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THE CORRELATION STRUCTURE OF THE OUTPUT PROCESS OF SOME SINGLE SERVER QUEUEING SYSTEMS[1]

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1. Introduction and summary. A queueing system can be regarded as transforming one point process into another (as pointed out for example in Kendall (1964), Section 6), namely, the *input* or *arrival process* with inter-arrival intervals $\{T_n\}$ is acted on by a system comprised of a queue discipline and a service (or, delay) mechanism, producing the *output* or *departure process* with inter-departure intervals $\{D_n\}$. The object of this paper is to study the correlation structure of the sequence $\{D_n\}$ (and this sequence we shall for convenience call the output process of the system) when the input process is a renewal process and when the service times $\{S_n\}$ (assumed to be independently and identically distributed, and independent of the input process) are such that the system can and does exist in its stationary state. In particular, we shall be concerned with conditions under which the process $\{D_n\}$ is uncorrelated, by which we mean that $\text{cov}(D_0, D_n) = E(D_0D_n) - (E(D_0))^2 = 0$ $(n = 1, 2, \cdots)$.

Schematically then, we study the mapping

$$\{T_n\} \xrightarrow{\{S_n\}/1} \{D_n\},$$

and as consequences of the formal theorems of the paper the following statements can be justified (T, S, and D denote typical members of $\{T_n\}$, $\{S_n\}$ and $\{D_n\}$).

(i) $\text{var}(D) \geqq \text{var}(S)$, with equality only in the trivial case where both $\{T_n\}$ and $\{S_n\}$ are deterministic.

(ii) Locally, the mapping can be made any of variance increasing, variance preserving, or variance decreasing (that is, all cases of $\text{var}(D) >, =, < \text{var}(T)$ are possible) by appropriate choice of $\{T_n\}$ and $\{S_n\}$. Globally however, the mapping is variance preserving, that is,

$$\text{var}(D_1 + \cdots + D_n)/\text{var}(T_1 + \cdots + T_n) \to 1 \qquad (n \to \infty).$$

(iii) When $\{T_n\}$ is a Poisson process, the process $\{D_n\}$ is uncorrelated if and only if it is a Poisson process (and this occurs if and only if the $\{S_n\}$ are negative exponential).

(iv) When the $\{S_n\}$ are negative exponential, $\{D_n\}$ is a renewal process if and only if it is a Poisson process (and this occurs if and only if $\{T_n\}$ is a Poisson process). However (cf. (iii)) $\{D_n\}$ can be uncorrelated without being a renewal

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process. If the $\{D_n\}$ are correlated then the terms cov (D_0, D_n) are of the same sign for all $n = 1, 2, \cdots$ and converge to zero monotonically.

(v) There exist $\{T_n\}$ and $\{S_n\}$ such that the serial covariances cov (D_0, D_n) are not of the same sign for all $n = 1, 2, \cdots$ (see remark after Theorem 7).

2. Notation and preliminaries. It is to be understood generally that n ranges over $\cdots, -1, 0, 1, \cdots$. We suppose that customer C_n arrives at the epoch t_n where $\{t_n\}$ are the successive epochs of a renewal process for which $T_n = t_{n+1} - t_n$ and Pr $(T_n \leqq x) = A(x)$ (all n), $A(0+) = 0$. Customers are attended by a single server in order of arrival. (The theorems of the paper are independent of this assumption of order-of-arrival service, but it simplifies the algebra.) C_n is served for a time S_n, Pr $(S_n \leqq x) = B(x)$ (all n), $B(0+) = 0$, where $\{S_n\}$ and $\{T_n\}$ are mutually independent sequences of independent positive random variables. Setting $U_n = S_n - T_n$ and Pr $(U_n \leqq x) = U(x)$ (all n and all real x), we assume that $E(U_n) < 0$ and $E(U_n^2) < \infty$ (we require the former condition to ensure stationarity of the system, while if $E(U_n^2) = \infty$ then the discussion below is pointless). At the arrival epoch $t_n - 0$ there are Q_n' customers in the system; C_n waits a time W_n while these Q_n' customers are being attended (if $Q_n' = 0$ then $W_n = 0$), and is then served a time S_n. At the departure epoch $t_n + W_n + S_n + 0$ there remain in the system Q_n customers. After C_n's departure the server is idle for a time V_{n+1}, with $V_{n+1} = 0$ if $Q_n > 0$ and $V_{n+1} = t_{n+1} - (t_n + W_n + S_n)$ if $Q_n = 0$. (This definition of an idle time associates with *every* customer C_n a non-negative random variable V_n. The sequence of idle periods as customarily defined (e.g. Prabhu (1965) p. 149) is obtained by deleting from $\{V_n\}$ all the elements which are zero.) In other words,

$$V_{n+1} = (W_n + U_n)^- = (T_n - S_n - W_n)^+ \tag{1}$$

where $x^- = (-x)^+ = \max(0, -x)$. Denoting by D_n the length of the inter-departure interval terminating with C_n's departure,

$$D_n = V_n + S_n. \tag{2}$$

Alternatively,

$$\begin{aligned} D_{n+1} &= t_{n+1} + W_{n+1} + S_{n+1} - (t_n + W_n + S_n) \\ &= W_{n+1} - W_n + S_{n+1} - S_n + T_n, \end{aligned} \tag{3}$$

from which (2) may be deduced since

$$W_n + U_n = (W_n + U_n)^+ - (W_n + U_n)^- = W_{n+1} - V_{n+1}.$$

Using stationarity and the fact that $E(U_n^2) < \infty$ implies $E(W_n) < \infty$ (Kiefer and Wolfowitz (1956)), (3) shows immediately that

$$E(D_m) = E(T_n) \qquad (\text{all} \quad m, n) \tag{4}$$

as is obvious intuitively by equating the mean number of arrivals to the mean number of departures in a long time interval. Coupling (2) with (4) then shows that

$$(5) \qquad E(V_n) = E(T_n) - E(S_n) = -E(U_n) = E((W_n + U_n)^-).$$

The main importance of (2) however lies in exhibiting a representation of D_n as the sum of two mutually independent non-negative random variables. That this is so follows from the independence of S_n of the entire history of the process prior to $t_n + W_n$, and V_n is determined by that history.

