambda_i \leqq \pi$, $i = 1, 2$; *then*

$$\sup_{m|\lambda_1 - \lambda_2| \geqq (\log m)^2} |r_N(\lambda_1, \lambda_2)| = O([\log m]^{-2}),$$

$$\limsup_{N\to\infty} \sup_{|\lambda_1-\lambda_2| \geqq m^{-1}} |r_N(\lambda_1, \lambda_2)| < 1.$$

The corollaries follow directly from the preceding lemma. For example, if λ_N is chosen to maximize $|\sigma_N^2(\lambda) - \|W\|_2^2|$ for $m^{-1} \log m \leqq \lambda \leqq \pi$, then

$$\sigma_N^2(\lambda_N) = (2/m) \sum_{v=1}^{m-1} w(vm^{-1})^2 + (2/m) \sum_{v=1}^{m-1} w(vm^{-1})^2 \cos (2v\lambda_N).$$

When $N \to \infty$ the first sum clearly tends to $\|W\|_2^2$; the second is $O([\log m]^{-1})$ by Lemma 3.2.

The random variables $Z_{N,1}(\lambda), \cdots, Z_{N,N}(\lambda)$, $0 \leqq \lambda \leqq \pi$, $N = 1, 2, \cdots$ have the desirable property of m-dependence, which we will now exploit. Let $k = k_N = [m(\log m)^4]$ where $[\cdot]$ denotes the greatest integer function. We may then write $N = nk + r$ where $0 \leqq r < k$. For $i = 1, \cdots, n$, $0 \leqq \lambda \leqq \pi$, and $N \geqq N_0$, let

$$U_{N,i}(\lambda) = k^{-\frac{1}{2}}(Z_{N,(i-1)k+1}(\lambda) + \cdots + Z_{N,ik-m}(\lambda)), \tag{3.4a}$$

$$V_{N,i}(\lambda) = m^{-\frac{1}{2}}(Z_{N,ik-m+1}(\lambda) + \cdots + Z_{N,ik}(\lambda)), \tag{3.4b}$$

$$V_{N,0}(\lambda) = Z_{N,nk+1}(\lambda) + \cdots + Z_{N,N}(\lambda).$$

Then clearly

$$Z_N(\lambda) = (nk/N)^{\frac{1}{2}}(U_N(\lambda) + (m/k)^{\frac{1}{2}}V_N(\lambda)) + N^{-\frac{1}{2}}V_{N,0}(\lambda) \tag{3.5}$$

where

$$U_N(\lambda) = n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n U_{N,i}(\lambda), \qquad 0 \leqq \lambda \leqq \pi, \qquad N \geqq N_0, \tag{3.6a}$$

$$V_N(\lambda) = n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n V_{N,i}(\lambda), \qquad 0 \leqq \lambda \leqq \pi, \qquad N \geqq N_0. \tag{3.6b}$$

Moreover, for N sufficiently large $U_N(\lambda)$ and $V_N(\lambda)$ are sums of independent, identically distributed random variables. Finally, we note that by Lemma 3.1

$$E\,|V_{N,i}(\lambda)|^4 \leqq B, \qquad E\,|U_{N,i}(\lambda)|^4 \leqq Bkm^{-1}. \tag{3.7}$$

This fact will be used repeatedly below. Let

$$\begin{aligned} U_{N,i}(\lambda)' &= U_{N,i}(\lambda): && \text{if} \quad |U_{N,i}(\lambda)| \leqq N^{0.3} \\ &= 0: && \text{if} \quad |U_{N,i}(\lambda)| > N^{0.3}; \\ V_{N,i}(\lambda)' &= V_{N,i}(\lambda): && \text{if} \quad |V_{N,i}(\lambda)| \leqq N^{0.3} \\ &= 0: && \text{if} \quad |V_{N,i}(\lambda)| > N^{0.3}; \end{aligned}$$

$$U_{N,i}(\lambda)'' = [U_{N,i}(\lambda)' - E(U_{N,i}(\lambda)')]/\mathrm{Var}\,(U_{N,i}(\lambda)');$$

