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MULTIVARIATE EXPONENTIAL-TYPE DISTRIBUTIONS

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1. Introduction and summary. Let $\mathbf{x}$ and $\boldsymbol{\theta}$ denote s-dimensional column vectors. The components $x_1, x_2, \cdots x_s$ of $\mathbf{x}$ are random variables jointly following an s-variate distribution and components $\theta_1, \theta_2, \cdots, \theta_s$ of $\boldsymbol{\theta}$ are real numbers. The random vector $\mathbf{x}$ is said to follow an s-variate Exponential-type distribution with the parameter vector (pv) $\boldsymbol{\theta}$, if its probability function (pf) is given by

$$f(\mathbf{x}, \boldsymbol{\theta}) = h(\mathbf{x}) \exp \{\mathbf{x}'\boldsymbol{\theta} - q(\boldsymbol{\theta})\}, \tag{1.1}$$

$\mathbf{x} \,\varepsilon\, R_s$ and $\boldsymbol{\theta} \,\varepsilon\, (\mathbf{a}, \mathbf{b}) \subset R_s$. R_s denotes the s-dimensional Euclidean space. The s-dimensional open interval $(\mathbf{a}, \mathbf{b})$ may or may not be finite. $h(\mathbf{x})$ is a function of $\mathbf{x}$, independent of $\boldsymbol{\theta}$, and $q(\boldsymbol{\theta})$ is a bounded analytic function of $\theta_1, \theta_2, \cdots \theta_s$, independent of $\mathbf{x}$.

We note that $f(\mathbf{x}, \boldsymbol{\theta})$, given by (1.1), defines the class of multivariate exponential-type distributions which includes distributions like multivariate normal, multinomial, multivariate negative binomial, multivariate logarithmic series, etc.

This paper presents a theoretical study of the structural properties of the class of multivariate exponential-type distributions. For example, different distributions connected with a multivariate exponential-type distribution are derived. Statistical independence of the components $x_1, x_2, \cdots, x_s$ is discussed. The problem of characterization of different distributions in the class is studied under suitable restrictions on the cumulants.

A canonical representation of the characteristic function of an infinitely divisible (id), purely discrete random vector, whose moments of second order are all finite, is also obtained.

$\varphi(\mathbf{t})$, $m(\mathbf{t})$, $k(\mathbf{t})$ denote, throughout this paper, the characteristic function (ch. f.), the moment generating function (mgf), and the cumulant generating function (cgf), respectively, of a random vector $\mathbf{x}$. The components t_i of the s-dimensional column vector $\mathbf{t}$ are all real.

2. Preliminary discussion. The mgf of a random vector $\mathbf{x}$ with pf $f(\mathbf{x}, \boldsymbol{\theta})$, defined by (1.1), can be obtained as

$$m(\mathbf{t}, \boldsymbol{\theta}) = \exp \{q(\boldsymbol{\theta} + \mathbf{t}) - q(\boldsymbol{\theta})\}, \mathbf{t} \,\varepsilon\, (\mathbf{a} - \boldsymbol{\theta}, \mathbf{b} - \boldsymbol{\theta}).$$

Therefore the cgf of $\mathbf{x}$ becomes

$$k(\mathbf{t}, \boldsymbol{\theta}) = q(\boldsymbol{\theta} + \mathbf{t}) - q(\boldsymbol{\theta}).$$

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Let (R_c) denote the following recurrence relation between the cumulants $\lambda_{r_1,r_2,\cdots r_s}(\boldsymbol{\theta})$ of $\mathbf{x}$ with pf $f(\mathbf{x}, \boldsymbol{\theta})$:

$$\lambda_{r_1,\cdots,r_i+1,r_{i+1},\cdots,r_s}(\boldsymbol{\theta}) = (\partial/\partial\theta_i)\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) \qquad (R_c)$$

for all $i = 1, 2, \cdots, s$ and all $r_i \geqq 0$ such that $\sum_1^s r_i \geqq 1$.

Patil [11] has shown that (R_c) holds between the cumulants of $\mathbf{x}$ if and only if $f(\mathbf{x}, \boldsymbol{\theta})$ is given by (1.1). He has also shown that the cgf of $\mathbf{x}$ in this case is given by

$$k(\mathbf{t}, \boldsymbol{\theta}) = q(\boldsymbol{\theta} + \mathbf{t}) - q(\boldsymbol{\theta}),$$

and that

$$\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) = \textstyle\prod_{i=1}^s (\partial^{r_i}/\partial\theta_i^{r_i})q(\boldsymbol{\theta}). \tag{2.1}$$

Since $q(\boldsymbol{\theta})$ is analytic in $\boldsymbol{\theta}$, it follows that the cumulants $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta})$, of all orders are analytic functions of $\boldsymbol{\theta}$. Therefore we can obtain the Taylor series expansion of $\lambda_{r_1,r_2,\cdots r_s}(\boldsymbol{\theta})$ in the neighborhood (nbd) $N(\boldsymbol{\theta}_0)$ of some $\boldsymbol{\theta}_0 \in (a, b)$as

$$\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) = \textstyle\sum_{i=0}^\infty \sum_{i_1+i_2+\cdots+i_s=i} \lambda_{r_1+i_1,r_2+i_2,\cdots,r_s+i_s} \cdot(\boldsymbol{\theta}_0)(\theta_1 - \theta_{01})^{i_1}(i_1\,!)^{-1} \cdots (\theta_s - \theta_{0s})^{i_s}(i_s\,!)^{-1} \tag{2.2}$$

for all $\boldsymbol{\theta} \in N(\boldsymbol{\theta}_0)$.