Even more follows from (1) and (2) on recalling that $\{W_n\}$ is a stationary Markov chain, with stationary distribution function (df) $W(x) = \Pr(W_n \leqq x)$ say. Then V_n is independent of W_{n-2} given W_{n-1}, and similarly W_{n+2} is independent of V_n and S_n (and hence of D_n) given W_{n+1}. Therefore for $n \geqq 2$,

$$\begin{aligned} &\Pr(D_0 \leqq x, D_n \leqq y) \\ (6) \qquad &= \int_{w=0-}^{\infty} \Pr(D_n \leqq y \mid W_{n-1} = w)\, d_w \Pr(W_{n-1} \leqq w, D_0 \leqq x) \\ &= \int_{w=0-}^{\infty} \Pr(D_1 \leqq y \mid W_0 = w) \\ &\cdot d_w \int_{v=0-}^{\infty} \Pr(W_{n-1} \leqq w \mid W_1 = v)\, d_v \Pr(W_1 \leqq v, D_0 \leqq x), \end{aligned}$$

where, recalling that $W_n > 0$ implies $V_n = 0$,

$$\begin{aligned} (7) \quad \Pr(W_1 \leqq v, D_0 \leqq x) &= \Pr(S_0 - T_0 \leqq v, V_0 + S_0 \leqq x \mid W_0 = 0) W(0) \\ &+ \int_{0+}^{\infty} \Pr(u + S_0 - T_0 \leqq v, S_0 \leqq x)\, dW(u). \end{aligned}$$

The principle of the argument leading to (6) and (7) is used below in the discussion of cov (D_0, D_n) in a system $GI/M/1$. It is also used by implication in the proof of Theorem 2.

3. The output process of $GI/G/1$. It follows from the independence of V_n and S_n in (2) that var $(D_n) \geqq$ var (S_n), with equality occurring only in the exceptional case where var $((W_n + U_n)^-) = 0$, so that $(W_n + U_n)^- =$ constant a.s. From the (assumed) stationarity of the system, $E(U_n) \leqq 0$, with equality only in the case that $U_n = S_n - T_n = 0$ a.s. and hence that $S_n =$ constant $= T_n$ a.s. When $E(U_n) < 0$, recalling (5) and that $W_{n+1} = (W_n + U_n)^+$, we have $W_n = 0$ a.s., and therefore $U_n^- = -U_n =$ constant a.s. Hence

THEOREM 1. *The inter-departure intervals $\{D_n\}$ in a stationary $GI/G/1$ queueing system have*

$$(8) \qquad \text{var}(D_n) \geqq \text{var}(S_n),$$

with equality if and only if both the service times and inter-arrival times are constant, in which case var $(D_n) = 0$.

The main part of this section is devoted to a proof of Theorem 2 below. The algebra leading to (9) is simple; the argument appears to be more involved than it ought. (Cox (Smith and Wilkinson (1965) pp. 436–437) asserted that (9) holds in $M/G/1$. Essentially the proof given below is a justification of the generalization (9) of his assertion.)

THEOREM 2. *The inter-departure intervals $\{D_n\}$ in a stationary $GI/G/1$ queueing* *system for which* $E(T_0^2) < \infty$ *and* $E(S_0^3) < \infty$ *have*

$$\text{var}(D_0) + 2\sum_{n=1}^{\infty} \text{cov}(D_0, D_n) = \text{var}(T_0). \tag{9}$$

PROOF. The moment conditions stated in the Theorem suffice to ensure that var (T_0) and var (W_n) are both finite (Kiefer and Wolfowitz (1956)). Using (3),

$$\sum_{j=1}^{r} D_{n+j} = W_{n+r} - W_n + S_{n+r} - S_n + \sum_{j=0}^{r-1} T_{n+j}. \tag{10}$$

On the right-hand side here, W_{n+r} is independent of S_{n+r} but dependent on the other terms, while all other terms are mutually independent. Since $\{D_n\}$ is a stationary sequence, it follows from (10) that

$$\begin{aligned} r \text{ var}(D_0) &+ 2\sum_{j=1}^{r-1} (r - j) \text{ cov}(D_0, D_j) \\ &= r \text{ var}(T_0) + 2\sum_{j=1}^{r} \text{cov}(T_0, W_j) + 2 \text{ var}(W_0) + 2 \text{ var}(S_0) \\ &\qquad - 2 \text{ cov}(W_0, W_r) - 2 \text{ cov}(S_0, W_r) \end{aligned}$$

whence by differencing with r replaced by $r + 1$,

$$\begin{aligned} &\text{var}(D_0) + 2\sum_{j=1}^{r} \text{cov}(D_0, D_j) \\ &\quad = \text{var}(T_0) + 2 \text{ cov}(T_0, W_{r+1}) - 2 \text{ cov}(W_0, W_{r+1}) + 2 \text{ cov}(W_0, W_r) \\ &\qquad - 2 \text{ cov}(S_0, W_{r+1}) + 2 \text{ cov}(S_0, W_r). \end{aligned} \tag{11}$$

Thus to prove (9), it suffices to show that as $r \to \infty$, cov (W_0, W_r), cov (T_0, W_r) and cov $(S_0, W_r - W_{r+1}) \to 0$.

