The Annals of Mathematical Statistics
1968, Vol. 39, No. 2, 433–441

CHARACTERIZATIONS OF INDEPENDENCE IN CERTAIN FAMILIES OF BIVARIATE AND MULTIVARIATE DISTRIBUTIONS

BY KUMAR JOGDEO[1]

Courant Institute of Mathematical Sciences, New York University

Introduction and summary. Among various tests of independence the ones based on the sample correlation coefficient and on the 2×2 contingency tables seem to be foremost in applications. Although the first of these tests the absence of linear relation, the motivation stems from the fact that in the case where the bivariate distribution is a member of the normal family uncorrelatedness is equivalent to independence. A natural question arises, whether there exists a wider family of bivariate distributions where independence is characterized by uncorrelatedness. The answer to this question is given in a recent paper of Lehmann (1966). In the case of multivariate distributions the similar question is more involved since pairwise independence, in general, is not enough for mutual independence. In the present paper a simple generalization of the notion of uncorrelatedness is shown to characterize independence in a family of multivariate distributions which is analogous to the bivariate family considered by Lehmann (1966).

When the data is available in the form of a 2×2 contingency table one might consider it as a simplified version of the data available on a pair of real random variables (X_1, X_2) or that the information available to the experimenter is only in the form of occurrence or nonoccurrence of the events $X_1 \leqq a$ and $X_2 \leqq b$ where the pair (a, b) is fixed. In both situations, one tests independence of the two *events* $[X_1 \leqq a]$ and $[X_2 \leqq b]$ although it may be desirable to test the independence of X_1 and X_2. Again, one might ask the question whether there exists a suitable family of bivariate distributions where the independence of the events of the above type characterizes the independence of the paired random variables. In the present paper such a family is given and a multivariate analogue of the same is shown to possess similar characterization of independence.

In a recent paper Esary, Proschan and Walkup (1967) have introduced a notion of association which has several applications. Although disjoint from their study, the results of the present paper are in the same set-up and are supplementary.

In order to give a precise summary, let (X_1, X_2) be a pair of real valued random variables with finite second moments. The pair is said to be *positively quadrant dependent* if

$$(0.1) \quad P[X_1 \leqq x_1, X_2 \leqq x_2] \geqq P[X_1 \leqq x_1]P[X_2 \leqq x_2], \quad \text{for all} \quad x_1, x_2,$$

Received 20 February 1967; revised 21 July 1967.

[1] This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under a Ford Foundation Grant for Probability and Statistics.

and *negatively quadrant dependent* if the inequality between the two sides of (0.1) is reversed. The bivariate distribution functions which satisfy the above restrictions define families $\mathfrak{F}_1$ and $\mathcal{G}_1$ respectively. In a recent paper, Lehmann (1966) showed that in $\mathfrak{F}_1 \cup \mathcal{G}_1$, the independence is characterized by uncorrelatedness. In the same paper he also defined a subclass $\mathfrak{F}_2 \subset \mathfrak{F}_1$ which described the *regression dependence* between X_1 and X_2, and gave several applications to tests of the hypotheses of independence.

Recently, Jogdeo and Patil (1967) showed that if $\mathfrak{F}_2$ is parametrized suitably then the independence of X_1, X_2 is characterized by the independence of two events $[X_1 \leqq a]$ and $[X_2 \leqq b]$, for *some* a and b, with the sole condition that the probabilities of these events be bounded away from 0 or 1. In particular, it was shown that if the dependence between X_1 and X_2 is described by a linear model

$$X_1 = \alpha + \beta X_2 + \sigma Z, \tag{0.2}$$

where X_2 and Z are independent, then the above characterization applies.

In the present paper the parametrization is replaced by making the condition of regression dependence symmetric in both variables. In particular, the class $\mathfrak{F}_3 \cup \mathcal{G}_3$ discussed by Lehmann (1966) is a subclass of the one considered presently.