We assume, henceforth, that the random vector $\mathbf{x}$ follows s-variate exponential-type distribution given by (1.1) and that (R_c) holds between its cumulants. The result (2.2) provides a leading argument in the proofs of most of the lemmas and theorems of this paper. Certain results of a similar nature for the uni-variate case are available in Bolger and Harkness [3].

LEMMA 2.1. *If $m(\mathbf{t}, \boldsymbol{\theta})$ is degenerate for some $\boldsymbol{\theta}_0 \in (\mathbf{a}, \mathbf{b})$, then $m(\mathbf{t}, \boldsymbol{\theta})$ is degenerate for all $\boldsymbol{\theta} \in (\mathbf{a}, \mathbf{b})$.*

COROLLARY. *If $\lambda_{r_1,r_2,\cdots r_s}(\boldsymbol{\theta})$ is positive for at least one $\boldsymbol{\theta}_0 \in (a, b)$, then $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta})$ is positive for all $\boldsymbol{\theta} \in (\mathbf{a}, \mathbf{b})$.*

LEMMA 2.2. *If $m(\mathbf{t}, \boldsymbol{\theta})$ corresponds to normal distribution for some $\boldsymbol{\theta}_0 \in (\mathbf{a}, \mathbf{b})$, then $m(\mathbf{t}, \boldsymbol{\theta})$ corresponds to normal distribution for all $\boldsymbol{\theta} \in (\mathbf{a}, \mathbf{b})$.*

3. Statistical independence of the components.

THEOREM 3.1. *Let $\mathbf{x} = (x_1, x_2)'$ be a bivariate exponential-type random vector with parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$, then x_1, x_2 are independent if and only if they are uncorrelated.*

PROOF. If x_1, x_2 are uncorrelated, we get their covariance $\mu_{11}(\boldsymbol{\theta}) = \lambda_{11}(\boldsymbol{\theta}) = 0$. Since the cumulants follow (R_c), $\lambda_{11}(\boldsymbol{\theta}) = 0$ implies that $\lambda_{r_1,r_2}(\boldsymbol{\theta}) = 0$ for all $r_1, r_2 \geqq 1$. Therefore the cgf

$$k(t_1, t_2) = \textstyle\sum_{r=1}^\infty \sum_{r_1+r_2=r} \lambda_{r_1,r_2}(\boldsymbol{\theta})i^r t_1^{r_1} t_2^{r_2}(r_1\,!\,r_2\,!)^{-1}$$

becomes

$$\begin{aligned} k(t_1, t_2) &= \textstyle\sum_{r_1=1}^\infty \lambda_{r_1,0}(\boldsymbol{\theta})i^{r_1}t_1^{r_1}(r_1\,!)^{-1} + \sum_{r_2=1}^\infty \lambda_{0,r_2}(\boldsymbol{\theta})i^{r_2}t_2^{r_2}(r_2\,!)^{-1} \\ &= k(t_1, 0) + k(0, t_2), \end{aligned}$$

from which follows the theorem.

THEOREM 3.2. *The random variables $x_1, x_2, \cdots, x_s$ following jointly an s-variate exponential-type distribution are mutually independent if and only if they are pairwise independent.*

PROOF. The cgf of the random vector $\mathbf{x}$ is given as

$$k(\mathbf{t}) = \sum_{r=1}^{\infty} \sum_{r_1+\cdots+r_s=r} \lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) i^r t_1^{r_1} t_2^{r_2} \cdots t_s^{r_s} (r_1!\, r_2! \cdots r_s!)^{-1}.$$

The pairwise independence of the variables $x_1, x_2, \cdots, x_s$ implies that $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) = 0$ where $r_i = 0$ or 1 for $i = 1, 2, \cdots, s$ such that $\sum_1^s r_i = 2$. Therefore, by (R_c) among the cumulants, we have $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) = 0$ for all $r_1, r_2, \cdots, r_s \geqq 0$ such that at least two of $r_1, r_2, \cdots, r_s$ are nonzero and that $\sum_1^s r_i \geqq 2$.

$$\therefore \quad k(\mathbf{t}) = \sum_{j=1}^{s} [\sum_{r_j \geqq 1} \lambda_{0,0,\cdots,r_j,0,\cdots,0}(\boldsymbol{\theta}) i^{r_j} t_j^{r_j} (r_j!)^{-1}].$$

The fact, that $(\partial^k/\partial\theta_i^k)\lambda_{0,0,\cdots r_j,0,\cdots,0}(\boldsymbol{\theta}) = 0$ for all $i \neq j$ and all $k \geqq 1$, implies that $\lambda_{0,0,\cdots r_j,0,\cdots,0}(\boldsymbol{\theta}) = \lambda_{0,0,\cdots r_j,0,\cdots,0}(\theta_j)$ for all $j = 1, 2, \cdots, s$.

$$\therefore \quad k(\mathbf{t}) = \sum_{j=1}^{s} [\sum_{r_j \geqq 1} \lambda_{0,0,\cdots r_j,0,\cdots,0}(\theta_j) i^{r_j} t_j^{r_j} (r_j!)^{-1}]$$

$$= k(t_1, 0, \cdots, 0) + k(0, t_2, 0, \cdots, 0) + \cdots + k(0, \cdots, 0, t_s) \tag{3.1}$$

$$\therefore \quad k(\mathbf{t}) = \sum_1^s k_i(t_i)$$

where $k_i(t_i)$ denotes the cgf of a univariate exponential-type distribution with parameter θ_i. From this follows the statement of the theorem.