$$V_{N,i}(\lambda)'' = [V_{N,i}(\lambda)' - E(V_{N,i}(\lambda)')]/\mathrm{Var}\,(V_{N,i}(\lambda)');$$

and let $U_N(\lambda)'$, $V_N(\lambda)'$, $U_N(\lambda)''$, $V_N(\lambda)''$ be $n^{-\frac{1}{2}}$ times their respective sums. (For example, $U_N(\lambda)'$ is defined exactly as was $U_N(\lambda)$ with $U_{N,i}(\lambda)'$ replacing $U_{N,i}(\lambda)$ for $i = 1, \cdots, n$.) Then in view of Lemma 2.1 and our choice of k, Theorem 3.1 would follow from

$$\max\nolimits_{0 \leqq \lambda \leqq 1} |V_{N,0}(\lambda)| = o_p(N^{1/2}(\log m)^{-1}), \tag{3.8a}$$

$$P(V_N(\lambda_{N,j})' \neq V_N(\lambda_{N,j}), \quad \text{for some} \quad j) \to 0, \tag{3.8b}$$

$$\max\nolimits_j |V_N(\lambda_{N,j})' - V_N(\lambda_{N,j})''| \leqq O(1) \max\nolimits_j |V_N(\lambda_{N,j})''| + o(1), \tag{3.8c}$$

$$\max\nolimits_j |V_N(\lambda_{N,j})''| = o_p(\log m), \tag{3.8d}$$

as $N \to \infty$ where $\lambda_{N,j} = \pi \cdot j/[m \log m]$, $j = 0, \cdots, [m \log m]$. (3.8a) follows easily from

$$E\{\max_\lambda |V_{N,0}(\lambda)|\} \leqq 2(m)^{-\frac{1}{2}} \sum_{v=1}^{m-1} E \mid \sum_{t=kn+1}^{N} \xi_t\xi_{t+v}|,$$
$$E |\sum_{t=kn+1}^{N} \xi_t\xi_{t+v}|^2 \leqq N - nk = r < k.$$

(3.8b) follows from (3.7) since by Markov's inequality

$$P(V_N'(\lambda_{N,j}) \neq V_N(\lambda_{N,j}) \quad \text{for some} \quad j) \leqq \sum_i\sum_j N^{-6/5}E\,|V_{N,i}(\lambda_{N,j})|^4 \leqq BN^{-1/5};$$

and (3.8c) follows similarly from (3.7). Finally, since for $\epsilon > 0$

$$P(\max_j |V_N(\lambda_{N,j})''| \geqq \epsilon \log m) \leqq \sum_j P(|V_N(\lambda_{N,j})''| \geqq 2(2\log m)^{\frac{1}{2}})$$

if N is sufficiently large, (3.8d) is an easy consequence of Lemma 3.3 part (i) (below). In Lemma 3.3 we have used $\Phi(\cdot)$ to denote the standardized, univariate normal distribution function and $\varphi_r(\cdot, \cdot)$ to denote the standardized, bivariate normal density with parameter r.

LEMMA 3.3. *Assume* (A_1), (A_4), *and* (A_5). *If* $0 < z_N \to \infty$ *and* $z_N = o(\log m)$ *as* $N \to \infty$, *then as* $N \to \infty$

(i) $P(|V_N(\lambda)''| \geqq z_N) \sim 2(1 - \Phi(z_N))$ *uniformly on* $[0, \pi]$, *and*

(ii) $$P(\pm U_N(\lambda_1)'' \geqq z_N, \pm U_N(\lambda_2)'' \geqq z_N)$$
$$\sim \int_{z_N}^{\infty} \int_{z_N}^{\infty} \varphi_{r_N(\lambda_1,\lambda_2)}(\pm y_1, \pm y_2)\, dy_1\, dy_2$$

uniformly on $S_N = \{(\lambda_1, \lambda_2): 0 \leqq \lambda_i \leqq \pi, i = 1, 2$ *and* $|\lambda_1 - \lambda_2| \geqq m^{-1}\}$. *Moreover, for* $p = 1, 2, \cdots$

(iii) $$P(\pm U_N(\lambda_i)'' \geqq z_N, i = 1, \cdots, p) \sim [(1 - \Phi(z_N))]^p$$

uniformly on $S_{N,p} = \{(\lambda_1, \cdots, \lambda_p): 0 \leqq \lambda_i \leqq \pi, i = 1, \cdots, p$ *and* $\min_{i\neq j} |\lambda_i - \lambda_j| \geqq m^{-1}(\log m)^2\}$.

