The Annals of Mathematical Statistics
1968, Vol. 39, No. 4, 1202–1209

THE WILCOXON TWO-SAMPLE STATISTIC ON STRONGLY MIXING PROCESSES[1]

BY R. J. SERFLING

Florida State University

1. Introduction and summary. On the basis of independent samples $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_n\}$ with distributions F and G, respectively, the hypothesis that $F \equiv G$ may be tested. Given the functional forms $F(x_1, \cdots, x_m)$ and $G(y_1, \cdots, y_n)$ of the sampling distributions except for values of certain parameters, the likelihood ratio approach, for example, can be used. In this case it is not crucial to assume that the samples are *random*, i.e., that $F(x_1, \cdots, x_m) = F(x_1) \cdots F(x_m)$ and $G(y_1, \cdots, y_n) = G(y_1) \cdots G(y_n)$, although such a simplification is useful whenever realistic. However, the nonparametric treatment of the problem has relied heavily on the assumption of random samples. Yet if the samples arise as realizations of two stochastic processes, the assumption of randomness is not realistic except in the case of renewal processes. Thus it is desirable to extend the scope of established nonparametric procedures to more general applications.

The present paper deals with the Wilcoxon two-sample statistic. Among the desirable features of this statistic, when defined on independent *random* samples, is its asymptotically normal distribution, which for large samples facilitates a test of the hypothesis that $F \equiv G$ and a calculation of the power for any alternative (F, G). It shall be seen that these aspects are true also when the samples arise from stochastic processes belonging to a wide class, including strictly stationary strongly mixing processes.

Assume that the samples $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_n\}$ are independent of each other, but let the random variables within a sample be possibly dependent. Assume that the functions $F(\cdot)$ and $G(\cdot)$ are continuous. The hypothesis $H: F \equiv G$ may be tested (conservatively) by testing the hypothesis $H_0 : \gamma = 0$, where $\gamma = 2P\{Y > X\} - 1$.

A representation of the Wilcoxon two-sample statistic is the U-statistic with sign function as kernel,

$$U = (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} s(Y_j - X_i), \tag{1.1}$$

where $s(u) = -1, 0, 1$ according as $u < 0, =0, >0$. Since $Es(Y - X) = \gamma$, the statistic U affords a natural basis for testing H_0.

Under appropriate conditions, the statistic $Z = m^{\frac{1}{2}}(U - \gamma)$ has a limiting normal distribution with mean 0 and variance

$$A^2 = 4 \lim_{k\to\infty} k^{-1} \operatorname{Var}\left[\sum_{i=1}^{k} G(X_i)\right] + 4c \lim_{k\to\infty} k^{-1} \operatorname{Var}\left[\sum_{i=1}^{k} F(Y_i)\right], \tag{1.2}$$

Received September 6, 1967.

[1] This research was conducted at the Research Triangle Institute, Durham, N.C., under Contract No. NAS8-20386 with the National Aeronautics and Space Administration.

as m and $n \to \infty$ such that m/n has a limit $c \neq 0$. The main conclusions of this nature are given in Theorems 3.1 and 3.2. Some areas of application are indicated in Section 4. The business of dealing with the quantity A^2 is discussed in Section 5.

The limiting behavior of Z is obtained by consideration of a statistic asymptotically equivalent in distribution but more amenable to the direct application of central limit theory, an approach put forth by Hoeffding [3] in dealing with a wide class of U-statistics as defined on a single sample of mutually independent rv's. The present contribution adapts the method to a single, but important, (two-sample) U-statistic with dependence allowed within samples.