4. Distributions derived from an s-variate exponential-type distribution. Let $\boldsymbol{\mu}$ and Λ denote the mean vector and the variance co-variance matrix of the random vector $\mathbf{x}$. Therefore $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_s)'$ and $\Lambda = (\lambda_{ij})$ with $\mu_i = \lambda_{0,\cdots,0,1,\cdots,0}(\boldsymbol{\theta})$ (with 1 in the ith position) $\lambda_{ij} = \lambda_{0,\cdots,0,1,0,\cdots,0,1,0,\cdots,0}(\boldsymbol{\theta})$ for $i \neq j$ (with 1 in the ith and jth positions) and $\lambda_{ii} = \lambda_{0,\cdots,0,2,0,\cdots,0}(\boldsymbol{\theta})$ for $i, j = 1, 2, \cdots, s$ (with 2 in the ith position). Let the vector $\mathbf{x}$ be partitioned into vectors $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)'$ where $\mathbf{x}_1 = (x_1, \cdots, x_p)'$, $p < s$, and $\mathbf{x}_2 = (x_{p+1}, \cdots, x_s)'$.

Let the corresponding partitions of $\boldsymbol{\mu}$ and Λ be $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)'$ and

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$

THEOREM 4.1. *If a random vector $\mathbf{x}$ with* pv $\boldsymbol{\theta}$ *follows s-variate exponential-type distribution with the mean vector $\boldsymbol{\mu}$ and the variance co-variance matrix Λ, then the marginal distribution of any set of components of $\mathbf{x}$ is again multivariate exponential-type with the mean vector and the variance co-variance matrix obtained by taking proper components of $\boldsymbol{\mu}$ and Λ.*

PROOF. Let $\mathbf{x}$, $\boldsymbol{\mu}$ and Λ be partitioned as indicated in the beginning of the section.

If $k(t_1, t_2, \cdots, t_s)$ denotes the cgf of $\mathbf{x}$, then the cgf of $\mathbf{x}_1$ is obtained as $k(t_1, t_2, \cdots, t_p, 0, \cdots, 0)$.

$$\therefore \quad k(t_1, t_2, \cdots, t_p, 0, \cdots, 0) = \sum_{r=1}^{\infty} [\sum_{r_1+\cdots+r_p=r} \lambda_{r_1,\cdots,r_p,0,\cdots,0}(\boldsymbol{\theta}) i^r t_1^{r_1} \cdots t_p^{r_p} (r_1! \cdots r_p!)^{-1}.$$

The cumulants $\lambda_{r_1,r_2,\cdots,r_p,0,\cdots,0}(\boldsymbol{\theta})$ obviously follow (R_c); therefore, the cgf $k(t_1, t_2, \cdots, t_p, 0, \cdots, 0)$ corresponds to the p-variate exponential-type marginal distribution of $\mathbf{x}_1$. It can be seen, by a similar argument, that the rest of the theorem holds true.

THEOREM 4.2. *If* $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)'$ *follows s-variate exponential-type distribution with* pv $\boldsymbol{\theta}$, *then the conditional distribution of* $\mathbf{x}_1$ *given* $\mathbf{x}_2$ *is p-variate exponential-type with* pv $\boldsymbol{\theta}_1$.

THEOREM 4.3. *Let* $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)'$ *have an s-variate exponential-type distribution with the mean vector* $\boldsymbol{\mu}$ *and the variance-co-variance matrix* Λ. *Then* $\mathbf{x}_1$ *is independent of* $\mathbf{x}_2$ *if and only if each covariance of a variable from* $\mathbf{x}_1$ *and a variable from* $\mathbf{x}_2$ *is zero.*

PROOF. Necessity of the condition is obvious. The sufficiency can be proved as follows. By assumption we get $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta}) = 0$ where $r_i = 0$ or 1, for $i = 1, 2, \cdots, s$ such that $\sum_1^p r_i = 1$ and $\sum_{p+1}^s r_i = 1$. Since the cumulants follow (R_c) the cgf of $\mathbf{x}$ becomes

$$\begin{aligned} k(\mathbf{t}) &= k(t_1, \cdots, t_p, 0, \cdots, 0) + k(0, \cdots, 0, t_{p+1}, \cdots, t_s) \\ &= k_1(\mathbf{t}_1) + k_2(\mathbf{t}_2) \end{aligned}$$

where $k_1(\mathbf{t}_1)$ and $k_2(\mathbf{t}_2)$ denote the cgf's of the marginal distributions of $\mathbf{x}_1$ and $\mathbf{x}_2$.

COROLLARY. *If* $\mathbf{x}$ *follows s-variate exponential-type distribution with* pv $\boldsymbol{\theta}$ *and if a set of components of* $\mathbf{x}$ *is uncorrelated with the other components, then the marginal distribution of the set is multivariate exponential-type with the parameter vector obtained by taking proper components of* $\boldsymbol{\theta}$.