(iv) $$P(\pm U_N(\lambda_i)'' \geqq z_N, i = 1, \cdots, p)$$
$$\sim (1 - \Phi(z_N))^{p-2} \int_{z_N}^{\infty} \int_{z_N}^{\infty} \varphi_{r_N(\lambda_1,\lambda_2)}(\pm y_1, \pm y_2)\, dy_1\, dy_2$$

uniformly on $S_{N,p}' = \{(\lambda_1, \cdots, \lambda_p): \lambda_2 - \lambda_1 \geqq m^{-1}, \lambda_i - \lambda_{i-1} \geqq m^{-1}[\log m]^2$, $i = 3, \cdots, p\}$.

PROOF. We will prove (ii) in the case that both the signs are +; the other cases are proved similarly. Since $U_N(\cdot)$ is continuous wp one, we may choose $\boldsymbol{\lambda}^N = (\lambda_1^N, \lambda_2^N)$ to maximize

$$P_N = |1 - P(U_N(\lambda_i)'' \geqq z_N, i = 1, 2)[\int_{z_N}^{\infty} \int_{z_N}^{\infty} \varphi_{r_N(\lambda_1,\lambda_2)}(y_1, y_2)\, dy_1\, dy_2]^{-1}|$$

for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in S_N$. Let $r_N = r_N(\lambda_1^N, \lambda_2^N)$; then we may select a subsequence $\{N_j\}$ for which R_{N_j} approaches its limit superior and

$$\rho_{N_j} = \mathrm{Cov}\,(U_{N_j}(\lambda_1^{N_j})'', U_{N_j}(\lambda_2^{N_j})'') \to \rho$$

as $j \to \infty$. Moreover, $\rho < 1$ by Corollary 3.2 since by (3.7)

(3.9) $$|\mathrm{Cov}\,(U_{N,1}(\lambda_1), U_{N,1}(\lambda_2)) - \mathrm{Cov}\,(U_{N,1}(\lambda_1)', U_{N,1}(\lambda_2)')| \leqq N^{-3/5}Bkm^{-1}$$

for $0 \leqq \lambda_1 \leqq \pi$, $i = 1, 2$.

From Theorem 5.1 of [11] on the large deviations of sums of independent, indentically distributed random vectors, we may infer that as $j \to \infty$

$$P(U_{N_j}(\lambda_i^{N_j})'' \geqq z_{N_j}, i = 1, 2) \sim \int_{z_{Nj}}^{\infty} \int_{z_{Nj}}^{\infty} \varphi_{\rho_{Nj}}(y_1, y_2)\, dy_1\, dy_2 .$$

Moreover, it follows from (5.5) of [11] and (our) (3.9) that

$$\int_{z_{Nj}}^{\infty} \int_{z_{Nj}}^{\infty} \varphi_{\rho_{Nj}}(y_1, y_2) \sim \int_{z_{Nj}}^{\infty} \int_{z_{Nj}}^{\infty} \varphi_{\tau_{Nj}}(y_1, y_2)\, dy_1\, dy_2 ,$$

as $j \to \infty$. Thus (ii) is established.

COROLLARY 3.3. $P(|U_N(\lambda)''| \geqq z_N) \sim 2(1 - \Phi(z_N))$ *as* $N \to \infty$ *uniformly on* $[0, \pi]$.

