The Annals of Mathematical Statistics
1968, Vol. 39, No. 3, 881–889

MOMENT CONVERGENCE OF SAMPLE EXTREMES

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1. Introduction and summary. Let Z_n be the maximum of n independent identically distributed random variables each having the distribution function $F(x)$. If there exists a non-degenerate distribution function (df) $\Lambda(x)$, and a pair of sequence a_n, b_n, with $a_n > 0$, such that

$$\lim_{n\to\infty} P\{a_n^{-1}(Z_n - b_n) \leqq x\} = \lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x) \tag{1.1}$$

on all points in the continuity set of $\Lambda(x)$, we say that $\Lambda(x)$ is an extremal distribution, and that $F(x)$ lies in its domain of attraction. The possible forms of $\Lambda(x)$ have been completely specified, and their domains of attraction characterized by Gnedenko [5]. These results and their applications are contained in the book by Gumbel [6]. A natural question is whether the various moments of $a_n^{-1}(Z_n - b_n)$ converge to the corresponding moments of the limiting extremal distribution. Sen [9] and McCord [8] have shown that they do for certain distribution functions $F(x)$, satisfying (1.1). Von Mises ([10] pages 271–294) has shown that they do for a wide class of distribution functions having two derivatives for all sufficiently large x. In Section 2, the question is answered affirmatively for all distribution functions $F(x)$ in the domain of attraction of any extremal distribution provided the moments are finite for sufficiently large n.

If there exists a sequence a_n such that

$$Z_n - a_n \to 0, \quad \text{i.p.} \tag{1.2}$$

we say that Z_n is stable in probability. If

$$Z_n/a_n \to 1, \quad \text{i.p.} \tag{1.3}$$

we say that Z_n is relatively stable in probability. Necessary and sufficient conditions are well known for stability and relative stability both in probability (see Gnedenko [5]) and with probability one (see Geffroy [4], and Barndorff-Nielsen [1]). In Section 3 necessary and sufficient conditions are found for mth absolute mean stability and relative stability.

The results of this work are valid for smallest values as well as for largest values.

2. Moment limits. By reparametrization, Gnedenko's limit laws [5] can be restated as follows. If

$$\lim_{n\to\infty} P\{(Z_n - b_n)/a_n \leqq x\} = G(x), \tag{2.1}$$

then $G(x)$ must be of the form

Received November 2, 1967.

$$(2.2)\qquad -\log G(x) = (1 + c(x - \beta)/\alpha)^{-1/c}, \qquad c \neq 0,$$
$$= \exp - (x - \beta)/\alpha, \qquad c = 0,$$

where $-\infty < \beta < \infty, 0 < \alpha < \infty$, and the domain of definition depends upon the parameters c, β, and α. Clearly

$$(2.3)\qquad a_n^{-m}E(Z_n - b_n)_-^m = a_n^{-m} \int_{-\infty}^0 (x - b_n)^m dF^n(x)$$
$$= a_n^{-m} \int_{-\infty}^0 (x - b_n)^m d\Lambda(\psi(x) - \log n),$$

where

$$(2.4)\qquad \exp -\psi(x) = -\log F(x).$$

and

$$(2.5)\qquad -\log \Lambda(x) = \exp -x,$$

$x_+ = x, x \geqq 0, = 0$, otherwise, and $x_- = x_+ - x$. We call these the positive and negative parts of x respectively. Now, we define

$$(2.6)\qquad Q(y) = \min\{x:\psi(x) \geqq y\}.$$

That is, $Q(y)$ is the inverse function for $\psi(x)$. Clearly, then, if $y \equiv \psi(x) - \log n$, it follows that $x = Q(y + \log n)$, for all points of increase of the function $\psi(x)$. Observe that the integral (2.3) taken over the sets of constancy of $\psi(x)$ vanishes. Consequently,

$$(2.7)\qquad a_n^{-m}E(Z_n - b_n)_-^m = a_n^{-m} \int_{-\infty}^0 (Q(y + \log n) - b_n)^m d\Lambda(y),$$

where $\Lambda(y)$ is given by (2.5). It is shown, in this section, that as $n \to \infty$, these moments approach those of the corresponding limiting distribution. But

$$E_G X_-^m \equiv \int_{-\infty}^0 x^m d\Lambda(c^{-1} \log (1 + c((x - \beta)/\alpha)))$$
$$= \int_{-\infty}^0 (\alpha R(y, c) + \beta)^m d\Lambda(y),$$

where $E_G(\cdot)$ is the expectation using the limiting distribution and

$$(2.8)\qquad R(y, c) = \int_0^y \exp cs\, ds = c^{-1}(\exp cy - 1),$$

is the inverse function for the function $-\log(-\log G(x))$ when $\alpha = 1$, and $\beta = 0$ (2.2). Similar relations hold, of course, for the moments of the positive part. The limits of integration are then 0 and ∞.