5. Characterization of a class of exponential-type distributions. In this section, first we obtain the characterization of a class of univariate exponential-type distributions and then extend the argument to the s-variate case. We may remark that we establish here a conjecture made by C. R. Rao [12] while studying certain discrete models arising out of methods of ascertainment.

THEOREM 5.1. *Let* $\lambda_1(\theta)$ *and* $\lambda_2(\theta)$ *denote the first two cumulants of an exponential-type random variable* x *with the parameter* θ, *then the relation* $\lambda_2(\theta)/\lambda_1(\theta) = 1/(1 + de^{\theta})$, *where d is some real number, holds true if and only if the corresponding distribution is either binomial, Poisson or negative binomial. Further, the distribution is binomial, Poisson or negative binomial according as d is positive, zero or negative.*

PROOF. Let the pf of x be given by

$$f(x, \theta) = h(x) \exp \{x\theta - q(\theta)\}. \tag{5.1}$$

The characterizing relation can, therefore, be written as

$$(d^2/d\theta^2)q(\theta)/(d/d\theta)q(\theta) \equiv q''(\theta)/q'(\theta) = 1/(1 + de^{\theta}). \tag{5.2}$$

We note that (5.2) holds true for binomial, Poisson and negative binomial with $d = +1$, 0 and -1, respectively.

Let $q''(\theta)/q'(\theta) = 1/(1 + de^\theta)$ be given. We consider two cases, $d = 0$ and $d \neq 0$.

CASE 1. If $d \neq 0$, $q(\theta) = (k/d) \log (1 + de^\theta) + c$ where $k > 0$ and c are constants. Therefore the ch.f. will be

$$(5.3) \qquad \begin{aligned} \varphi(t, \theta) &= e^{i(k/2d)\cdot t}(pe^{\frac{1}{2}it} + qe^{-\frac{1}{2}it})^{k/d} \\ &= \varphi_1(t)\cdot\varphi_2(t) \end{aligned}$$

where $p = de^\theta/(1 + de^\theta)$ and $q = 1/(1 + de^\theta)$ such that $p + q = 1$, and $\varphi_1(t) = e^{i(k/2d)\cdot t}$ denotes the ch.f. corresponding to a degenerate distribution. Following Lukacs [8], we note that

(a) When p and q both are positive, $\varphi_2(t)$ is a ch.f. if and only if k/d is a positive integer. Therefore d is positive and $\varphi(t, \theta)$ denotes the ch.f. of a binomial distribution.

(b) When p and q are of opposite signs, $\varphi_2(t)$ is a ch.f. if and only if k/d is a negative real number. Therefore d is negative and $\varphi(t, \theta)$ denotes the ch.f. of a negative binomial distribution.

CASE 2. If $d = 0$, $q(\theta) = ke^\theta + c'$ where $k > 0$ and c' are constants. Therefore the ch.f. $\varphi(t, \theta) = \exp\{-ke^\theta + ke^{\theta+it}\}$ which is the ch.f. of a Poisson distribution.

We, now, obtain the multivariate analogue of the result proved in Theorem 5.1.

THEOREM 5.2. *Let* $\mathbf{x}$ *be an s-dimensional exponential-type random vector with* pv $\boldsymbol{\theta}$. *Then the relation,*

$$\sum_{i=1}^{s} \sum_{j=1}^{s} \lambda_{ij}/\sum_1^s \mu_i = 1/(1 + d(\sum_1^s e^{\theta_i}))$$

where d is some finite real number, holds true if and only if the corresponding distribution is multinomial, multiple Poisson or multivariate negative binomial. Further, it is multinomial, multiple Poisson or multivariate negative binomial according as d is positive, zero or negative, respectively.

PROOF. Follows on the lines similar to those of Theorem 5.1.

6. Characterization of multivariate normal distribution. An exponential-type random vector with pv $\boldsymbol{\theta}$ is said to follow s-variate Normal distribution if its ch.f. is given by

$$\varphi(\mathbf{t}, \boldsymbol{\theta}) = \exp\{i\mathbf{t}'\boldsymbol{\mu} - \tfrac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\}$$

where the components μ_i of $\boldsymbol{\mu}$ are functions of $\boldsymbol{\theta}$ and $\Sigma = (\sigma_{ij})$, with σ_{ij} constants, is positive definite or semidefinite. Therefore the cgf of $\mathbf{x}$ is $k(\mathbf{t}, \boldsymbol{\theta}) = i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}$.

THEOREM 6.1. *An s-variate exponential-type distribution is s-variate normal if and only if all the cumulants of order* 3 *are zero.*

PROOF. Straight forward.

REMARK. We define an s-variate distribution to be symmetric if all the cumulants of odd order $\geqq 3$ are zero. We note, here, that in the class of s-variate exponential-type distributions s-variate normal distribution is the only symmetric distribution.

THEOREM 6.2. *Let* $f(\mathbf{x}, \boldsymbol{\theta})$ *denote the* (pf) *of a bivariate exponential-type random vector* x *and* pv $\boldsymbol{\theta}$. *Then* $f(x, \theta)$ *is the* pf *of a bivariate normal distribution if and only if following conditions are satisfied:*

(i) *The regression of one of the two variables on the other is linear.*

(ii) *The marginal distribution of any one of the two variables is normal.*

PROOF. See the proof of the next theorem.