In Section 2, the results stated above are generalized to multivariate distributions. Since it is well known that the pairwise independence is not enough for mutual independence, the conditions which characterize mutual independence take various forms and interpretations. The basic characteristic of the class of n-variate distributions which yields simple characterizations may be described as follows. If A_i denotes the event $X_i \leqq x_i$ (or $\geqq x_i$) then $P(\cap A_i)$ is either $\geqq$ or $\leqq \prod P(A_i)$ *uniformly* in x_i. For example, in the family of trivariate distributions which satisfy

$$P[X_1 \leqq x_1, X_2 \leqq x_2, X_2 \leqq x_3] \leqq \textstyle\prod_{i=1}^{3} P[X_i \leqq x_i] \quad \text{for all} \quad x_1, x_2, x_3, \tag{0.3}$$

the independence is characterized by

$$\begin{aligned} EX_iX_j &= EX_iEX_j; \qquad i \neq j, \quad i = 1, 2, 3, \\ EX_1X_2X_3 &= EX_1EX_2EX_3. \end{aligned} \tag{0.4}$$

If the condition (0.3) is made stronger by requiring

$$h(x_k; x_i, x_j) = P[X_i \leqq x_i, X_j \leqq x_j \mid X_k = x_k], \quad i \neq j \neq k; \quad i = 1, 2, 3, \tag{0.5}$$

to be monotone in x_k for every x_i, x_j fixed then the independence of X_1, X_2 and X_3 is equivalent to that of the three events $[X_1 \leqq a]$, $[X_2 \leqq b]$ and $[X_3 \leqq c]$, for some a, b, c such that the probabilities of these events are bounded away from 0 and 1.

1. Bivariate families. Let (X_1, X_2) be a pair of real valued random variables having finite variances and let F, F_1 and F_2 denote the joint and the marginal distributions respectively. Following the notation of Lehmann (1966) let

$$\begin{aligned} \mathfrak{F}_1 &= \{F : F(x_1, x_2) \geqq F_1(x_1)F_2(x_2), \text{ for all } x_1, x_2\}. \\ \mathcal{G}_1 &= \{F : F(x_1, x_2) \leqq F_1(x_1)F_2(x_2), \text{ for all } x_1, x_2\}. \end{aligned} \tag{1.1}$$

These classes describe positive and negative quadrant dependence between X_1 and X_2. It is clear that $\mathcal{F}_1$ and $\mathcal{G}_1$ can also be defined by other inequalities which are equivalent to the one used above. It was pointed out by Lehmann (1966) that the above classes remain the same regardless of whether the distribution functions are defined to be continuous on the left or right.

By using a lemma of Hoeffding it was shown by Lehmann (1966) that if the distribution function F of (X_1, X_2) is a member of $\mathcal{F}_1 \cup \mathcal{G}_1$ (also written as $(X_1, X_2) \varepsilon \mathcal{F}_1 \cup \mathcal{G}_1$) then the uncorrelatedness of X_1, X_2 is equivalent to their independence. It was also shown that a subclass $\mathcal{F}_2 \subset \mathcal{F}_1$ (and $\mathcal{G}_2 \subset \mathcal{G}_1$) plays an important role in applications to testing hypotheses of independence. The class $\mathcal{F}_2$, describing the regression dependence is given by the following condition: for every x_2,

$$(1.2) \qquad h(u, x_2) = P[X_2 \leqq x_2 \mid X_1 = u] \text{ is nonincreasing in } u.$$

Note that the defining property of $\mathcal{F}_2$ is not symmetric in X_1, X_2 while that of $\mathcal{F}_1$ is. In the following, it is shown that if the condition (1.2) is also required when X_1, X_2 are interchanged, the characterization of independence is very simple. The above class can be said to describe mutual positive regression dependence. A similar definition holds for the negative dependence.