It is known (Theorem 1 of Daley (1968)) that cov $(W_0, W_r) \to 0$ $(r \to \infty)$, and to demonstrate the convergence of the other terms, we use the intuitively obvious Lemma A (cf. Serfling (1967) for comment).

LEMMA A. *If* $f(x)$ *and* $g(x)$ *are non-decreasing functions of* x $(-\infty < x < \infty)$, *and* X *is a random variable such that* $Ef \equiv E(f(X))$, $Eg \equiv E(g(X))$, *and* $Efg \equiv E(f(X)g(X))$ *exist, then*

$$\text{cov}(f(X), g(X)) \equiv Efg - EfEg \geqq 0.$$

If either $f(X)$ *or* $g(X)$ *is* a.s. *constant, then equality holds.*

To show that $E(T_0W_r) \to E(T_0)E(W_0)$ (which is equivalent to showing that cov $(T_0, W_r) \to 0$), define for $r = 1, 2, \cdots$ and all $x \geqq 0$

$$A_r(x) = \int_{W_r \leqq x} T_0 \, d \,\text{Pr} = E(T_0 \chi[W_r \leqq x])$$

where $\chi[B]$ is the indicator function of the event B. Then for $y \geqq 0$,

$$A_{r+1}(y) = \int_{0-}^{\infty} U(y - x) \, dA_r(x) = \int_{-\infty}^{y} A_r(y - x) \, dU(x), \tag{12}$$

and each $A_r(x)$ is a non-decreasing non-negative function of x with $A_r(x) \uparrow E(T_0)$ $(x \to \infty)$ for all r. Thus $\{A_r(x)/E(T_0)\}$ is a sequence of df's. related by (12) and therefore by Lindley's work (1952) $A_r(x)/E(T_0)$ converges as $r \to \infty$ to the unique probabilistic solution $W(x)$ of the equation

$$\begin{aligned} W(x) &= \int_{-\infty}^{y} W(y - x) \, dU(x) \qquad &(y \geqq 0), \\ &= 0 \qquad &(y < 0), \end{aligned} \tag{13}$$

and $W(\cdot)$ is known to be the df of the stationary waiting time sequence $\{W_n\}$. Lemma A is used in deducing the inequality below:

$$\begin{aligned} A_1(x) &= E(T_0\chi[W_1 \leqq x]) = E(T_0\chi[W_0 + S_0 - T_0 \leqq x]) \\ &= E(E(T_0\chi[W_0 + S_0 - T_0 \leqq x] \mid W_0, S_0)) \\ &\geqq E(E(T_0) \Pr (W_0 + S_0 - T_0 \leqq x \mid W_0, S_0)) \\ &= E(T_0)W(x). \end{aligned}$$

Now by (12) and (13),

$$A_{r+1}(y) - E(T_0)W(y) = \int_{-\infty}^{y} (A_r(y - x) - E(T_0)\, W(y - x))\, dU(x).$$

Here, $U(\cdot)$ is a non-decreasing function and $A_1(y) \geqq E(T_0)W(y)$ for all $y \geqq 0$, so by induction, $A_r(y) \geqq E(T_0)W(y)$ for every $r = 1, 2, \cdots$. Since $E(T_0W_r) < \infty$,

$$\begin{aligned} E(T_0W_r) &= \int_0^\infty x\, dA_r(x) = \int_0^\infty (E(T_0) - A_r(x))\, dx \\ &\to \int_0^\infty (E(T_0) - E(T_0)W(x))\, dx \qquad \text{by dominated convergence} \\ &= E(T_0)E(W_0), \end{aligned}$$

which completes our proof that $\operatorname{cov}(T_0, W_r) \to 0$.

Next, consider

$$\begin{aligned} \operatorname{cov}(S_0, W_{r+1}) - \operatorname{cov}(S_0, W_r) & \\ &= E(S_0(W_{r+1} - W_r)) = E(S_0((W_r + U_r)^+ - W_r)) \\ &= E(E[S_0((W_r + U_r)^+ - W_r) \mid W_0, T_0, U_1, \cdots, U_r]) \end{aligned}$$

where $W_r = (\cdots((S_0 + W_0 - T_0)^+ + U_1)^+ + \cdots + U_{r-1})^+$ is a non-decreasing function of S_0, and $(W_r + U_r)^+ - W_r$ is a non-increasing function of W_r, and hence a non-increasing function of S_0. Therefore by Lemma A, the conditional expectation is bounded above by

$$E(S_0)E((W_r + U_r)^+ - W_r \mid W_0, T_0, U_1, \cdots, U_r),$$

which has expectation zero, so

$$0 \geqq E(S_0(W_{r+1} - W_r)) = \operatorname{cov}(S_0, W_{r+1}) - \operatorname{cov}(S_0, W_r).$$

But $\{\operatorname{cov}(S_0, W_r)\}$ is a bounded sequence, so $\operatorname{cov}(S_0, W_r)$ converges monotonically to a finite limit as $r \to \infty$, and hence

$$\operatorname{cov}(S_0, W_r) - \operatorname{cov}(S_0, W_{r+1}) \to 0 \quad (r \to \infty)$$

which completes the proof of the theorem.