THEOREM 6.3. *The* pf $f(\mathbf{x}, \boldsymbol{\theta})$ *of an s-variate exponential-type random vector* $\mathbf{x}$, *with* pv $\boldsymbol{\theta}$, *is the* pf *of an s-variate normal distribution if and only if following conditions are satisfied:*

(iii) *The regression of one of the variables on the rest is linear.*

(iv) *Rest of the* $(s - 1)$ *variables are distributed normally in pairs.*

PROOF. The necessity of the conditions is obvious. By condition (iii) we get

$$E(x_1 \mid x_2, x_3, \cdots, x_s) = \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_s x_s + \beta.$$

$\alpha_2, \cdots, \alpha_s$ are nonzero real numbers and β is some real number. In general putting $\beta = 0$ and taking the expectation wrt $x_2, x_3, \cdots, x_s$, we get

$$\lambda_{1,0,\cdots,0}(\boldsymbol{\theta}) = \alpha_2\lambda_{0,1,0,\cdots,0}(\boldsymbol{\theta}) + \alpha_3\lambda_{0,0,1,0,\cdots,0}(\boldsymbol{\theta}) + \cdots + \alpha_s\lambda_{0,\cdots,0,1}(\boldsymbol{\theta}). \tag{6.1}$$

By condition (iv), we know that

$$\lambda_0, r_2, \cdots, r_s(\boldsymbol{\theta}) = 0 \quad \text{for} \quad \textstyle\sum_2^s r_i \geqq 3 \quad \text{with} \quad r_2, \cdots, r_s \geqq 0. \tag{6.2}$$

It can be verified, with the help of (6.1), (6.2) and the fact that the cumulants $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta})$ follow (R_c), that all the cumulants $\lambda_{r_1,r_2,\cdots,r_s}(\boldsymbol{\theta})$ of order $\geqq 3$ are zero. The statement of the theorem follows from Theorem 6.1.

THEOREM 6.4. *The* pf $f(\mathbf{x}, \boldsymbol{\theta})$ *of a bivariate exponential-type random vector* $\mathbf{x}$ *with* pv $\boldsymbol{\theta}$ *is the* pf *of a bivariate normal distribution if and only if following conditions are satisfied:*

(v) *The regression of one of the two variables on the other is linear.*

(vi) *The distribution of* $x_1 + x_2$ *is univariate normal.*

PROOF. The necessity of the conditions is obvious. To prove their sufficiency, we note that condition (v) implies that the cumulants $\lambda_{r_1,r_2}(\boldsymbol{\theta})$ of $\mathbf{x}$ satisfy $\lambda_{m+1,n}(\boldsymbol{\theta}) = \alpha\lambda_{m,n+1}(\boldsymbol{\theta})$ where α is the regression coefficient. Now let

$$k(t_1, t_2) = \textstyle\sum_{r=1}^{\infty} [\sum_{r_1+r_2=r} \lambda_{r_1,r_2}(\boldsymbol{\theta}) i^r t_1^{r_1} t_2^{r_2} (r_1 ! \, r_2 !)^{-1}]$$

denote the cgf of $\mathbf{x}$. Also, the cgf $k(t)$ of $x_1 + x_2$ is obtained from $k(t_1, t_2)$ by substituting $t_1 = t_2 = t$.

$$\therefore \quad k(t) = \textstyle\sum_{r=1}^{\infty} [\sum_{r_1+r_2=r} \lambda_{r_1,r_2}(\boldsymbol{\theta}) i^r t^r (r_1 ! \, r_2 !)^{-1}].$$

By condition (vi), we know that the cumulant of order 2 of $x_1 + x_2$ is positive and all the cumulants of $x_1 + x_2$ of order $\geqslant 3$ are zero.

$$\therefore \quad \lambda_{30}(\boldsymbol{\theta}) + 3\lambda_{21}(\boldsymbol{\theta}) + 3\lambda_{12}(\boldsymbol{\theta}) + \lambda_{03}(\boldsymbol{\theta}) = 0,$$

$$\therefore \quad (1 + a)^3\lambda_{30}(\boldsymbol{\theta}) = 0 \qquad \text{where} \quad a \neq -1.$$

Therefore $\lambda_{30}(\boldsymbol{\theta}) = 0$ and consequently all the cumulants of order 3 of $f(\mathbf{x}, \boldsymbol{\theta})$ are zero. Hence, by Theorem 6.1, it follows that $f(\mathbf{x}, \boldsymbol{\theta})$ is the pf of a bivariate normal distribution.

7. Infinitely divisible distributions. Dwass and Teicher [4] have discussed the following canonical representation of an id ch.f. as follows:

The function $\varphi(\mathbf{t}) = \varphi(t_1, t_2, \cdots, t_s)$ is an id ch.f. if and only if it can be written as

$$(7.1) \quad \varphi(\mathbf{t}) = \exp \{i\mathbf{t}'\boldsymbol{\gamma} - \tfrac{1}{2}\mathbf{t}'\Sigma\mathbf{t} + \textstyle\int_{R_s}(e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x}(1 + |x|^2)^{-1}(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x})\}$$

where $\boldsymbol{\gamma}$ and $\mathbf{t}$ denote the s-dimensional column vectors with the components $\gamma_1, \gamma_2, \cdots, \gamma_s$ and $t_1, t_2, \cdots, t_s$, real numbers. Σ denotes the positive definite or semidefinite matrix and $\mu_G(A) = \int_A dG(\mathbf{x})$ is a finite Lebesgue Stieltjes measure on the Borel sets of R_s such that $\mu_G(A) = 0$ for $A = \{\mathbf{x} \mid |x| = 0\}$. $|x|$ denotes the Euclidean length of the s-dimensional vector $\mathbf{x}$. The representation (7.1) is unique. The first factor in (7.1) is obviously the ch.f. of an s-variate normal distribution and the second factor corresponds to the ch.f. of an s-variate Poisson type distribution.