It was shown by Gnedenko [5], that the same limit law (2.1) holds for a different sequence of constants a_n' and b_n' iff

$$(2.9)\qquad \lim_{n\to\infty} a_n'/a_n = 1, \qquad \lim_{n\to\infty} (b_n' - b_n)/a_n = 0.$$

Clearly, such a change in the normalizing constants will not alter the asymptotic properties of the moments.

THEOREM 2.1. *If m is any real positive number, if $E(Z_n)_-^m < \infty$ for sufficiently* *large* n, *and if* (2.1) *holds, then*

$$\lim_{n\to\infty} a_n^{-m} E(Z_n - b_n)_-^m = \int_{-\infty}^0 (-x)^m \, dG(x),$$
$$\lim_{r\to\infty} a_n^{-m} E(Z_n - b_n)_+^m = \int_0^\infty x^m \, dG(x),$$

provided the latter are finite.

Before proceeding to the proof of the theorem, three lemmas are proven.

LEMMA 2.1. *For any pair of sequences* a_n *and* b_n *satisfying* (2.1), *there exists a real* λ *such that*

$$(2.10) \qquad \lim_{n\to\infty} n^{-\lambda} a_n = \lim_{n\to\infty} n^{-\lambda} b_n = 0.$$

PROOF. According to [5], the sequence a_n satisfies the relation

$$\lim_{n\to\infty} a_{kn}/a_n = k^\gamma,$$

for any positive integer k, and some real positive γ. According to Dynkin [3], it follows that the first part of (2.10) holds if $\lambda > \gamma$. Now the second part is is proved. From [5], we know that a_n, and b_n satisfy the relation $\lim_{n\to\infty} n(1 - F(a_n x + b_n)) = -\log G(x)$. Let A be any real number. Suppose that for some x_0, $(1 - F(x_0)) > Ax_0^{-k}$, where k is a positive real number. Then $P\{X > x_0\} > Ax_0^{-k}$, and $E|X|^k > A$. Clearly, then, if

$$\limsup_{x\to\infty} x^k(1 - F(x)) = \infty, \quad \text{then } E|X|^k = \infty .$$

But from [5], it is known that if (2.1) holds, there must be some $k > 0$, for which $E|X|^k < \infty$, hence for which $\limsup_{x\to\infty} x^k(1 - F(x)) < \infty$. But then for some real positive c_1, c_2, any fixed x, and for sufficiently large n, $(a_n x + b_n)^k \leqq c_1(1 - F(a_n x + b_n))^{-1}$, and $(1 - F(a_n x + b_n))^{-1} \leqq n/c_2$. So $(a_n x + b_n)^k \leqq c_1 n/c_2$, which proves the lemma.

LEMMA 2.2. If $E(Z_n)_-^m < \infty$ for some n, then it holds for all larger n, and if in addition (2.1) holds, then

$$(2.11) \qquad \lim_{n\to\infty} a_n^{-m} E(Z_n - b_n)_-^m I_d = 0,$$

where

$$(2.12) \qquad \begin{aligned} I_d &= 1, \quad \text{if } Z_n \leqq d, \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

provided d is such that $F(d) < 1$.

PROOF. The expression (2.11) is equal to

$$(2.13) \quad na_n^{-m} \int_{-\infty}^d (x - b_n)^m F^{n-1}(x) \, dF(x)$$
$$\leqq na_n^{-m} c_m \int_{-\infty}^d (|x|^m + b_n^m) F^{n-1}(x) \, dF(x),$$

for some real constant c_m, by the c_r inequality (Loève [7] page 155). But the second part of (2.13) is $a_n^{-m} b_n^m F^n(d) \to 0$, as $n \to \infty$, since by Lemma 2.1,

$a_n^{-m} b_n^m$ approaches ∞ at most as rapidly as some power of n. Furthermore $na_n^{-m} \int_{-\infty}^{d} |x|^m F^{n-1}(x)\, dF(x) = na_n^{-m} \int_{-\infty}^{d} |x|^m F^{(n-n_0)}(x) F^{n_0-1}(x)\, dF(x) \leqq a_n^{-m} (F(d))^{n-n_0} \int_{-\infty}^{d} |x|^m dF^{n_0}(x) \to 0$, as $n \to \infty$. This proves the lemma.