4. The distribution of $\{D_n\}$ in $GI/M/1$. In this brief section we appeal to known results in finding $E(e^{-\theta D_n})$ and $\operatorname{var}(D_n)$ in the case of a stationary single-server queueing system with recurrent input and negative exponential service time df $B(x) = 1 - e^{-\mu x}$. The traffic intensity, τ is given by

$$\tau = E(S_0)/E(T_0) = (\mu \int_0^\infty x\, dA(x))^{-1}.$$

Then for $0 < \tau < 1$ the df $W(x)$ of the stationary waiting time sequence $\{W_n\}$ is given by

$$W(x) = \Pr (W_n \leqq x) = 1 - \delta e^{-\mu(1-\delta)x} \qquad (x \geqq 0) \tag{14}$$

(e.g. Prabhu (1965), p. 44) where δ is the unique root in $0 < \delta < 1$ of

$$\delta = \alpha(\mu(1 - \delta)) = \int_0^\infty e^{-\mu(1-\delta)x}\, dA(x) \tag{15}$$

and $\alpha(\theta) = E(e^{-\theta T_0})$ $(Rl(\theta) \geqq 0)$ is the Laplace-Stieltjes transform of the inter-arrival time df $A(\cdot)$. For $x \geqq 0$,

$$\Pr (U_n \leqq -x) = \Pr (U_n < -x) = \int_x^\infty (1 - e^{-\mu(t-x)})\, dA(t), \tag{16}$$

and

$$\begin{aligned} \Pr (V_{n+1} > x) &= \Pr ((W_n + U_n)^- > x) = \Pr (W_n + U_n < -x) \\ &= \int_{0-}^\infty \Pr (U_n < -x - y)\, dW(y) \\ &= \int_x^\infty (1 - e^{-\mu(1-\delta)(t-x)})\, dA(t). \end{aligned} \tag{17}$$

Thus

$$\psi(\theta) \equiv E(e^{-\theta V_n}) = [\delta\theta - \mu(1 - \delta)\alpha(\theta)][\theta - \mu(1 - \delta)]^{-1}, \tag{18}$$

and so, using (2), we have established

THEOREM 3. *In a stationary $GI/M/1$ queueing system the inter-departure intervals $\{D_n\}$ have*

$$E(e^{-\theta D_n}) = \mu(\mu + \theta)^{-1}\cdot[\delta\theta - \mu(1 - \delta)\alpha(\theta)][\theta - \mu(1 - \delta)]^{-1} \qquad (Rl(\theta) > 0). \tag{19}$$

Differentiation of (18) leads to

$$E(V_n^2) = \psi''(0) = E(T_0^2) - 2(\mu E(T_0) - 1)\mu^{-2}(1 - \delta)^{-1},$$

and hence

COROLLARY 3.1

$$\operatorname{var} (D_n) = \operatorname{var} (T_0) - (\tau^{-1} - \delta^{-1})2\delta(E(S_0))^2(1 - \delta)^{-1}. \tag{20}$$

5. Conditions for independence of $\{D_n\}$ in $GI/M/1$. Finch (1959) showed that the output process of a stationary $M/G/1$ queueing system is a renewal process only if the service time df is negative exponential, in which case the output process is known (Burke (1956)) to be a Poisson process (with the same rate parameter as the input process). We have been unable to find in the literature a proof of the following result.

THEOREM 4. *The output process of a stationary $GI/M/1$ queueing system is a* *renewal process if and only if the input process is a Poisson process, in which case the output process is a Poisson process.*

PROOF. By Burke's result already quoted, we have only to prove the necessity of the condition. We outline the steps of the proof, omitting most of the algebraic detail.

If the output process is a renewal process, then $\{D_n\}$ is a sequence of mutually independent and identically distributed non-negative random variables, and therefore we seek to prove that

$$(21) \qquad E(D_2 e^{-\theta D_0}) = E(D_2)E(e^{-\theta D_0}) \quad \textit{identically in} \quad Rl(\theta) > 0.$$

Recalling (cf. the end of Section 2) that $\{W_n\}$ is an embedded Markov chain for the process, we have (whether or not $\{D_n\}$ are independent)

$$(22) \qquad \begin{aligned} E(D_2 e^{-\theta D_0}) &= E(E(D_2 e^{-\theta D_0} \mid W_1, D_0)) \\ &= E(e^{-\theta D_0} E(D_2 \mid W_1, D_0)) = E(e^{-\theta D_0} E(D_2 \mid W_1)) \\ &= \int_{w=0-}^{\infty} E(D_2 \mid W_1 = w)\, d_w E(e^{-\theta D_0}; W_1 \leqq w) \end{aligned}$$

where

$$\begin{aligned} E(e^{-\theta D_0}; W_1 \leqq w) &= \int_{W_1 \leqq w} e^{-\theta D_0}\, d\Pr \\ &= \int_{S_0 - T_0 \leqq w, W_0 = 0} e^{-\theta S_0} e^{-\theta V_0}\, d\Pr + \int_{W_0 + S_0 - T_0 \leqq w, W_0 > 0} e^{-\theta S_0}\, d\Pr \\ &= \int_{W_0 + S_0 - T_0 \leqq w} e^{-\theta S_0}\, d\Pr - \int_{S_0 - T_0 \leqq w} e^{-\theta S_0}(1 - e^{-\theta V_0})\, d\Pr \\ &= \int_0^\infty e^{-\theta s} \mu e^{-\mu s}\, ds \int_{(s-w)^+}^{\infty} (1 - \delta e^{-\mu(1-\delta)(t+w-s)})\, dA(t) \\ &\quad - (1 - \psi(\theta)) \int_0^\infty e^{-\theta s} \mu e^{-\mu s}\, ds \int_{(s-w)^+}^{\infty} dA(t) \end{aligned}$$

which on reduction gives

$$(23) \quad \begin{aligned} E(e^{-\theta D_0}; W_1 \leqq w) &= \mu\psi(\theta)(\mu + \theta)^{-1}(1 - e^{-(\mu+\theta)w}\alpha(\mu + \theta)) \\ &\quad - \mu\delta(\mu\delta + \theta)^{-1}(\delta e^{-\mu(1-\delta)w} - e^{-(\mu+\theta)w}\alpha(\mu + \theta)). \end{aligned}$$