COROLLARY 3.4. *There is a* $\delta > 0$ *for which*

$$\max_{|\lambda_1 - \lambda_2| \geqq m^{-1}} P(|U_N(\lambda_i)''| \geqq z_N, i = 1, 2)(1 - \Phi(z_N))^{-1} \leqq Be^{-\delta z}N^2$$

for N *sufficiently large.*

PROOF. By Corollary 3.2 there is an $r < 1$ for which $|r_N(\lambda_1, \lambda_2)| \leqq r$ for all $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \varepsilon S_N$ and N sufficiently large. Since by Lemma 2 of [1]

$$\begin{aligned} &\int_{z_N}^{\infty} \int_{z_N}^{\infty} \varphi_{|r_N(\lambda_1, \lambda_2)|}(y_1, y_2)\, dy_1\, dy_2 \\ (3.10) \qquad &\sim (2\pi z_N^2)^{-1}\{(1 + |r_N(\lambda_1, \lambda_2)|)^3/[1 - |r_N(\lambda_1, \lambda_2)|]\}^{\frac{1}{2}} \\ &\qquad\qquad \cdot \exp\{-z_N^2/(1 + |r_N(\lambda_1, \lambda_2)|)\} \\ &\leqq B_r z_N^{-2} \exp(-\tfrac{1}{2}z_N^2 - \tfrac{1}{4}(1 - r)z_N^2), \end{aligned}$$

the corollary follows. The asymptotic equality in (3.10) may also be deduced from equation (5.5) of [11].

THEOREM 3.2. *Under the hypotheses of Theorem* 3.1

$$\max_{0 \leqq \lambda \leqq \pi} (N/2m \log m)^{\frac{1}{2}} |2\pi g_N(\lambda) - 1|/\|W\|_2 \to 1 \quad \textit{in probabiilty as} \quad N \to \infty.$$

PROOF. By Theorem 3.1, Lemma 3.2, and (3.9) we may write

$$\begin{aligned} (3.11) \qquad (N/m)^{\frac{1}{2}}(2\pi g_N(\lambda) - 1) &= U_N(\lambda) + r_N(\lambda)' \\ &= U_N(\lambda)''\sigma_N(\lambda)' + r_N(\lambda)'' + r_N(\lambda)' \end{aligned}$$

where $\max_i \{|r_N(\lambda_{N,i})'| + |r_N(\lambda_{N,i})''|\} = o_p([\log m]^{-1})$ and $\sigma_N(\lambda)'' \to \|W\|_2$ as $N \to \infty$ uniformly on $[m^{-1} \log m, \pi]$. Thus, by Lemma 2.1, it will suffice to show that for arbitrary $\epsilon > 0$,

$$(3.12a) \quad \lim P(\max_j |U_N(\lambda_{N,j})''| \sigma_N(\lambda_{N,j})' \geqq (1 + \epsilon)\|W\|_2(2 \log m)^{\frac{1}{2}}) = 0,$$

$$(3.12b) \quad \lim P(\max_j U_N(\lambda_{N,i})''\sigma_N(\lambda_{N,j})' \leqq (1 - \epsilon)\|W\|_2(2 \log m)^{\frac{1}{2}}) = 0.$$

To establish (3.12a) let S be the set of integers j for which $1 \leqq j \leqq p = [m \log m]$ and $\lambda_{N,j} \geqq m^{-1} \log m$. Then if $\epsilon = 2\epsilon' > 0$ is given, we find from Corollaries 3.1 and 3.3 that for N sufficiently large

$$\begin{aligned}
P(\max_{j\varepsilon S} |U_N(\lambda_{N,j})''| \sigma_N(\lambda_{N,j})' &\geqq (1+\epsilon)\|W\|_2(2\log m)^{\frac{1}{2}}) \\
&\leqq \sum_{j\varepsilon S} P(|U_N(\lambda_{N,j})''| \geqq (1+\epsilon')(2\log m)^{\frac{1}{2}}) \\
&\leqq 4m \log m(1-\Phi((1+\epsilon')(2\log m)^{\frac{1}{2}})) = o(1)
\end{aligned}$$

as $N \to \infty$, and

$$\begin{aligned}
P(\max_{j\notin S} |U_N(\lambda_{N,j})''| \sigma_N(\lambda_{N,j})' &\geqq (1+\epsilon)\|W\|_2(2\log m)^{\frac{1}{2}}) \\
&\leqq \sum_{j\notin S} P(|U_N(\lambda_{N,j})''| \geqq c(2\log m)^{\frac{1}{2}}) \qquad (3.13)\\
&\leqq 2(\log m)^2(1-\Phi(c(2\log m)^{\frac{1}{2}})) = o(1)
\end{aligned}$$

as $N \to \infty$ where $c^2 > 0$ is a lower bound for $\|W\|_2^2/\sigma_N^2(\lambda)'$. This establishes (3.12a). (3.12b) may be established by essentially the same arguments that are used in [2] to establish an analogous assertion. Full details are given in [10].