Definition 1.1. The bivariate distribution function of a pair (X_1, X_2) is said to be a member of the class $\mathcal{H}(2)$ if the following conditions are satisfied:

(i) for every x_2,

$$h(u, x_2) = P[X_2 \leqq x_2 \mid X_1 = u] \text{ is monotone in } u,$$

(ii) for every x_1,

$$g(v, x_1) = P[X_1 \leqq x_1 \mid X_2 = v] \text{ is monotone in } v.$$

Note that the functions h and g are not required to be monotone in the same direction for all x_2 and x_1. Thus the class $\mathcal{H}(2)$ is wider than the one describing mutual regression dependence. In fact $\mathcal{H}(2)$ has no inclusive relation with $\mathcal{F}_1 \cup \mathcal{G}_1$. The class $\mathcal{F}_3 \cup \mathcal{G}_3$ defined in Lehmann (1966) is a subclass of $\mathcal{H}(2)$.

Theorem 1. *If $(X_1, X_2) \varepsilon \mathcal{H}(2)$ then the existence of a pair of real numbers (a, b) satisfying*

$$(1.3) \qquad 0 < P[X_1 \leqq a] < 1, \qquad 0 < P[X_2 \leqq b] < 1$$

and

$$(1.4) \qquad P[X_1 \leqq a, X_2 \leqq b] = P[X_1 \leqq a]P[X \leqq b]$$

implies that X_1 and X_2 are independent. Consequently, if $(X_1, X_2) \varepsilon \mathcal{H}_2 \cap (\mathcal{F}_1 \cup \mathcal{G}_1)$ then the defining inequalities (1.1) are either equalities for all x_1, x_2 or strict inequalities for all x_1, x_2 (excepting of course those making both sides 0 or 1).

Proof. The independence of $[X_1 \leqq a]$ and $[X_2 \leqq b]$ gives

$$(1.5) \qquad P[X_2 \leqq b \mid X_1 \leqq a] = P[X_2 \leqq b \mid X_1 > a],$$

which is the same as

$$(1.6)\quad (1/P[X_1 \leqq a]) \int_{-\infty}^{a+} h(u, b)\, dF_1(u) = (1/P[X_1 > a]) \int_{a+}^{\infty} h(u, b)\, dF_1(u).$$

Both the sides of (1.6) are weighted averages of the monotone function $h(\cdot, b)$, and can be equal only if $h(\cdot, b)$ is a constant function. Hence (1.6) implies that the random variable X_1 is independent of the event $[X_2 \leqq b]$. In particular, for every x_1, the events $[X_1 \leqq x_1]$ and $[X_2 \leqq b]$ are independent. Now using these pairs and the function $g(v, x_1)$ in the same way as above, it can be easily seen that the events $[X_1 \leqq x_1]$ and $[X_2 \leqq x_2]$ are independent for all x_1, x_2.

This proves the first assertion of the theorem and the second is an immediate consequence of this.

INTERPRETATION FOR TESTING HYPOTHESIS OF INDEPENDENCE.

Lehmann (1966, Corollary 2 to Theorem 3) shows that the quadrant test of Blomquist and the usual tests for 2×2 tables are unbiased for the alternatives in $\mathfrak{F}_2$. These tests, in reality however, test the independence of the indicators of the type discussed above and not of the component random variables of the bivariate distribution. Thus from Lehmann's result it follows that the power function of these tests will remain at α, the level of significance, for all those alternatives where some specific indicators are independent but the component random variables are not. In view of the above theorem it is clear that when alternatives are in $\mathfrak{F}_2 \cap \mathfrak{H}_2$ such a boundary consists only of the hypothesis and the tests are strictly unbiased.

2. Multivariate families. While generalizing the characterizations of independence from bivariate to the multivariate situation one has to bear in mind that in most cases the pairwise independence is not enough for total or mutual independence. Consequently, it is natural to expect conditions involving all the component random variables simultaneously.