Referring to (22) we also require

$$(24) \qquad \begin{aligned} E(D_2 \mid W_1 = w) &= E((w + U_1)^- + S_2) \\ &= \mu^{-1} + \int_0^\infty \Pr((w + U_1)^- > x)\, dx \\ &= \mu^{-1} + \int_w^\infty (t - w - (1 - e^{-\mu(t-w)})\mu^{-1})\, dA(t). \end{aligned}$$

Now combine (23) and (24) as in (22), and recall that $E(D_2) = E(T_0)$ and $E(e^{-\theta D_0}) = \mu\psi(\theta)/(\mu + \theta)$. Then after some algebra we get

$$\begin{aligned} E(D_2 e^{-\theta D_0}) - E(D_2)E(e^{-\theta D_0}) &= (\mu\psi(\theta)(\mu + \theta)^{-1} - \mu\delta(\mu\delta + \theta)^{-1}) \\ &\cdot [\alpha(\mu)\mu^{-1} + \alpha(\mu + \theta)((\alpha(\mu) - \alpha(\mu + \theta))\theta^{-1} - (1 - \alpha(\mu + \theta))(\mu + \theta)^{-1})]. \end{aligned}$$

Thus, independence of D_0 and D_2 implies that at least one of the factors on the right hand side is zero identically in $Rl(\theta) > 0$. If the second factor is zero, then

$\alpha(\mu)/\mu = \alpha(\mu + \theta)/(\mu + \theta)$ identically in $\theta > 0$, which is impossible when $\alpha(\cdot)$ is the Laplace-Stieltjes transform of the df of a non-negative random variable. Therefore the first factor is identically zero, and using (18) this implies that $\alpha(\theta) = \mu\delta/(\mu\delta + \theta)$ which shows that the distribution of T_n is negative exponential, and hence that the input process, if a renewal process, is necessarily a Poisson process. The theorem is proved.

6. The serial covariance of $\{D_n\}$ in $GI/M/1$. We now discuss the covariance structure of $\{D_n\}$ in a stationary $GI/M/1$ queueing system. Continuing the notation of the two previous sections, the main result is Theorem 5 and its Corollaries.

THEOREM 5. *The inter-departure intervals $\{D_n\}$ in a stationary $GI/M/1$ queueing system have*

$$\text{(25)} \qquad \operatorname{cov}(D_0, D_n) = (\tau^{-1} - \delta^{-1})E(S_0)[E(D_n \mid W_0 = 0) - E(D_0)].$$

COROLLARY 5.1. $\operatorname{cov}(D_0, D_n) \to 0$ *monotonically* $(n = 1, 2, \cdots)$.

COROLLARY 5.2. *The necessary and sufficient condition that the sequence $\{D_n\}$ of inter-departure intervals in a stationary $GI/M/1$ queueing system should be uncorrelated is that the traffic intensity $\tau = E(S_0)/E(T_0)$ should satisfy the equation $\tau = \delta = \alpha(\mu(1 - \delta))$.*

PROOF. The general idea behind the following derivation of (25) is to use an equation resembling (22). First we find (cf. the derivation of (23))

$$\text{(26)} \qquad \begin{aligned} E(D_0 ; W_1 \leqq v) &= \int_{W_1 \leqq v} D_0 \, d \Pr \\ &= E(V_0) \Pr(S_0 - T_0 \leqq v) + (1 - e^{-\mu v}\alpha(\mu))\mu^{-1} \\ &\quad - (\delta e^{-\mu(1-\delta)v} - e^{-\mu v}\alpha(\mu))(\mu\delta)^{-1}. \end{aligned}$$

Now

$$\begin{aligned} \Pr(S_0 - T_0 \leqq v) &= \Pr(U_0 \leqq v) = \Pr((W_0 + U_0)^+ \leqq v \mid W_0 = 0) \\ &= \Pr(W_1 \leqq v \mid W_0 = 0) = 1 - e^{-\mu v}\alpha(\mu) \qquad (v \geqq 0)' \end{aligned}$$

and on using (14) as well, (26) can be written as

$$\text{(27)} \qquad \begin{aligned} E(D_0 ; W_1 \leqq v) &= (\tau^{-1} - \delta^{-1})E(S_0)[\Pr(W_1 \leqq v \mid W_0 = 0) \\ &\quad - \Pr(W_1 \leqq v)] + E(D_0) \Pr(W_1 \leqq v). \end{aligned}$$

Thus (cf. (22)) for $n \geqq 2$,

$$E(D_0 D_n) = (\tau^{-1} - \delta^{-1})E(S_0)[E(D_n \mid W_0 = 0) - E(D_n)] + E(D_0)E(D_n).$$

$\operatorname{cov}(D_0, D_n) = E(D_0 D_n) - E(D_0)E(D_n)$, so (25) is proved except for $n = 1$.

$$\text{(28)} \qquad \begin{aligned} E(D_0 D_1) &= E((V_0 + S_0)(V_1 + S_1)) \\ &= E(D_0)E(S_1) + E((V_0 + S_0)(T_0 - S_0 - W_0)^+) \\ &= E(D_0)E(S_0) + E(V_0)E((T_0 - S_0)^+) \\ &\quad + E(S_0(T_0 - S_0 - W_0)^+) \\ &= \alpha(\mu)E(S_0)(E(T_0) - E(S_0)\delta^{-1}) + E(D_0)E(D_1). \end{aligned}$$

Now by (24), $E(D_1 \mid W_0 = 0) = E(T_0) + \alpha(\mu)/\mu$, so (25) is shown in the case $n = 1$, and the theorem is proved.