The final theorem of this section gives the limiting distribution of a restricted maximum. Indeed, let

$$M_N = a_N^{-1}(\max_{0\leqq j\leqq m} (N/m)^{\frac{1}{2}}|(2\pi g_N(\lambda_{N,j}^*) - 1)/\|W\|_2| - b_N)$$

for $N \geqq N_0$, where a_N, b_N, and $\lambda_{N,i}^*$ are given by (2.1) and (2.2). Then we have

THEOREM 3.3. *Under the hypotheses of Theorem* 3.1

$$\lim_{N\to\infty} P(M_N < x) = \exp(-\exp(-x))$$

for $-\infty < x < \infty$.

PROOF. By Theorem 3.1, Corollary 3.1, and (3.11), and (3.13), it will suffice to prove the theorem with

$$M_N^* = a_N^{-1}(\max_{\log m\leqq j\leqq m} |U_N(\lambda_{N,j}^*)''| - b_N)$$

replacing M_N. For every integer $l \geqq 1$, we have by Bonferrai's inequalities

$$\sum_{p=1}^{2l} (-1)^{p+1} T_{N,p}(x) \leqq P(M_N^*(x) \geqq x) \leqq \sum_{p=1}^{2l+1} (-1)^{p+1} T_{N,p}(x) \qquad (3.14)$$

where

$$T_{N,p}(x) = \sum_{(N,p)} P(|U_N(\lambda_{N,i_j}^*)''| \geqq a_N x + b_N,\ j = 1, \cdots, p)$$

and $\sum_{(N,p)}$ denotes summation over all subsets of size p drawn from $\{\lambda_{N,[\log m]}^*, \cdots, \lambda_{N,m}^*\}$. (3.14) has been used in [8] in a similar connection. Moreover, in view of Lemma 3.3 (parts (iii) and (iv)) and Corollary 3.3, it follows essentially as in [8] that

$$T_{N,p} \to \exp(-px)/p! \quad \text{as} \quad N \to \infty$$

for each fixed p, so that the theorem follows from the arbitrariness of l.

4. Reduction to white noise. The proof of Theorems 2.1 and 2.2 will be completed by showing that under the appropriate hypotheses as $N \to \infty$

$$\max_{|\lambda| \leqq \pi} |[f(\lambda) - E(f_N(\lambda))]/f(\lambda)| = o([m/N \log N]^{\frac{1}{2}}), \tag{4.1}$$

$$\max_{|\lambda| \leqq \pi} |[f_N(\lambda) - E(f_N(\lambda))]/f(\lambda) - (2\pi g_N(\lambda) - 1)| = o_p([m/N \log N]^{\frac{1}{2}}). \tag{4.2}$$

LEMMA 4.1. *Let* $W(\cdot)$ *satisfy* (A_5); *if either* (i) (A_3) *and* $N \log N = o(m_N^3)$ *as* $N \to \infty$, *or* (ii) (A_3') *and* $N \log N = o(m^5)$, *as* $N \to \infty$, *then* (4.1) *holds.*

PROOF. The left side (4.1) is dominated by

$$B \max_{|\lambda| \leqq \pi} |f(\lambda) - m \int_{-\infty}^{\infty} W(m(\lambda - u))f(u)\, du| + B \max_{|\lambda| \leqq \pi} |m \int_{-\infty}^{\infty} W(m(\lambda - u))(f(u) - E(I_N(u)))\, du| = R_1 + R_2.$$

If (A_3) is satisfied, then clearly

$$R_1 \leqq \max B_{|\lambda| \leqq \pi} \int_{-\infty}^{\infty} |f(\lambda) - f(\lambda - um^{-1})| W(u)\, du \leqq Bm^{-1} \int_{-\infty}^{\infty} |u| W(u)\, du;$$

and if (A_3') is satisfied we may expand $(f(\lambda) - f(\lambda - um^{-1}))$ in a Taylor Series to obtain $R_2 \leqq Bm^{-2} \int_{-\infty}^{\infty} u^2 W(u)\, du$ from the symmetry of $W(\cdot)$. Since it is well-known ([6]) that

$$\max_{|\lambda| \leqq \pi} |f(\lambda) - E(I_N(\lambda))| \leqq B \log N/N$$

if $f(\cdot)$ satisfies a uniform Lipschitz condition, the lemma follows.