LEMMA 2.3. If (2.1) holds, and $\epsilon > 0$ is arbitrarily chosen, there exists a real y_0, and an integer n_0, such that if $y + \log n \geqq y_0$, and $n \geqq n_0$, then

$$\alpha(1-\epsilon)R(y, c-\epsilon) + \beta - \epsilon \leqq a_n^{-1}(Q(y+\log n) - b_n) \leqq \alpha(1-\epsilon)^{-1}R(y, c+\epsilon) + \beta + \epsilon, \tag{2.14}$$

where $R(y, c)$ is given by (2.8).

PROOF. We begin by observing that the result is invariant under changes of location and scale. That is, for purposes of the proof, we can assume without loss of generality that $\alpha = 1$, and $\beta = 0$. Hence, we assume that

$$\lim_{n\to\infty} a_n^{-1}(Q(y+\log n) - b_n) = R(y, c). \tag{2.15}$$

Let y_1 be any fixed real number, positive or negative. Then for all sufficiently large n,

$$R(y, c-\epsilon) - \epsilon \leqq a_n^{-1}(Q(y+\log n) - b_n) \leqq R(y, c+\epsilon) + \epsilon, \tag{2.16}$$

simultaneously for all y between 0 and y_1.

From (2.14), it is clear that $R(y, c)$ is continuous at $y = 0$, and so

$$\lim_{n\to\infty} a_n^{-1}(Q(\log n) - b_n) = 0,$$

and we can, without loss of generality, replace b_n with $Q(\log n)$ in what follows.

Now, let us consider the case of an arbitrary negative value of y and an n such that the conditions of the lemma are satisfied. Let l be the largest integer for which $y < -l \log 2$. Then

$$\begin{aligned} &a_n^{-1}(Q(y+\log n) - Q(\log n)) \\ &\quad = a_n^{-1}\textstyle\sum_{k=1}^{l} (Q(\log n - k\log 2) - Q(\log n - (k-1)\log 2)) \\ &\qquad + a_n^{-1}(Q(y+\log n) - Q(\log n - l\log 2)). \end{aligned} \tag{2.17}$$

Dynkin [3] has shown that if $\lim_{n\to\infty} a_{nk}/a_n = k^c$, then for any $\epsilon > 0$, there exists an n_1, such that for all $n \geqq n_1$, and $k > 1$, $(1-\epsilon)k^{c-\epsilon} \leqq a_{nk}/a_n \leqq (1+\epsilon)k^{c+\epsilon}$. We can, of course, choose n_0 to be as large as n_1. If k is less than 1, but $nk > n_0$, clearly, $(1-\epsilon)(1/k)^{c-\epsilon} \leqq a_n/a_{nk} \leqq (1+\epsilon)(1/k)^{c+\epsilon}$. So $(1+\epsilon)^{-1}k^{c+\epsilon} \leqq a_{nk}/a_n \leqq (1-\epsilon)^{-1}k^{c-\epsilon}$. Furthermore by (2.14), $\int_0^{-\log 2} \exp(c-\epsilon)s\, ds \leqq a_{n2^{-(k-1)}}^{-1}(Q(\log n - k\log 2) - Q(\log n - (k-1)\log 2)) \leqq \int_0^{-\log 2} \cdot \exp(c+\epsilon)s\, ds$. Changing signs, $\int_{-\log 2}^{0} \exp(c+\epsilon)s\, ds \leqq a_{n2^{-(k-1)}}^{-1}(Q(\log n - (k-1)\log 2) - Q(\log n - k\log 2)) \leqq \int_{-\log 2}^{0} \exp(c-\epsilon)s\, ds$. But $a_n^{-1}(Q(\log n - (k-1)\log 2) - Q(\log n - k\log 2)) = a_{n2^{-(k-1)}}^{-1}(Q(\log n - (k-1)\log 2) - Q(\log n - k\log 2))\,(a_{n2^{-(k-1)}}/a_n)$. So

$$\begin{aligned} &(1+\epsilon)^{-1}2^{-(k-1)(c+\epsilon)} \int_{-\log 2}^{0} \exp(c+\epsilon)s\, ds \\ &\quad \leqq a_n^{-1}(Q(\log n - (k-1)\log 2) - Q(\log n - k\log 2)) \\ &\quad \leqq (1-\epsilon)^{-1}2^{-(k-1)(c-\epsilon)} \int_{-\log 2}^{0} \exp(c-\epsilon)s\, ds. \end{aligned} \tag{2.18}$$