Define:

$$W = m^{-\frac{1}{2}}\sum_{i=1}^{m} [f_{10}(X_i) - \gamma] + m^{\frac{1}{2}}n^{-1}\sum_{j=1}^{n} [f_{01}(Y_j) - \gamma], \tag{1.3}$$

where $f_{10}(t) = Es(Y - t) = 1 - 2G(t)$ and $f_{01}(t) = Es(t - X) = 2F(t) - 1$. Since $Ef_{10}(X) = Ef_{01}(Y) = \gamma$, we have $EW = E(Z - W) = 0$. In Section 2 we find conditions such that $E(Z - W)^2 \to 0$, in which case it follows by Chebyshev's inequality that $(Z - W) \to 0$ in probability and hence that the statistics Z and W have the same limiting distribution (if any).

The application of central limit theory to W is through the sums $\sum_1^m f_{10}(X_i)$ and $\sum_1^n f_{01}(Y_j)$, or equivalently through $m^{-\frac{1}{2}}\sum_1^m G(X_i)$ and $n^{-\frac{1}{2}}\sum_1^n F(Y_j)$. If each of these independent normed sums has a limiting normal distribution, then W is asymptotically normal, as m and $n \to \infty$ such that $m/n \to c \neq 0$. Relevant central limit theorems for sums of dependent variables are utilized in Section 3.

2. Conditions under which $E(Z - W)^2 \to 0$. Let $g(x, y) = [s(y - x) - \gamma] - [f_{10}(x) - \gamma] - [f_{01}(y) - \gamma]$. Then

$$Z - W = m^{-\frac{1}{2}}n^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n} g(X_i, Y_j) \tag{2.1}$$

and

$$E(Z - W)^2 = m^{-1}n^{-2}\Delta, \tag{2.2}$$

with

$$\Delta = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=1}^{m}\sum_{b=1}^{n} \Delta(i, j, a, b) \tag{2.3}$$

and

$$\Delta(i, j, a, b) = Eg(X_i, Y_j)g(X_a, Y_b). \tag{2.4}$$

First we shall obtain some useful representations for $\Delta(i, j, a, b)$. Let F_{ia} denote the distribution function of (X_i, X_a) and G_{jb} the distribution function of (Y_j, Y_b).

LEMMA 2.1. *$\Delta(i, j, a, b)$ has the following representations*:

$$\begin{aligned}\Delta(i, j, a, b) \\ = 4\{E[F_{ia}(Y_j, Y_b) - F(Y_j)F(Y_b)] - \operatorname{Cov}[G(X_i), G(X_a)]\},\end{aligned} \tag{2.5}$$

$$(2.6)\quad \Delta(i, j, a, b) = 4\{E[G_{jb}(X_i, X_a) - G(X_i)G(X_a)] - \text{Cov}\,[F(Y_i), F(Y_b)]\}.$$

PROOF. Since $Eg(X, y) = 0 = Eg(x, Y)$,

$$(2.7)\quad \begin{aligned} \Delta(i, j, a, b) &= Es(Y_b - X_a)s(Y_j - X_i) - E(f_{10}(X_i) - \gamma)(f_{10}(X_a) - \gamma) \\ &\quad - E(f_{01}(Y_j) - \gamma)(f_{01}(Y_b) - \gamma) - \gamma^2 \\ &= Es(Y_b - X_a)s(Y_j - X_i) - 4\,\text{Cov}\,[G(X_i), G(X_a)] \\ &\quad - 4\,\text{Cov}\,[F(Y_j), F(Y_b)] - \gamma^2. \end{aligned}$$

Now

$$(2.8)\quad \begin{aligned} Es(Y_b - X_a)s(Y_j - X_i) &= Es[(Y_b - X_a)(Y_j - X_i)] \\ &= 2P\{(Y_b - X_a)(Y_j - X_i) > 0\} - 1 \\ &= 2P\{Y_b < X_a, Y_j < X_i\} \\ &\quad + 2\{PY_b > X_a, Y_j > X_i\} - 1. \end{aligned}$$

It can be found easily that

$$P\{Y_b < X_a, Y_j < X_i\} = P\{Y_b > X_a, Y_j > X_i\} - \gamma,$$

so, by (2.7) and (2.8), we obtain

$$(2.9)\quad \begin{aligned} \Delta(i, j, a, b) &= 4P\{Y_b > X_a, Y_j > X_i\} - 4\,\text{Cov}\,[G(X_i), G(X_a)] \\ &\quad - 4\,\text{Cov}\,[F(Y_j), F(Y_b)] - 4\theta^2. \end{aligned}$$