P. Lévy [7] has shown that every id s-dimensional random vector $\mathbf{x} \sim \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_2$ represents the discontinuous part of $\mathbf{x}$ and $\mathbf{x}_1$ represents the continuous part and depends upon Gaussian law, besides being independent of $\mathbf{x}_2$. Therefore the ch.f. of an id purely discrete random vector $\mathbf{x}$ can be written as

$$(7.2) \quad \varphi(\mathbf{t}) = \exp \{\textstyle\int_{R_s} (e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x}(1 + |x|^2)^{-1}(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x})\}.$$

We shall now obtain the canonical representation of the ch.f. of id, purely discrete bivariate random vector, whose moments of second order are all finite. Then we extend the result to the s-variates, $s > 2$.

THEOREM 7.1. *The function* $\varphi(t_1, t_2)$ *is the* ch.f. *of a bivariate, purely discrete,* id *random vector* $\mathbf{x}$ *with all the moments of second order finite if and only if it can be written as*

$$\varphi(t_1, t_2) = \exp \{it'\beta + \textstyle\int_{R_2} (e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x})\, dF(\mathbf{x})/(x_1 + x_2)^2\}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ *with* β_1, β_2 *real.* $\mu_F(A) = \int_A dF(\mathbf{x})$ *is a finite Lebesgue Stieltjes measure on the Borel sets of* R_2 *such that* $\mu_F(A) = 0$ *for* $A = \{\mathbf{x} \mid |x| = 0\}$. *The representation is unique.*

PROOF. By assumption the moments of second order of $\mathbf{x}$ are finite. Therefore the cumulants $\lambda_{20}, \lambda_{11}, \lambda_{02}$ are finite and the cgf $k(t_1, t_2)$ can be partially differentiated twice. Assuming that $\varphi(t_1, t_2)$ has the representation (7.2) with $s = 2$, we form the second central difference quotient

$$\Delta_2^{h_1}k(0, 0)/(2h_1)^2$$

wrt t_1 and conclude that

$$(7.3) \qquad \lim_{h_1 \to 0} |\Delta_2^{h_1}k(0, 0)/(2h_1)^2| < \infty.$$

Noting the fact that $\varphi(t_1, t_2)$ admits the representation (7.2) and using (7.3), we obtain

$$\int_{R_2} x_1^2(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x}) < \infty.$$

Similarly, using the fact that λ_{11} and λ_{02} are finite, we obtain

$$\int_{R_2} x_1x_2(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x}) < \infty \quad \text{and} \quad \int_{R_2} x_2^2(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x}) < \infty.$$

Thus, it can be seen that the integrals,

$$\int_{R_2}(1 + |x|^2)\, dG(\mathbf{x}) \qquad \text{and} \qquad \int_{R_2}(x_1^2 + 2x_1x_2 + x_2^2)(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x}),$$

are finite. Also, the integrals $\int_{R_2} x_1\, dG(\mathbf{x})$ and $\int_{R_2} x_2\, dG(\mathbf{x})$, are finite. Therefore writing $i\mathbf{t}'\boldsymbol{\beta} = \int_{R_2} i\mathbf{t}'\mathbf{x}\, dG(\mathbf{x})$, where β_1, β_2 are real numbers, the representation (7.2) of $\varphi(t_1, t_2)$ can be written as

$$\varphi(t_1, t_2) = \exp\{i\mathbf{t}'\boldsymbol{\beta} + \int_{R_2}(e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x})\, dF(\mathbf{x})/(x_1 + x_2)^2\}$$

where $\mu_F(A) = \int_A dF(\mathbf{x}) = \int_A (x_1^2 + 2x_1x_2 + x_2^2)(1 + |x|^2)|x|^{-2}\, dG(\mathbf{x})$ is a bounded, non-negative, countably additive set function which is uniquely determined by $\varphi(t_1, t_2)$ and is such that

$$\mu_F(A) = 0 \qquad \text{for} \qquad A = \{\mathbf{x} \mid |x| = 0\}.$$

REMARK 1. The function $\varphi(\mathbf{t})$ is the ch.f. of an s-dimensional, purely discrete, random vector, whose moments of second order are all finite, if and only if it can be written in the form

$$(7.4) \qquad \varphi(\mathbf{t}) = \exp\{i\mathbf{t}'\boldsymbol{\beta} + \int_{R_s}(e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x})\, dF(\mathbf{x})/(\textstyle\sum_1^s x_i)^2$$

where $\boldsymbol{\beta}$ is the s-dimensional column vector with β_i real for $i = 1, 2, \cdots, s$ and $\mu_F(A)$ is a finite Lebesgue Stieltjes measure on the Borel sets of R_s such that $\mu_F(A) = 0$ for $A = \{\mathbf{x} \mid |x| = 0\}$ and that it is determined uniquely by $\varphi(\mathbf{t})$.