However,

$$2^{-(k-1)(c+\epsilon)} \int_{-\log 2}^{0} \exp (c+\epsilon)s\, ds$$
$$= \int_{-\log 2}^{0} \exp (c+\epsilon)(s-(k-1)\log 2)\, ds \tag{2.19}$$
$$= \int_{-k\log 2}^{-(k-1)\log 2} \exp (c+\epsilon)s\, ds.$$

A similar result holds, of course, for the term on the right side of (2.17). So

$$(1+\epsilon)^{-1-l\log 2} \int_y^{l\log 2} \exp (c+\epsilon)s\, ds - \epsilon + (1+\epsilon)^{-1} \sum_{k=1}^{l} \int_{-k\log 2}^{-(k-1)\log 2} \exp (c+\epsilon)s\, ds$$
$$= (1+\epsilon)^{-1} \int_y^0 \exp (c+\epsilon)s\, ds - \epsilon$$
$$\leqq -a_n^{-1}(Q(y+\log n) - Q(\log n))$$
$$\leqq (1-\epsilon)^{-1} \int_y^0 \exp (c-\epsilon)s\, ds + \epsilon.$$

Changing the sign, we get the inequality (2.14), since $(1-\epsilon) \leqq (1+\epsilon)^{-1}$.

Now assume that y is positive. Then

$$a_n^{-1}(Q(y+\log n) - Q(\log n))$$
$$= a_n^{-1} \sum_{k=1}^{l} (Q(\log n + k\log 2) - Q(\log n + (k-1)\log 2)) \tag{2.20}$$
$$+ a_n^{-1}(Q(y+\log n) - Q(\log n + l\log 2),$$

where l is the largest integer less than $y/\log 2$. Clearly

$$a_n^{-1}(Q(\log n + k\log 2) - Q(\log n + (k-1)\log 2))$$
$$= a_{n2^{k-1}}^{-1}(Q(\log n + k\log 2)$$
$$- Q(\log n + (k-1)\log 2))(a_{n2^{k-1}}/a_n).$$

So

$$(1-\epsilon)2^{(k-1)(c-\epsilon)} \int_0^{\log 2} \exp (c-\epsilon)s\, ds$$
$$= (1-\epsilon) \int_0^{\log 2} \exp (c-\epsilon)(s+(k-1)\log 2)\, ds$$
$$= (1-\epsilon) \int_{(k-1)\log 2}^{k\log 2} \exp (c-\epsilon)s\, ds$$
$$\leqq a_n^{-1}(Q(\log n + k\log 2) - Q(\log n + (k-1)\log 2))$$
$$\leqq (1+\epsilon) \int_{(k-1)\log 2}^{k\log 2} \exp (c+\epsilon)s\, ds.$$

Recalling (2.16) and (2.18), then, and the fact that $(1+\epsilon) \leqq (1-\epsilon)^{-1}$, the result follows.

Proof of Theorem 2.1. Note that if $F(d) = 1$, $\psi(d) - \log n = \infty$. Hence the value d of x, corresponding to any finite value of y, is such that $F(d) < 1$. It follows, by Lemma 2.2, that for any such y_0, $\lim_{n\to\infty} a_n^{-m} \int_{-\infty}^{y_0-\log n} (Q(y+\log n) - b_n)^m\, d\Lambda(y) = 0$. It follows, by Lemma 2.3 that

$$\liminf_{n\to\infty} a_n^{-m} \int_{-\infty}^{0} (Q(y+\log n) - b_n)^m\, d\Lambda(y)$$
$$\geqq \int_{-\infty}^{0} (\alpha(1-\epsilon)R(y, c-\epsilon) + \beta - \epsilon)^m\, d\Lambda(y),$$

$$\limsup_{n\to\infty} a_n^{-m} \int_{-\infty}^{0} (Q(y + \log n) - b_n)^m \, d\Lambda(y)$$
$$\leqq \int_{-\infty}^{0} (\alpha(1-\epsilon)^{-1}R(y, c+\epsilon) + \beta + \epsilon)^m \, d\Lambda(y).$$

From the obvious continuity of the integrand, and the fact that $\epsilon > 0$ was arbitrarily chosen, it follows that $\lim_{n\to\infty} a_n^{-m} \int_{-\infty}^{0} (Q(y + \log n) - b_n)^m \, d\Lambda(y) = \int_{-\infty}^{0} (\alpha R(y_1 c) + \beta)^m \, d\Lambda(y)$. Clearly the corresponding result holds for the moments of the positive part. Thus the theorem is proved.