To fix the ideas, trivariate families are considered in detail, so that the results for the general multivariate families are transparent enough, and will not be stated separately. A straightforward generalization of the family $\mathfrak{F}_1$ (see (1.1)) is made by the following theorem.

Suppose $\mathbf{X} = (X_1, X_2, X_3)$ is a triplet of real valued random variables with the third absolute moments finite.

DEFINITION 2.1. The triple $\mathbf{X}$ is said to belong to $\mathfrak{L}(3)$ if

$$(2.1)\quad P[X_1 \leqq x_1, X_2 \leqq x_2, X_3 \leqq x_3] \geqq \prod_{i=1}^{3} P[X_i \leqq x_i] \quad \text{for} \quad x_1, x_2, x_3.$$

THEOREM 2. *If* $(X_1, X_2, X_3) \varepsilon \mathfrak{L}(3)$ *then* X_1, X_2, X_3 *are independent if and only if* (i) $EX_iX_j = EX_iEX_j$; $i \neq j$, $i = 1, 2, 3$, *and* (ii) *one of the pairs, say* (X_1, X_2), *is 'conditionally uncorrelated,'* i.e.

$$E[X_1X_2 \mid X_3] = E[X_1 \mid X_3]E[X_2 \mid X_3].$$

In applications 'conditional uncorrelatedness' may be interpreted as nonexistence of what is known as 'spurious correlation.'

PROOF. The inequality (2.1) clearly implies that all the pairs are members of $\mathfrak{F}_1$ and thus uncorrelatedness makes the random variables pairwise independent. Now, from (2.1) it is obvious that,

$$P[X_1 \leqq x_1, X_2 \leqq x_2 \mid X_3 \leqq x_3] \geqq P[X_1 \leqq x_1]P[X_2 \leqq x_2], \quad \text{for all} \quad x_1, x_2, x_3. \tag{2.2}$$

Using the pairwise independence, (2.2) may be written as

$$P[X_1 \leqq x_1, X_2 \leqq x_2 \mid X_3 \leqq x_3] \geqq P[X_1 \leqq x_1 \mid X_3 \leqq x_3]P[X_2 \leqq x_2 \mid X_3 \leqq x_3], \quad \text{for all } x_1, x_2, x_3. \tag{2.3}$$

Since (X_1, X_2) is given to be conditionally uncorrelated the relation between the two sides of (2.3) must be that of equality, which shows that

$$P[X_1 \leqq x_1, X_2 \leqq x_2, X_3 \leqq x_3] = \textstyle\prod_{i=1}^{3} P[X_i \leqq x_i], \quad \text{for all} \quad x_1, x_2, x_3. \tag{2.4}$$

The theorem stated above needs verification of a condition, which may not always be feasible. In order to look for a more useful criterion, it is natural to consider a generalization of the lemma (due to Hoeffding) of which Lehmann's result is an immediate consequence. In the following, this lemma (as reported by Lehmann (1966)) is briefly stated together with the basic steps of its proof. Let

$$I(u, x) = 1, \quad \text{if} \quad x \leqq u; \qquad = 0 \quad \text{otherwise}. \tag{2.5}$$

LEMMA 1. (Hoeffding) *Let (X_1, X_2) be a pair of real valued random variables with finite second moments and let (Y_1, Y_2) be another pair having the same distribution but independent of (X_1, X_2). Then*

$$\begin{aligned} 2[EX_1X_2 - EX_1EX_2] &= E(Y_1 - X_1)(Y_2 - X_2) \\ &= E \textstyle\iint_{-\infty}^{\infty} [I(u_1, X_1) - I(u_1, Y_1)][I(u_2, X_2) - I(u_2, Y_2)]\, du_1\, du_2 \\ &= 2 \textstyle\iint_{-\infty}^{\infty} \{P[X_1 \leqq u_1, X_2 \leqq u_2] - P[X_1 \leqq u_1]P[X_2 \leqq u_2]\}\, du_1\, du_2. \end{aligned} \tag{2.6}$$

The following remarks show that the central idea of the above lemma, although very simple, may be exploited fruitfully.