But

$$\begin{aligned} P\{Y_b > X_a, Y_j > X_i\} - \theta^2 &= EF_{ia}(Y_j, Y_b) - \theta^2 \\ &= E[F_{ia}(Y_j, Y_b) - F(Y_j)F(Y_b)] \\ &\quad + \text{Cov}\,[F(Y_j), F(Y_b)], \end{aligned}$$

so that (2.9) reduces to (2.5).

Now from (2.7) we see that $\Delta(i, j, a, b)$ does not change under the transformation $X_i \leftrightarrow Y_j$, $X_a \leftrightarrow Y_b$, $F \leftrightarrow G$. Hence (2.6) follows by analogy with (2.5).

REMARK. Formulas (2.5) and (2.6) show that weak dependence in *either* (not necessarily both) of the pairs (X_i, X_a) and (Y_j, Y_b) suffices for $\Delta(i, j, a, b)$ to be small. This fact is essential to the use of the following result.

LEMMA 2.2. *Suppose, for some non-negative function $r(k)$ with $\sum_0^\infty r(k) < \infty$, that*

$$|\Delta(i, j, a, b)| \leq r(\max\,[|i - a|, |j - b|])$$

for all (i, j, a, b). Then $\Delta = o(mn^2)$, as m and $n \to \infty$ such that m/n has a limit $c \neq 0$.

PROOF. We have

$$\begin{aligned}|\Delta| &\leqq \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=1}^{m} \sum_{b=1}^{n} r(\max [|i-a|, |j-b|]) \\ &= \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{b=1}^{n} r(|j-b|) + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=1}^{m} r(|i-a|) + mnr(0) \\ &\quad + \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=1, a\neq i}^{m} \sum_{b=1, b\neq j}^{n} r(\max [|i-a|, |j-b|]).\end{aligned}$$

The first two terms on the right are $O(mn)$ since $\sum_0^\infty r(k) < \infty$. The last term may be written

$$4 \textstyle\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{a=i+1}^{m} \sum_{b=j+1}^{n} r(\max [|i-a|, |j-b|]),$$

or, equivalently,

$$4 \textstyle\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{a=1}^{i} \sum_{b=1}^{j} r(\max [a, b]). \tag{2.10}$$

It remains to show that (2.10) is $o(mn^2)$. Now

$$\begin{aligned}\textstyle\sum_{j=1}^{n-1} \sum_{b=1}^{j} r(\max [a, b]) &= \textstyle\sum_{j=1}^{a} \sum_{b=1}^{j} r(\max [a, b]) \\ &\quad + \textstyle\sum_{j=a}^{n-1} \sum_{b=1}^{j} r(\max [a, b]) - ar(a) \\ &= \textstyle\sum_{j=1}^{a} jr(a) - ar(a) + \sum_{j=a}^{n-1} [(a-1)r(a) \\ &\quad + \textstyle\sum_{b=a}^{j} r(b)] \\ &= [\tfrac{1}{2} a(a+1) - a + (a-1)(n-a)]r(a) \\ &\quad + \textstyle\sum_{j=a}^{n-1} \sum_{b=a}^{j} r(b) \\ &= (n - \tfrac{1}{2}a)(a-1)r(a) + \textstyle\sum_{j=a}^{n-1} (n-j)r(j).\end{aligned}$$

Putting $R(a) = \sum_{k=a}^{\infty} r(k)$, it follows that

$$\textstyle\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{a=1}^{i} \sum_{b=1}^{j} r(\max [a, b]) \leqq mn \sum_{a=1}^{m} ar(a) + mn \sum_{a=1}^{m} R(a). \tag{2.11}$$