REMARK 2. Noting the fact that $\int_{R_s} (1 + |x|^2)\, dG(\mathbf{x})$ is finite, we define $\mu_k(A) = \int_A dk(\mathbf{x}) = \int_A (1 + |x|^2)\, dG(\mathbf{x})$. Therefore the canonical representation of the ch.f. of an id, purely discrete random vector, with all the moments of second order finite, becomes

$$\varphi(\mathbf{t}) = \exp\{i\mathbf{t}'\boldsymbol{\beta} + \int_{R_s} e^{i\mathbf{t}'\mathbf{x}} - 1 - i\mathbf{t}'\mathbf{x})\, dk(\mathbf{x})/|x|^2\}$$

where $\mu_k(A)$ is a finite Lebesgue Stieltjes measure on the Borel sets of R_s such that $\mu_k\{\mathbf{x} \mid |\mathbf{x}| = 0\}$ and such that it is uniquely determined by $\varphi(\mathbf{t})$.

We note that this form is analogous to the Kolmogorov cononical representa-discussed by Lukacs [9], p. 90.

THEOREM 7.2. *If* $\varphi(\mathbf{t})$ *is the* ch.f. *of an* id, *purely discrete random vector* $\mathbf{x}$, *whose moments of second order are all finite, then*

$$-[\textstyle\sum \lambda_{r_1,r_2,\cdots,r_s}]^{-1}[\sum_1^s \partial^2/\partial t_i^2 + \sum_{i\neq j} \partial^2/\partial t_i\partial t_j] \log \varphi(\mathbf{t})$$

is also a ch.f. $\sum \lambda_{r_1,r_2,\cdots,r_s}$ *denotes the summation over all the cumulants of* $\mathbf{x}$ *of second order.*

PROOF. Follows immediately from the canonical representation (7.4) of $\varphi(\mathbf{t})$.

8. Characterization of bivariate Poisson-type distribution. The ch.f. of a bivariate Poisson vector $\mathbf{x}$ with parameters α_1, α_2, α_{12} is given by

$$\varphi_{\mathbf{x}}(\mathbf{t}) = \exp\{\alpha_1(e^{it_1} - 1) + \alpha_2(e^{it_2} - 1) + \alpha_{12}(e^{it_1+it_2} - 1)\}.$$

For $\alpha_1 = \alpha_2 = 0$, $\varphi_{\mathbf{x}}(\mathbf{t}) = \exp\{\alpha_{12}(e^{it_1+it_2} - 1)\}$ represents the ch.f. of two Poisson random variables with means, variances and co-variances equal to α_{12}. Consider, in such a case, the linear transformation $\mathbf{y} = A\mathbf{x} - \mathbf{b}$ where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

with a_1, a_2, b_1, b_2 real numbers such that $a_1a_2 > 0$.

$$(8.1) \qquad \therefore \quad \varphi_{\mathbf{y}}(\mathbf{t}) = \exp\{\alpha_{12}(e^{it_1a_1+it_2a_2} - i\mathbf{t}'\mathbf{c} - 1)\}$$

where

$$\mathbf{c} = \alpha_{12}^{-1}\cdot\mathbf{b}.$$

We define $\mathbf{y}$ to be bivariate Poisson-type if its ch.f. is given by (8.1). The mgf of $\mathbf{y}$ will be

$$m(\mathbf{t}) = \exp\{\alpha_{12}(e^{t_1a_1+t_2a_2} - \mathbf{t}'\mathbf{c} - 1)\}.$$

LEMMA 8.1. *If $m(\mathbf{t}, \boldsymbol{\theta})$ is the* mgf *of a bivariate Poisson-type distribution, for $\boldsymbol{\theta}_0 \in (\mathbf{a}, \mathbf{b})$, then $m(\mathbf{t}, \boldsymbol{\theta})$ is the* mgf *of a bivariate Poisson-type distribution for all $\boldsymbol{\theta} \in (\mathbf{a}, \mathbf{b})$.*

PROOF. Given

$$m(\mathbf{t}, \boldsymbol{\theta}_0) = \exp\left\{(\lambda_{11}/a_1a_2)(e^{t_1a_1+t_2a_2} - 1) + \mathbf{t}'\begin{pmatrix}\lambda_{10} - \lambda_{11}/a_2 \\ \lambda_{01} - \lambda_{11}/a_1\end{pmatrix}\right\}$$

where the cumulants λ_{11}, λ_{10}, λ_{01} are functions of $\boldsymbol{\theta}_0$ and a_1, a_2 are nonzero real numbers. We also know that

$$\lambda_{r_1,r_2(\boldsymbol{\theta}_0)} = a_1^{r_1-1}a_2^{r_2-1}\lambda_{11}(\boldsymbol{\theta}_0) \neq 0 \quad \text{for all} \quad r_1, r_2 \geqq 0$$

such that $r_1 + r_2 \geqq 2$. Therefore if we show that

$$\lambda_{r_1,r_2}(\boldsymbol{\theta}) = a_1^{r_1-1}a_2^{r_2-1}\lambda_{11}(\boldsymbol{\theta}_0) \quad \text{for all} \quad \boldsymbol{\theta} \in (\mathbf{a}, \mathbf{b}),$$

then the statement of the lemma will follow. This can be proved by induction.