3. Moment stability and relative stability. We begin by defining the upper limit $x_\infty \equiv \sup\{x : F(x) < 1\}$. It is well known [5] that for stability in probability, it is necessary and sufficient that either

$$\text{(3.1)} \qquad \forall \epsilon > 0, \qquad \lim_{x\to\infty} (1 - F(x+\epsilon))/(1 - F(x)) = 0,$$

or $x_\infty < \infty$. By a power series expansion of $-\log F(x)$ about $F(x) = 1$, it is clear that for any distribution function $F(x)$,

$$\text{(3.2)} \qquad \lim_{x\to x_\infty} (-\log F(x))/(1 - F(x)) = 1.$$

So the condition (3.1) can be replaced by the equivalent $\forall \epsilon > 0$, $\lim_{x\to x_\infty} \log F(x+\epsilon)/\log F(x) = 0$. Recalling the definition (2.4) of $\psi(x)$, this can be rewritten $\forall \epsilon > 0$, $\lim_{x\to x_\infty} (\psi(x+\epsilon) - \psi(x)) = \infty$. Using the definition (2.6) of $Q(y)$, clearly,

$$\text{(3.3)} \qquad \lim_{y\to\infty} (Q(y+\epsilon) - Q(y)) = 0,$$

for all ϵ, including negative ones.

The result (1.2) will hold simultaneously for two pairs of sequences a_n and a_n' iff $\lim_{n\to\infty} (a_n - a_n') = 0$. Clearly this does not affect the result (3.4).

THEOREM 3.1. *If m is any real positive number, then*

$$\text{(3.4)} \quad \lim_{n\to\infty} E(Z_n - a_n)_-^m = 0, \qquad \lim_{n\to\infty} E(Z_n - a_n)_+^m = 0,$$

iff $E(Z_n)_-^m < \infty$, *for sufficiently large* n, *and*

$$\text{(3.5)} \qquad Z_n - a_n \to 0, \quad \text{i.p.}$$

Before proving the theorem, three lemmas are proven.

LEMMA 3.1. *Let $\epsilon > 0$ be arbitrarily chosen. If $Q(y)$ satisfies* (3.3), *there exists a y_0, such that for $y + \tau \geqq y_0$,*

$$\text{(3.6)} \qquad -\epsilon(|\tau| + 1) \leqq Q(y+\tau) - Q(y) \leqq \epsilon(|\tau| + 1).$$

PROOF. Clearly for any fixed value of τ, say τ_0, for sufficiently large y, the inequality holds for all τ between 0 and τ_0. First, let τ be negative. Then $Q(y+\tau) - Q(y) = \sum_{k=1}^{l} (Q(y-k) - Q(y-k+1)) + Q(y+\tau) - Q(y-l)$, where l is so chosen that $l \leqq \tau \leqq l + 1$. But for sufficiently large y, and all k,

$$\text{(3.7)} \qquad -\epsilon \leqq Q(y - (k-1)) - Q(y-k) \leqq \epsilon.$$

By summation, and the use of the original inequality (3.6), for the last term, the result follows for negative τ. By a simple modification of the reasoning, it follows for positive τ. So the lemma is proved.

LEMMA 3.2. *Let a_n be any sequence satisfying* (3.5). *Then*

$$\lim_{n\to\infty} (\log n - a_n) = \infty. \tag{3.8}$$

PROOF. First assume that $x_\infty = \infty$. Let $c_1 > 0$ be some number. It clearly follows from the preceding lemma that for all sufficiently y, $Q(y) \leqq c_1 y$, and hence $\psi(x) \geqq x/c_1$ for all sufficiently large x. Thus $F(x) \geqq \exp - e^{-x/c_1}$, and so by (3.2), there exists a positive finite constant c_2, such that for all sufficiently large x, $1 - F(x) \leqq c_2 e^{-x/c_1}$. It is known [5], that $\lim_{n\to\infty} n(1 - F(a_n - \epsilon)) = \infty$. Hence $\lim_{n\to\infty} nc_2 e^{-(a_n-\epsilon)/c_1} = \infty$. So $\lim_{n\to\infty} ne^{-a_n} = \infty$, which is equivalent to (3.8). If $x_\infty < \infty$, a_n can be taken to be equal to x_∞, and the result is immediate.