THEOREM 4.1. *Assume* (A_1)–(A_5). *If either* (i) $N \log N = o(m_N^4)$ *or* (ii) (A_3') *and* $N \log N = o(m_N^8)$ *then* (4.2) *holds.*

PROOF. Since the left side of (4.2) is dominated by

$$\begin{aligned} R_1 + R_2 = {} & B \max_{|\lambda| \leqq \pi} |m \int W(m[\lambda - u]) \\ & \cdot \{(I_N(u) - E(I_N(u)) - f(u)(J_N(u) - E(J_N(u)))\}\, du| \\ & + B \max_\lambda |m \int W(m[\lambda - u])[f(\lambda) - f(u)][J_N(u) - E(J_N(u))]\, du| \end{aligned}$$

where $J_N(\cdot)$ is the periodogram of the $\{\xi_t\}$ process, it will suffice to show that $R_i = o_p((m/N \log N)^{\frac{1}{2}})$, $i = 1, 2$. Consider first $R_1 = \max_{|\lambda| \leqq \pi} R_1(\lambda)$. Using (1.1), (1.2), and (1.9), we may (after some manipulation) write

$$R_1(\lambda) = (2\pi N)^{-1} \sum \sum_{r,s=-\infty}^{\infty} a_r a_s\, d_{rs}(\lambda)$$

where

$$d_{rs}(\lambda) = \sum \sum_{v_1, v_2 = 1}^{N} - \sum_{v_1 = r+1}^{r+N} \sum_{v_2 = s+1}^{s+N} W((v_1 - v_2)/m) e^{-i(v_1 - v_2)\lambda} \cdot (\xi_{v_1 - r} \xi_{v_2 - s} - R_\xi(v_1 - v_2 - r + s)).$$

Let $C_{r,s,N}$ denote the set of lattice points in the two sums not common to both sums, then

$$d_{rs}(\lambda) = \sum \sum_{C_{r,s,N}} w((v_1 - v_2)/m) e^{-i(v_1 - v_2)\lambda} (\xi_{v_1 - r} \xi_{v_2 - s} - R_\xi(v_1 - v_2 - r + s)).$$

Let $v = v_1 - v_2$ and $u = v_2$ then

$$d_{rs}(\lambda) = \sum_{v = \min(-N+1, -N+r+1-s)}^{\max(N-1, N+r-s-1)} e^{-iv\lambda} w(vm^{-1}) \cdot \sum_{D_{r,s,N}} (\xi_{v+u-r} \xi_{u-s} - R_\xi(v - r + s))$$

where $D_{r,s,v}$ is the set of integers in the projection onto the v_1 axis of that part of the line $v_1 - v_2 = v$ which intersects $C_{r,s,N}$.

Now

$$E \max_\lambda d_{rs}(\lambda) \leqq 2\sum_{\min}^{\max} |w(vm^{-1})| E| \sum_{D_{r,s,v}} \xi_{v+u-r}\xi_{u-s} - R_\xi(v + r - s)|,$$

and due to the independence of the ξ_j,

$$E|\sum_{D_{r,s,v}} \xi_{v+u-r}\xi_{u-s} - R_\xi(v + r - s)|^2 \leqq 2N \qquad |r| \text{ or } |s| > N$$
$$\leqq |r| + |s| \quad \text{otherwise}$$

if $v \neq r - s$. If $v = r - s$ then a constant term appears involving $E\xi_j^4$. Therefore