For $x \geqq 0$ set $W_n(x) = \Pr(W_n \leqq x \mid W_0 = 0)$, so that for each fixed x, $W_n(x) \downarrow W(x)(n \to \infty)$ (Lindley (1952)).

$$(29) \qquad \begin{aligned} E(D_{n+1} \mid W_0 = 0) &= E(S_{n+1}) + E((W_n + U_n)^- \mid W_0 = 0) \\ &= E(S_0) + \int_0^\infty dx \int_{-\infty}^{-x} W_n(-x-y)\, dU(y), \end{aligned}$$

so for $n \geqq 1$, in $GI/M/1$,

$$\begin{aligned} &\operatorname{cov}(D_0, D_{n+1}) - \operatorname{cov}(D_0, D_n) \\ &\quad = (\tau^{-1} - \delta^{-1})E(S_0) \int_0^\infty dx \int_{-\infty}^{-x} (W_n(-x-y) - W_{n-1}(-x-y))\, dU(y) \end{aligned}$$

which by the monotonic behaviour of $\{W_n(\cdot)\}$ shows that the right hand side is either never positive or never negative, and hence proves Corollary 5.1.

It is easy to see from (28) that $\operatorname{cov}(D_0, D_1) = 0$ if and only if $\tau = \delta$, and inspection of (25) then implies the conclusion of Corollary 5.2.

We remark that (3) can be used to write from (29)

$$(30) \qquad E(D_{n+1} \mid W_0 = 0) - E(D_0) = E(W_{n+1} - W_n \mid W_0 = 0),$$

and consequently, a generating function for $\operatorname{cov}(D_0, D_n)$ (given in (25)) can be found via the known distribution for W_n given $W_0 = 0$ (e.g. Takács (1962), p. 121).

7. The serial covariance of $\{D_n\}$ in $M/G/1$. It is well known that in a single-server queueing system with arbitrary service time df $B(\cdot)$ and Poisson input process with rate parameter λ the sequence of queue size random variables $\{Q_n\}$ at the departure epochs constitutes an embedded Markov chain. Jenkins (1966a), (1966b) used the chain $\{Q_n\}$ to study $\operatorname{cov}(D_0, D_1)$ and $\operatorname{cov}(D_0, D_2)$ in such a stationary $M/G/1$ queue, and we derive below the generating function for $\operatorname{cov}(D_0, D_n)(n = 1, 2, \cdots)$ as a preliminary to establishing Theorem 7.

THEOREM 6. *The inter-departure intervals $\{D_n\}$ in a stationary $M/G/1$ queueing system have*

$$(31) \qquad \begin{aligned} &\lambda^2(1-\tau)^{-1} \sum_{n=1}^\infty \operatorname{cov}(D_0, D_n) z^n \\ &\quad = (w - z)(1-z)^{-1}(1-w)^{-1} + (zw' - w)((1-z)ww')^{-1} \end{aligned} \qquad (|z| < 1)$$

where $w' = dw/dz$, $w \equiv w(z)$ is the root of smallest modulus of

$$(32) \qquad w = z\beta(\lambda(1-w)) = z\int_0^\infty e^{-\lambda(1-w)x}\, dB(x),$$

and $\beta(\theta) = E(e^{-\theta S_0})$.

THEOREM 7. *The inter-departure intervals $\{D_n\}$ of a stationary $M/G/1$ queueing system are uncorrelated if and only if the service distribution is negative exponential,* i.e., *the system is $M/M/1$.*

REMARK. Combining this last result with equation (9) in Theorem 2, it follows that apart from an $M/M/1$ system, any other stable $M/G/1$ system for which var $(S_0) = (E(S_0))^2$ (and hence var $(D_0) =$ var $(T_0))$ has cov $(D_0, D_n) > 0$ for at least one positive integer n, and <0 for at least one other positive integer. This justifies assertion (v) in our introductory remarks.

PROOF. The inter-departure interval D_n terminates at the epoch $t_n + W_n + S_n$, at which epoch the queue size is Q_n, so since $\{Q_n\}$ is an embedded Markov chain, the distribution of D_{n+1} is completely determined given Q_n. Consequently the sequence $\{(Q_n, D_n)\}$ is also an embedded Markov chain. This observation enables us in principle to find an expression for the joint distribution of D_0 and D_n for such a stationary system. For convenience we take $n = 1, 2, \cdots$, and in introducing the notation (33) below, $i, j, k = 0, 1, \cdots$ and $x, y \geqq 0$. We define for a stationary $M/G/1$ queueing system

$$(33) \qquad \pi_i = \Pr(Q_0 = i), \qquad p_{ij}(x) = \Pr(Q_1 = j, D_1 \leqq x \mid Q_0 = i),$$
$$p_{jk}^{(n)} = \Pr(Q_n = k \mid Q_0 = j), \qquad p_k(y) = \Pr(D_1 \leqq y \mid Q_0 = k).$$

Then by the Markovian nature of $\{Q_n\}$,

$$(34) \quad F_{ijk}^{(n)}(x, y) = \Pr(Q_0 = i, Q_1 = j, D_1 \leqq x, Q_n = k, D_{n+1} \leqq y)$$
$$= \pi_i p_{ij}(x) p_{jk}^{(n-1)} p_k(y)$$

where $p_{jk}^{(0)} = \delta_{jk}$, the Kronecker delta. Indeed, the expression is simpler for $n = 1$, reducing to zero unless $k = j$ when

$$(35) \qquad F_{ij}^{(1)}(x, y) = \pi_i p_{ij}(x) p_j(y),$$

but we shall have no special need to refer to this particular case of (34): Jenkins (1966a), (1966b) has shown how to exploit it. The joint distribution of D_0 and D_n is then given by

$$(36) \qquad F^{(n)}(x, y) = \Pr(D_0 \leqq x, D_n \leqq y)$$
$$= \sum_{i=0}^{\infty} \sum_{j=(i-1)^+}^{\infty} \sum_{k=(j-n+1)^+}^{\infty} F_{ijk}^{(n)}(x, y).$$