REMARK 1. In order to use the same technique for triplets, or vectors having an odd number of components, a slight change is needed. To illustrate this suppose (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are independent identically distributed triplets. Then obviously

$$E(X_1 - Y_1)(X_2 - Y_2)(X_3 - Y_3) = 0. \tag{2.7}$$

However, if one chooses (Y_1, Y_2, Y_3) to have the same distribution as $(-X_1, X_2, X_3)$ then it will be seen (Theorem 3) that a useful expression results.

REMARK 2. The definition of $I(u, x)$ could be changed to suit the type of inequality desired. Thus in (2.5) $x \leqq u$ may be replaced by $x < u$ or $x > u$, etc.

These two remarks are basic for the following characterizations of independence. Recall that $\mathbf{X} \varepsilon \mathcal{L}(3)$ means the inequality (2.1) holds.

THEOREM 3. *If* $(X_1, X_2, X_3) \varepsilon \mathcal{L}(3)$ *then* $EX_iX_j = EX_iEX_j, i \neq j, i = 1, 2, 3,$ *and* $EX_1X_2X_3 = EX_1EX_2EX_3$ *implies that* X_1, X_2, X_3 *are independent.*

PROOF. First observe that if $\mathbf{X}$ satisfies (2.1), then the same inequality holds if some or all inequalities $X_i \leqq x_i$ are replaced by strict inequalities $X_i < x_i$, provided the change is made on both sides of (2.1). This may be seen by a limiting argument, the same way as done by Lehmann (1966) in Lemma 1 of his paper.

Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be a triplet independent of $\mathbf{X} = (X_1, X_2, X_3)$ and having the same distribution as $(-X_1, X_2, X_3)$. After some simplification it is seen that

$$(2.8) \quad E(X_1 - Y_1)(X_2 - Y_2)(X_3 - Y_3) = 2\{EX_1X_2X_3 + EX_1EX_2X_3 - EX_2EX_1X_3 - EX_3EX_1X_2\}.$$

The left side of (2.8) may also be written as

$$(2.9) \quad E(X_1 - Y_1)(X_2 - Y_2)(X_3 - Y_3) = E\int\int\int_{-\infty}^{\infty} \prod_{i=1}^{3} [I(u_i, Y_i) - I(u_i, X_i)]\, du_1\, du_2\, du_3,$$

where the indicator function I is defined by (2.5). Taking the expectation inside the integral one gets a triplefold integral whose kernel $K(u_1, u_2, u_3)$ may be written as follows after adopting the following notation. Let

$$(2.10) \quad A_i = [X_i \leqq u_i], \qquad i = 1, 2, 3 \quad \text{and} \quad B_1 = [X_1 \geqq -u_1].$$

Then

$$(2.11) \quad K(u_1, u_2, u_3) = \{P(B_1A_2A_3) + P(B_1)P(A_2A_3) - P(A_2)P(B_1A_3) - P(A_3)P(B_1A_2)\} - \{P(A_1A_2A_3) + P(A_1)P(A_2A_3) - P(A_2)P(A_1A_3) - P(A_3)P(A_1A_2)\},$$

and

$$(2.12) \quad E(X_1 - Y_1)(X_2 - Y_2)(X_3 - Y_3) = \int\int\int_{-\infty}^{\infty} K(u_1, u_2, u_3)\, du_1\, du_2\, du_3.$$

Now, $(X_1, X_2, X_3) \varepsilon \mathcal{L}(3)$ so that uncorrelatedness of all the pairs implies that they are pairwise independent and thus

$$(2.13) \quad K(u_1, u_2, u_3) = \{P(B_1A_2A_3) - P(B_1)P(A_2)P(A_3)\} - \{P(A_1A_2A_3) - P(A_1)P(A_2)P(A_3)\}.$$