$$\begin{aligned} E(\log N/mN)^{\frac{1}{2}} \max_\lambda \sum\sum_{r,s=-\infty}^{\infty} a_r a_s\, d_{rs}(\lambda) & \\ \leqq B((m \log N)/N)^{\frac{1}{2}}\{\sum_{s=-\infty}^{\infty} \sum_{|r| \leqq N} |a_r a_s|(|r| + |s|)^{\frac{1}{2}} & \\ + \sum_{s=-\infty}^{\infty} \sum_{|r|>N} |a_r a_s| N^{\frac{1}{2}}\} & \\ \leqq B(m \log N/N)^{\frac{1}{2}}\{\sum_{-N}^{N} r^{\frac{1}{2}} a_r + N^{\frac{1}{2}} \sum_{|r|>N} a_r\} & \\ \leqq B\,(m \log N)^{\frac{1}{2}} N^{-\beta} = o(1). & \end{aligned}$$

Thus $R_1 = O_p([m/N \log N]^{\frac{1}{2}})$ as $N \to \infty$. An argument similar to the above can be found in [3], p. 191.

Now consider $R_2 = \max_{|\lambda| \leqq \pi} R_2(\lambda)$. By the Schwartz inequality and (A_3) we have

$$\begin{aligned} R_2(\lambda)^2 &\leqq B \int_{-\infty}^{\infty} u^2 m^{-2} |J_N(\lambda - um^{-1}) - E(J_N(\lambda - um^{-1}))|^2 W(u)\, du \\ &= Bm^{-2} \sum_{|l| \leqq m} w''(lm^{-1}) e^{il\lambda} \sum_{v_1 - v_2 = l, |v_1| < N, i=1,2} T_N(v_1)T_N(v_2) \\ &\qquad - E(T_N(v_1)T_N(v_2)) \end{aligned}$$

where $T_N(\cdot)$ is the covariance estimate for the $\{\xi_t\}$ process. Since by Lemma 1 in [3], p. 186

$$E\{\sum_{v_1 - v_2 = l, |v_1| < N, i=1,2} T_N(v_1)T_N(v_2) - E(T_N(v_1)T_N(v_2))\}^2 \leqq B/N \tag{4.3}$$

we find $E|R_2|^2 \leqq B/mN^{\frac{1}{2}}$ which is $O((m/N \log N))$ under condition (i). Thus (4.2) holds under condition (i). Now let (ii) be satisfied. Then

$$\begin{aligned} R_2(\lambda) &\leqq B|f'(\lambda)m^{-1} \int_{-\infty}^{\infty} u[J_N(\lambda - um^{-1}) - E(J_N(\lambda - um^{-1}))]W(u)\, du| \\ &\quad + Bm^{-2} \int_{-\infty}^{\infty} |J_N(\lambda - um^{-1}) - E(J_N(\lambda - um^{-1}))| u^2 W(u)\, du \\ &= R_2'(\lambda) + R_2''(\lambda). \end{aligned}$$

Now

$$R_2'(\lambda) \leqq Bm^{-1}|\sum_{0<|v|<m} w'(vm^{-1})(T_N(v) - E(T_N(v)))e^{iv\lambda}|$$

so that $E(\max_{|\lambda|<\pi} |R_2'(\lambda)|) \leqq BN^{-\frac{1}{2}}$. And by the Schwartz inequality

$$\begin{aligned} R_2''(\lambda)^2 &\leqq Bm^{-4} \int_{-\infty}^{\infty} |J_N(\lambda - um^{-1}) - E(J_N(\lambda - um^{-1}))|^2 u^2 W(u)\, du \\ &= Bm^{-4} \sum_{|l| \leqq m} w''(lm^{-1}) e^{iv\lambda} \\ &\quad \cdot \{\sum_{v_1 - v_2 = l, |v_i| \leqq N, i=1,2} T_N(v_1)T_N(v_2) - E(T_N(v_1)T_N(v_2))\} \end{aligned}$$

so that by (4.3) $E(\max_{|\lambda| \leqq \pi} |R_2''(\lambda)|^2) \leqq Bm^{-3}N^{-\frac{1}{2}}$ which is $O(m/N \log N)$ under (ii). Thus Theorem 4.1 (and therefore Theorems 2.1 and 2.2) are established.

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