Our interest in (36) is that, coupled with (34), it shows that in order to find $E(D_0 D_n) = E(D_1 D_{n+1})$, it suffices to find $E(D_{n+1} \mid Q_n = k) = E(D_1 \mid Q_0 = k)$ and $E(D_1 ; Q_1 = j \mid Q_0 = i)$. Now

$$(37) \quad p_k(y) = \Pr(D_1 \leqq y \mid Q_0 = k) = B(y) \qquad (k > 0),$$
$$= \int_0^y (1 - e^{-\lambda(y-u)})\, dB(u) \qquad (k = 0),$$

so

$$(38) \qquad \lambda E(D_1 \mid Q_0 = k) = \lambda E(S_0) + \delta_{0k} = \tau + \delta_{0k}$$

since in $M/G/1$ with the present notation, the traffic intensity $\tau = \lambda E(S_0)$. Similarly,

$$p_{ij}(x) = \int_0^x e^{-\lambda u} (\lambda u)^{j+1-i} ((j + 1 - i)!)^{-1}\, dB(u) \qquad (j \geqq i - 1 \geqq 0),$$
$$(39) \quad p_{0j}(x) = \int_0^x \lambda e^{-\lambda t}\, dt \int_0^{x-t} e^{-\lambda u} (\lambda u)^j (j!)^{-1}\, dB(u)$$
$$= \int_0^x (e^{-\lambda u} - e^{-\lambda x}) (\lambda u)^j (j!)^{-1}\, dB(u),$$

so

$$\lambda E(D_1\,;\,Q_1 = j \mid Q_0 = i) = (j + 2 - i)p_{i,j+1} \qquad (j \geqq i - 1 \geqq 0) \tag{40}$$
$$= (j + 1)p_{0,j+1} + p_{0j} \qquad (i = 0, j \geqq 0),$$

where $p_{ij} = p_{ij}^{(1)}$. It is now required to simplify

$$\lambda^2 E(D_0 D_n) = \sum_{i=0}^{\infty} \pi_i \sum_{j=(i-1)^+}^{\infty} \lambda E(D_1\,;\,Q_1 = j \mid Q_0 = i) \cdot \sum_{k=(j-n+1)^+}^{\infty} p_{jk}^{(n-1)} \lambda E(D_1 \mid Q_0 = k). \tag{41}$$

By (38) the summation on k becomes

$$\sum_{k=(j-n+1)^+}^{\infty} p_{jk}^{(n-1)}(\tau + \delta_{0k}) = \tau + p_{j0}^{(n-1)}. \tag{42}$$

Referring to (40), we observe that by dominated convergence,

$$\sum_{j=0}^{\infty} (j + 1)p_{0,j+1} = \sum_{j=0}^{\infty} \int_0^{\infty} e^{-\lambda u} \lambda u (\lambda u)^j (j!)^{-1}\, dB(u) = \lambda E(S_0) = \tau$$

and

$$\sum_{j=i-1}^{\infty} (j + 2 - i)p_{i,j+1} = \sum_{j=i-1}^{\infty} \int_0^{\infty} e^{-\lambda u} \lambda u (\lambda u)^{j+1-i} ((j + 1 - i)!)^{-1}\, dB(u) = \tau.$$

Thus

$$\lambda^2 E(D_0 D_n) = \tau^2 + \pi_0(\tau + p_{00}^{(n)}) + \pi_0 \sum_{j=0}^{\infty} (j + 1)p_{0,j+1}p_{j0}^{(n-1)} + \sum_{i=1}^{\infty} \pi_i \sum_{j=i-1}^{\infty} (j + 2 - i)p_{i,j+1}p_{j0}^{(n-1)}. \tag{43}$$

Here, $\tau^2 + \pi_0\tau = \tau$; also, it is known (e.g. Takács (1962), p. 71) that

$$\sum_{n=0}^{\infty} p_{j0}^{(n)} z^n = (w(z))^j (1 - w(z))^{-1} \qquad (|z| < 1) \tag{44}$$

where $w \equiv w(z)$ is as stated in Theorem 6, so from the last sum in (43), for $|z| < 1$,

$$\sum_{n=1}^{\infty} \sum_{j=i-1}^{\infty} (j + 2 - i)p_{i,j+1}p_{j0}^{(n-1)} z^n$$
$$= z \sum_{j=i-1}^{\infty} \int_0^{\infty} e^{-\lambda u} \lambda u (\lambda u w)^{j+1-i} w^{i-1} ((j + 1 - i)!(1 - w))^{-1}\, dB(u)$$
$$= z w^{i-1}(1 - w)^{-1} \int_0^{\infty} \lambda u e^{-\lambda(1-w)u}\, dB(u)$$
$$= z w^{i-1}(1 - w)^{-1}(z^{-1} - w(z^2 w')^{-1})$$

where the last equality is deduced by differentiation in (32). Similar expressions exist for the other terms in n in (43). Also, for $|\zeta| < 1$,

$$\sum_{i=0}^{\infty} \pi_i \zeta^i = (1 - \tau)(1 - \zeta)\beta(\lambda(1 - \zeta))(\beta(\lambda(1 - \zeta)) - \zeta)^{-1}$$

(e.g. Prabhu (1965), p. 41), so now combining these remarks in (43) (remembering also that $\lambda E(D_0) = 1$),

$$\lambda^2 \sum_{n=1}^{\infty} \operatorname{cov}(D_0, D_n) z^n = -(1 - \tau)z(1 - z)^{-1} + \pi_0 w(1 - w)^{-1} + (\pi_0 z(1 - w)^{-1} + \sum_{i=1}^{\infty} \pi_i z w^{i-1}(1 - w)^{-1})(z^{-1} - w(z^2 w')^{-1})$$

from which (31) now follows (since $\pi_0 = 1 - \tau$ and $\beta(\lambda(1 - w(z))) = w(z)/z$). Theorem 6 is proved.