Again, $(X_1, X_2, X_3) \varepsilon \mathcal{L}(3)$ implies that the second term on the right side of (2.13) is nonnegative and also as remarked earlier,

$$(2.14) \quad P[X_1 < v_1, X_2 \leqq u_2, X_3 \leqq u_3] \geqq P[X_1 < v_1]P[X_2 \leqq u_2]P[X_3 \leqq u_3] \quad \text{for all} \quad v_1, u_2, u_3.$$

Subtracting both sides of (2.14) from $P(A_2A_3)$ and putting $v_1 = -u_1$, it can be seen that the first term on the right side of (2.13) is nonpositive. Thus $(X_1, X_2, X_3) \varepsilon \mathcal{L}(3)$ implies that $K(u_1, u_3, u_3)$ is nonpositive. On the other hand, uncorrelatedness and the condition $EX_1X_2X_3 = EX_1EX_2EX_3$ clearly implies that the left side of (2.12) is zero. Hence $K(u_1, u_2, u_3)$, which is a sum of two nonpositive terms, is zero a.e. (w.r.t. Lebesgue measure). Consequently, $P(A_1A_2A_3)$ equals $P(A_1)P(A_2)P(A_3)$ a.e. and both of these being distributions functions, they have to agree everywhere proving thereby the independence of X_1, X_2 and X_3.

REMARK 3. From the definition of class $\mathcal{L}(3)$ it may be said that the random variables have *positive relation*. However, from the proof of the above theorem it is obvious that this *relation* is not qualified by the sign of the difference $EX_1X_2X_3 - EX_1EX_2EX_3$. In fact, if X_i are uncorrelated, the sign of this difference is always nonpositive.

The conclusion of Theorem 3 can be seen to be valid for a class of distribution functions larger than $\mathcal{L}(3)$. This can be done, as suggested in Remark 2, by changing one or all inequalities $X_i \leqq x_i$ in (2.1) by $X_i \geqq x_i$. To do this formally let $\mathfrak{M}(3)$ denote the class of trivariate distributions such that

$$(2.15) \quad P[X_1\Delta_1 x_1, X_2\Delta_2 x_2, X_3\Delta_3 X_3]\Delta \textstyle\prod_{i=1}^{3} P[X_i\Delta_i x_i], \quad \text{for all} \quad x_1, x_2, x_3,$$

where Δ, with or without subscripts, denotes either of the inequalities $\geqq$ or $\leqq$. The following theorem is an implication of the above remark.

THEOREM 4. *If* $(X_1, X_2, X_3) \varepsilon \mathfrak{M}(3)$ *then* $EX_iX_j = EX_iEX_j$, $i \neq j$, $i = 1, 2, 3$, *and* $EX_1X_2X_3 = EX_1EX_2EX_3$ *implies that* X_1, X_2 *and* X_3 *are independent.*

It should be noted that even in $\mathfrak{M}(3)$ the pairwise independence is not equivalent to the total independence. This may be seen from the following example which is a version of the celebrated example of Bernstein (cf. Tucker (1962)).

EXAMPLE 1 (Bernstein). Suppose that the three faces of a regular tetrahedron are painted with the colors red, white and green, while the fourth has stripes of all three colors. The tetrahedron is rolled, and the color(s) on the face at the base is (are) noted. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the indicators of the three colors i.e. X_i has value 1 if the ith color seen, 0 if not. Then it may be verified that

$$(2.16) \quad P[X_1 \leqq x_1, X_2 \leqq x_2, X_3 \leqq x_3] \leqq \textstyle\prod_{i=1}^{3} P[X_i \leqq x_i] \quad \text{for all} \quad x_1, x_2, x_3.$$

Thus $\mathbf{X} \varepsilon \mathfrak{M}(3)$ and the random variables are pairwise independent; however, they are not mutually independent.

Now, we seek the generalization of the class $\mathcal{H}(2)$ (see Definition 1.1) to the families of multivariate distributions.