LEMMA 3.3. *Let d be any real number such that $F(d) < 1$. Let a_n be such that* (2.1) *holds. Then*

$$\lim_{n\to\infty} E(Z_n - a_n)_-^m I_d = 0, \tag{3.9}$$

where I_d is given by (2.12).

PROOF. First, observe that Lemma 3.2 implies that a_n approaches infinity more slowly than ρ^n for any $\rho > 1$. Hence $\lim_{n\to\infty} a_n^m F^n(d) = 0$. But, by the c_r inequality ([7] page 155),

$$\begin{aligned}
&E(Z_n - a_n)_-^m I_d \\
&= \int_{-\infty}^{d} (x - a_n)^m dF^n(x) \\
&\leqq nc_m \int_{-\infty}^{d} (|x|^m + a_n^m) F^{n-1}(x)\, dF(x) \\
&= nc_m \int_{-\infty}^{d} |x|^m F^{(n-n_0)}(x) F^{n_0-1}(x)\, dF(x) + nc_m a_n^m \int_{-\infty}^{d} F^{n-1}(x)\, dF(x) \\
&\leqq c_m (F^{(n-n_0)}(d) \int_{-\infty}^{d} |x|^m dF^{n_0}(x) + na_n^m F^n(d)) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus the lemma is proved.

PROOF OF THEOREM 3.1. First the sufficiency of the conditions for (3.3) are proved, assuming that $x_\infty = \infty$. Using the definitions (2.4) and (2.6) of $\psi(x)$, and $Q(y)$, $E(Z_n - a_n)_-^m = \int_{-\infty}^{0} (x - a_n)^m dF^n(x) = \int_{-\infty}^{0} (x - a_n)^m d\Lambda \cdot (\psi(x) + \log n) = \int_{-\infty}^{0} (Q(y + \log n) - a_n)^m d\Lambda(y)$, where $y = \psi(x) + \log n$, since the portion of the integral taken over the sets of constancy of $\psi(x)$ vanishes. Therefore, what is to be proved is that

$$\lim_{n\to\infty} \int_{-\infty}^{0} (Q(y + \log n) - Q(\log n))^m d\Lambda(y) = 0.$$

Clearly, for any finite y_0 the corresponding value of d is such that $F(d) < 1$. So, by Lemma 3.3, $\lim_{n\to\infty} \int_{-\infty}^{y_0 - \log n} (Q(y + \log n) - Q(\log n))^m d\Lambda(y) = 0$, for any such y_0. By Lemma 3.1, however, it follows that if $\epsilon > 0$ is arbitrarily chosen, there exists such a y_0, for which

$$\liminf_{n\to\infty} \int_{y_0 - \log n}^{0} (Q(y + \log n) - Q(\log n))^m d\Lambda(y) \geqq \epsilon \int_{-\infty}^{0} (y - 1)\, d\Lambda(y),$$

$$\limsup_{n\to\infty} \int_{y_0 - \log n}^{0} (Q(y + \log n) - Q(\log n))^m d\Lambda(y) \leqq -\epsilon \int_{-\infty}^{0} (y - 1)\, d\Lambda(y).$$

Since ϵ was arbitrarily chosen, the theorem follows for the first part of (3.4). For the second part, the proof is the same with an appropriate but obvious modification. If $x_\infty < \infty$, clearly, we can let $a_n = x_\infty$, and the result follows immediately by Lemma 3.3, since d can be taken to be any value less than x_∞. The converse follows immediately from the Markov inequality ([7], page 163).

Now, we consider the problem of relative stability. It is known [5] that, in order that

$$Z_n/a_n \to 1, \quad \text{i.p.} \tag{3.10}$$

it is necessary and sufficient that, either

$$\forall \epsilon > 1, \qquad \lim_{x\to\infty} (1 - F(x\epsilon))/(1 - F(x)) = 0, \tag{3.11}$$

or $x_\infty < \infty$. Let $X_i^* = \log (X_i)_+$, and $Z_n^* = \max_{1\leqq k \leqq n} X_i^*$. For the present, it is assumed that $x_\infty > 0$. In this case, (3.10) is equivalent to $(Z_n)_+/a_n \to 1$, i.p. since with probability one, $Z_n = (Z_n)_+$ for all sufficiently large n. It is clear that the results concerning convergence both in probability and in the mth mean, are unaltered by the replacement of a_n by a_n', iff $\lim_{n\to\infty} (a_n/a_n') = 1$.