The sufficiency of the conditions of Theorem 7 follow from Burke's result quoted in Section 5, so we have only to prove their necessity, which by (31) implies that identically in $|z| < 1$,

$$ww'(w - z) + (zw' - w)(1 - w) = 0. \tag{45}$$

By means of the transformation $\theta = \lambda(1 - w)$ and $z = (\lambda - \theta)/\lambda\beta(\theta)$ (45) becomes the differential equation (46) in $\beta \equiv \beta(\theta)$ and θ, valid at least for $0 < \theta < \lambda$:

$$\theta(d\beta/d\theta) + \beta(1 - \beta) = 0, \tag{46}$$

so that for some constant A and $0 < \theta < \lambda$,

$$\beta(\theta) = A(A + \theta)^{-1}. \tag{47}$$

Since $\lim_{\theta\to 0+} (-\lambda\beta'(\theta)) = \tau$, $A = \lambda/\tau > 0$. $\beta(\cdot)$ being the Laplace-Stieltjes transform of a df, and $A/(A + \theta)$ being analytic in $Rl(\theta) > 0$, we can extend the range of definition of $\beta(\theta)$ in (47) from $0 < \theta < \lambda$ to the half-plane $Rl(\theta) > 0$, and this implies that $B(\cdot)$ is the negative exponential df with mean τ/λ. Theorem 7 is proved.

Recalling that the intervals between occurrences of a renewal process are independent and therefore uncorrelated, Theorem 7 therefore includes Finch's (1959) result that the only renewal process which can occur as the output process of a stationary $M/G/1$ system is the Poisson process.

8. Concluding remarks. Jenkins (1966a) observed that in the class of stationary $M/G/1$ systems for which the traffic intensity τ is given and the service distribution is a gamma distribution, the serial correlation $\operatorname{cov}(D_0, D_1)/\operatorname{var}(D_0)$ of successive inter-departure intervals is a maximum in the limiting case of constant service time. This conclusion is in fact true without the restriction on the class of service distributions, specifically, *in a stationary $M/G/1$ queueing system with traffic intensity τ and arrival rate λ,*

$$\sup_{B(\cdot)\in\mathcal{B}} [\operatorname{cov}(D_0, D_1)/\operatorname{var}(D_0)] = (e^{-\tau} + \tau - 1)(1 + \tau)^{-1} \tag{48}$$

where $\mathcal{B}$ is the family of df's *on* $(0, \infty)$ *with mean* $\lambda\tau$ *and finite second moment:*

$$\mathcal{B} = \{B(\cdot)\colon B(\cdot) \text{ a df}, B(0+) = 0, \textstyle\int_0^\infty x\, dB(x) = \lambda\tau, \int_0^\infty x^2\, dB(x) < \infty\};$$

furthermore, this bound is attained when

$$B(x) = 1 \quad (x \geqq \lambda\tau) \tag{49}$$
$$= 0 \quad (x < \lambda\tau).$$

We give an outline of the proof of this assertion, starting from

$$\operatorname{var}(D_0) = \operatorname{var}(S_0) + (1 - \tau^2)\lambda^{-2},$$
$$\operatorname{cov}(D_0, D_1) = (1 - \tau)(\beta^2(\lambda) - \beta(\lambda) - \lambda\beta'(\lambda))(\lambda^2\beta(\lambda))^{-1}.$$

Set $\lambda = 1$ without loss of generality, and observe that in

$$(50)\quad \text{cov}\,(D_0, D_1)/\text{var}\,(D_0) = (1 - \tau)(1 - \tau^2 + \text{var}\,(S_0))^{-1} \cdot (\beta^2(1) - \beta(1) - \beta'(1))(\beta(1))^{-1}$$

we can write

$$1 - \beta(1) = \int_0^\infty e^{-x}(1 - B(x))\,dx, \quad -\beta'(1) = \int_0^\infty e^{-x}(1 - x)(1 - B(x))\,dx.$$

The first factor in (50) is a maximum and equal to $1/(1 + \tau)$ when $B(\cdot)$ is given by (49). The denominator of the second factor is a minimum when $1 - \beta(1)$ is a maximum, which occurs for $B(\cdot) \in \mathcal{B}_1$ when $B(\cdot)$ is as given in (49), where $\mathcal{B}_1$ now denotes the class $\mathcal{B}$ defined above with $\lambda = 1$; to show that the numerator of the second factor is a maximum when $B(\cdot)$ is as given in (49), write

$$\mathcal{B}_1 = \bigcup_{0 < c \leq \tau} \mathcal{B}^c, \quad \mathcal{B}^c = \mathcal{B}_1 \cap \{B(\cdot): \int_0^\infty e^{-x}\,dB(x) = e^{-c}\}.$$

It can now be shown that

$$\sup_{B(\cdot)\in\mathcal{B}^c} (\beta^2(1) - \beta(1) - \beta'(1)) = e^{-c}(e^{-c} - 1 + c) = g(c),$$

and that on $0 < c \leqq \tau$, $g(c)$ attains its maximum at $c = \tau$, this occurring when $B(\cdot)$ is as given in (49).

It is possible to choose $B(\cdot)$ so that cov (D_0, D_1)/var (D_0) is arbitrarily close to minus one (for example, $dB(x) = (\Gamma(\nu))^{-1}e^{-x}x^{\nu-1}\,dx (0 < x < \infty)$ for positive ν sufficiently small).

We have not been able to formulate satisfactorily the analogous statements for the output process of a $GI/M/1$ system.

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