DEFINITION 2.2. A triple (X_1, X_2, X_3) is said to be a member of $\mathcal{H}(3)$ if for every x_i, x_j

$$P[X_i\Delta_i x_i, X_j\Delta_j x_j \mid X_k = x_k]$$

is monotone in x_k; $i \neq j \neq k$, $i = 1, 2, 3$.

THEOREM 5. *If* $\mathbf{X} \varepsilon \mathcal{H}_3$ *then the independence of* X_1, X_2 *and* X_3 *is equivalent to that of the three events* $[X_1 \leqq a]$, $[X_2 \leqq b]$ *and* $[X_3 \leqq c]$, *for some* a, b, c *such that the probabilities of these events are bounded away from* 0 *and* 1.

PROOF. Note that

$$(2.17)\quad P[X_1 \leqq a,\ X_2 \leqq b \mid X_3 \leqq c] = P[X_1 \leqq a,\ X_2 \leqq b \mid X_3 > c],$$

and for every x_1, x_2 the function

$$(2.18)\quad h(u; x_1, x_2) = P[X_1 \leqq x_1, X_2 \leqq x_2 \mid X_3 = u]$$

is monotone in u. From (2.17) it follows that

$$(2.19)\quad (1/P[X_3 \leqq c]) \int_{-\infty}^{c+} h(u; a, b)\, dF_{X_3}(u) = (1/P[X_3 > c]) \int_{c+}^{\infty} h(u; a, b)\, dF_{X_3}(u),$$

and monotonicity of the function $h(u; a, b)$ implies that it is constant in u. However, this is equivalent to X_3 being independent of the event $[X_1 \leqq a, X_2 \leqq b]$ or to say that $[X_3 \leqq x_3]$ is independent of the same event *for all* x_3. Repeating the same procedure now after conditioning by X_1 and then by X_2 the theorem is established. (For members of $\mathcal{H}(3)$ having other sets of inequalities the proof is similar.)

It is clear that class $\mathcal{H}(3)$ does not bear any inclusive relation with $\mathfrak{M}(3)$. However, the defining condition of $\mathcal{H}(3)$ (see Definition 2.2) is more stringent.

Another type of family, where the same characterization holds is a two parameter family. This is a multivariate analog of the bivariate parametric family considered in [2].

Suppose β_1 and β_2 are parameters of a family of trivariate distributions such that $\beta_1 \neq 0$ implies that for every x_1,

$$(2.20)\quad P[X_1 \leqq x_1 \mid X_2 = x_2] \text{ is nonconstant and monotone in } x_2,$$

and $\beta_2 \neq 0$ implies that for every x_1, x_2,

$$(2.21)\quad P[X_1 \leqq x_1,\ X_2 \leqq x_2 \mid X_3 = x_3] \text{ is nonconstant and monotone in } x_3.$$

Further $\beta_1 = 0$ implies the independence of X_1, X_2 while $\beta_2 = 0$ implies the same for (X_1, X_2) and X_3. An example where such a parametrization holds is the following linear model:

$$(2.22)\quad \begin{aligned} X_1 &= \alpha_1 + \beta_1 X_2 + \sigma_1 Z_1 \\ X_2 &= \alpha_2 + \beta_2 X_3 + \sigma_2 Z_2, \end{aligned}$$

where X_2 is independent of Z_1 and the three random variables X_3, Z_2, Z_1 are mutually independent. By the same method of proof as in Theorem 5 it can be seen that for such a parametric family the same characterization holds.

REFERENCES

[1] ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–74.

[2] JOGDEO, KUMAR and PATIL, G. P. (1967). Characterization of independence in family of bivariate distributions with the regression dependence. *Abstract. Ann. Math. Statist.* **38** 642.

[3] LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–53.

[4] TUCKER, H. G. (1962). *An Introduction to Probability and Mathematical Statistics*. Academic Press, New York.