LEMMA 8.2. *Let $\varphi(\mathbf{t}, \boldsymbol{\theta}_0)$ denote the* ch.f. *of a bivariate, purely discrete,* id *random vector $\mathbf{x}$ and let the following relation*

$$\lambda_{r_1,r_2}(\boldsymbol{\theta}_0) = a_1^{r_1-1}\cdot a_2^{r_2-1}\lambda_{11}(\boldsymbol{\theta}_0) \neq 0 \quad for \quad 2 \leqq r_1 + r_2 \leqq 4,$$

hold true; then $\varphi(\mathbf{t}, \boldsymbol{\theta}_0)$ represents the ch.f. *of a bivariate Poisson-type random vector.*

PROOF. We know by hypothesis that

$$(9.2) \quad \bar{\varphi}(\mathbf{t}, \boldsymbol{\theta}_0) = -(\boldsymbol{\xi}\Lambda\boldsymbol{\xi}(\boldsymbol{\theta}_0))^{-1}[\partial^2/\partial t_1^2 + 2\partial^2/\partial t_1\partial t_2 + \partial^2/\partial t_2^2] \log \varphi(\mathbf{t}, \boldsymbol{\theta}_0)$$

where

$$\xi'\Lambda\xi(\theta_0) = (1 \quad 1)\begin{pmatrix}\lambda_{20}(\theta_0) & \lambda_{11}(\theta_0)\\ \lambda_{11}(\theta_0) & \lambda_{02}(\theta_0)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

represents the ch.f. of a certain distribution function $\bar{F}(\mathbf{x}, \theta_0)$.

Let $\mathbf{\mu}$ and Σ denote the mean vector and the variance co-variance matrix, respectively of $\bar{F}(\mathbf{x}, \theta_0)$. It can be verified

$$\lambda_{r_1,r_2}(\theta_0) = a_1^{r_1-1}a_2^{r_2-1}\lambda_{11}(\theta_0), \qquad 2 \leqq r_1 + r_2 \leqq 4,$$

that

$$\mathbf{\mu} = \begin{pmatrix}a_1\\a_2\end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}.$$

Thus $\bar{\varphi}(\mathbf{t}, \theta_0)$ represents the ch.f. of a degenerate distribution with one mass point $\mathbf{a} = (a_1, a_2)'$; and the lemma follows after same argument.

THEOREM 8.1. *If* $\varphi(\mathbf{t}, \theta_0)$ *is the* ch.f. *of a bivariate, purely discrete,* id *random vector* $\mathbf{x}$ *whose cumulants follow the relation,* $\lambda_{r_1,r_2}(\theta_0) = a_1^{r_1-1}a_2^{r_2-1}\lambda_{11}(\theta_0) \neq 0$ *for* $2 \leqq r_1 + r_2 \leqq 4$, *then* $\varphi(\mathbf{t}, \theta)$ *is the* ch.f. *of a bivariate Poisson-type random vector for all* $\theta \varepsilon (\mathbf{a}, \mathbf{b})$.

PROOF. Follows from the Lemmas 8.1 and 8.2.

THEOREM 8.2. *If* $\lambda_{r_1,r_2}(\theta) = a_1^{r_1-1}\cdot a_2^{r_2-1}\lambda_{11}(\theta) \neq 0$ *for* $(r_1, r_2) = (2, 0), (0, 2)$ *and* $(2, 2)$ *and for all* $\theta \varepsilon (\mathbf{a}, \mathbf{b})$, *where* $\lambda_{r_1,r_2}(\theta)$ *denote the cumulants of a bivariate exponential-type random vector* $\mathbf{x}$, *then* $\mathbf{x}$ *follows bivariate Poisson-type distribution.*

PROOF. We can show by induction that

$$\lambda_{r_1+1,r_2+1}(\theta) = a_1^{r_1}\cdot a_2^{r_2}\lambda_{11}(\theta) \quad \text{for} \quad r_1, r_2 \geqq 1,$$

$$\lambda_{r_1+1,0}(\theta) = a_1^{r_1}a_2^{-1}\lambda_{11}(\theta) \quad \text{for} \quad r_1 \geqq 1,$$

$$\lambda_{0,r_2+1}(\theta) = a_2^{r_2}a_1^{-1}\lambda_{11}(\theta) \quad \text{for} \quad r_2 \geqq 1.$$

$$\therefore \quad \lambda_{r_1,r_2}(\theta) = a_1^{r_1-1}\cdot a_2^{r_2-1}\lambda_{11}(\theta) \quad \text{for all} \quad r_1, r_2 \geqq 0$$

such that $r_1 + r_2 \geqq 2$. Therefore the cgf of $\mathbf{x}$ becomes

$$k(\mathbf{t}, \theta) = \lambda_{11}(\theta)(a_1a_2)^{-1}(e^{it'a} - 1) + it'\begin{pmatrix}\lambda_{10}(\theta) - \lambda_{11}(\theta)a_2^{-1}\\ \lambda_{01}(\theta) - \lambda_{11}(\theta)a_1^{-1}\end{pmatrix}.$$

Therefore $\varphi(\mathbf{t}, \theta) = \exp\{k(\mathbf{t}, \theta)\}$ determines the distribution of $\mathbf{x}$ to be bivariate Poisson-type.

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