Theorem 3.2. *If $x_\infty > 0$, then, in order that*

$$\lim_{n\to\infty} E(Z_n)_-^m/a_n^m = 0, \quad \text{and} \quad \lim_{n\to\infty} E(Z_n)_+^m/a_n^m = 1, \tag{3.12}$$

it is necessary and sufficient that $E(Z_n)_-^m$ be finite for sufficiently large n, and that (3.10) *hold.*

Proof. To begin with, the first part of (3.12) is proved. Clearly $a_n^{-m}E(Z_n)_-^m = a_n^{-m} \int_{-\infty}^0 x^m \, dF^n(x) = na_n^{-m} \int_{-\infty}^0 x^m F^{(n-n0)}(x) F^{n0-1}(x) \, dF(x) \leqq a_n^{-m} \cdot F^{(n-n0)}(0) \int_{-\infty}^0 x^m \, dF^{n0}(x) \to 0$, as $n \to \infty$, since $F(0) < 1$.

To prove the second part, we employ the logarithmic transformation. Then it is sufficient to prove that

$$\lim_{n\to\infty} E(\exp m(Z_n^* - a_n^*)) = 1.$$

By assumption we can assume that for any d such that $F(d) < 1$,

$$\begin{aligned}
&E(\exp m(Z_n^* - a_n^*))I_d \\
&\quad = \int_{-\infty}^d \exp \; m(x - a_n^*) \, dF^n(x) = (\exp - ma_n^*) \int_{-\infty}^d \exp \; mx \, dF^n(x) \\
&\quad = n(\exp - ma_n^*) \int_{-\infty}^d \exp \; mxF^{n-1}(x) \, dF(x) \\
&\quad < n(\exp - ma_n^*) \int_{-\infty}^d \exp \; mxF^{(n-n0)}(x) F^{(n0-1)}(x) \, dF(x) \\
&\quad \leqq (\exp - ma_n^*) F^{(n-n0)}(d) \int_{-\infty}^d \exp \; mx \, dF^{n0}(x) \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

since a_n^* is non-decreasing. The remaining part of the expectation is given by $\int_d^\infty \exp \; m(x - a_n^*) \, dF^n(x) = \int_d^\infty \exp \; m(x - a_n^*) \, d\Lambda(\psi(x) - \log n)$. As observed above, a_n^* can be replaced by $Q(\log n)$ without loss of generality. So the preceding term is $\int_{y_0 - \log n}^\infty \exp \; m(Q(y + \log n) - Q(\log n)) \, d\Lambda(y)$, where $y = \psi(x) - \log n$, $y_0 = \psi(d)$, and $\psi(x)$, and $Q(y)$ are given by (2.4) and (2.6), using the starred arguments. So by Lemma 3.1, if $\epsilon > 0$ is arbitrarily chosen,

$$\liminf_{n\to\infty} \int_d^\infty \exp m(x - a_n^*)\, dF^n(x) \geqq (\exp - \epsilon) \int_{-\infty}^\infty \exp - \epsilon|y|\, d\Lambda(y),$$

$$\limsup_{n\to\infty} \int_d^\infty \exp m(x - a_n^*)\, dF^n(x) \leqq (\exp \epsilon) \int_{-\infty}^\infty \exp \epsilon|y|\, d\Lambda(y).$$

So, since $\epsilon > 0$ was arbitrarily chosen, sufficiency is proved. Necessity follows from the Markov inequality. Thus the theorem is proved.

The following corollary results from the L_r Convergence Theorem, part 3 (Loève [7], page 163), and the Markov inequality.

COROLLARY 3.1. *If $x_\infty > 0$ the result of Theorem 3.2 holds* iff

$$\lim_{n\to\infty} E|a_n^{-1}Z_n - 1|^m = 0.$$

4. Acknowledgments. The author is indebted to Professor Simeon M. Berman for many helpful comments, which have led to considerable clarification of some of the proofs. The comments of a referee are also gratefully acknowledged.

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