Since $R(a) \to 0$ as $a \to \infty$, the rightmost term of (2.11) is $o(m^2 n)$ as m and $n \to \infty$. Moreover, taking an integer-valued function $g(m)$ which satisfies $g(m) \to \infty$ and $g(m) = o(m)$ as $m \to \infty$, we may write

$$\begin{aligned}\textstyle\sum_{a=1}^{m} ar(a) &= \textstyle\sum_{a=1}^{g} ar(a) + \sum_{a=g+1}^{m} ar(a) \leqq g \sum_0^\infty r(k) + mR(g) \\ &= O(g) + mo(1) = o(m), \quad \text{as} \quad m \to \infty.\end{aligned}$$

Hence the right-hand side of (2.11) is $o(m^2 n)$ as m and $n \to \infty$, so (2.10) is $o(mn^2)$ as m and $n \to \infty$ such that $m/n \to c \neq 0$. This completes the proof.

By Lemmas 2.1 and 2.2, along with (2.2), the following result holds.

THEOREM 2.1. *Suppose that for some non-negative function* $h(k)$ *with* $\sum_0^\infty h(k) < \infty$,

$$|E[F_{ia}(Y_j, Y_b) - F(Y_j)F(Y_b)] - \operatorname{Cov} [G(X_i), G(X_a)]| \leqq h(|i-a|) \tag{2.12}$$

and

$$|E[G_{jb}(X_i, X_a) - G(X_i)G(X_a)] - \operatorname{Cov} [F(Y_j), F(Y_b)]| \leqq h(|j-b|). \tag{2.13}$$

Then Z and W have the same limiting distribution (if any) as m and $n \to \infty$ such that $m/n \to c \neq 0$.

3. Strongly mixing stochastic processes. Following Ibragimov [4], let us consider two notions of regularity for the dependence in a sequence $\{\xi_i\}_{-\infty}^{\infty}$. In defining them, let $\mathfrak{M}_a^b$ denote the σ-algebra generated by events of the form $\{(\xi_{i_1}, \cdots, \xi_{i_k}) \varepsilon E\}$, where $-\infty \leqq a - 1 < i_1 < \cdots < i_k < b + 1 \leqq \infty$ and E is a k-dimensional Borel set.

CONDITION (I). For any event $B \varepsilon \mathfrak{M}_{a+k}^{\infty}$, with probability 1,

$$|P(B|\mathfrak{M}_{-\infty}^{a}) - P(B)| \leqq \phi(k) \downarrow 0 \qquad (k \to \infty). \tag{3.1}$$

CONDITION (II). For any events $A \varepsilon \mathfrak{M}_{-\infty}^{a}$ and $B \varepsilon \mathfrak{M}_{a+k}^{\infty}$,

$$|P(AB) - P(A)P(B)| \leqq \alpha(k) \downarrow 0 \qquad (k \to \infty). \tag{3.2}$$

The latter condition is known as *strong mixing* or *uniform mixing*. Ibragimov shows [4] that condition (I) implies condition (II) with $\alpha(k) \leqq \phi(k)$. A special case of these conditions is *m-dependence*, given by (II) with $\alpha(k) = 0$ for $k > m$.

LEMMA 3.1. *If the sequences $\{X_i\}_{-\infty}^{\infty}$ and $\{Y_i\}_{-\infty}^{\infty}$ each satisfy regularity Condition (II) with $\sum \alpha(k) < \infty$, then the statistics Z and W have the same limiting distribution (if any) as m and $n \to \infty$ such that $m/n \to c \neq 0$.*

PROOF. We shall show that the conditions of Theorem 2.1 hold with $h(k) = 5\,\alpha(k)$. Applying condition (II) to the events $\{X_i \leqq s\}$ and $\{X_a \leqq t\}$, we obtain

$$|F_{ia}(s, t) - F(s)F(t)| \leqq \alpha(|i - a|). \tag{3.3}$$

Since the right-hand side of (3.3) does not depend upon (s, t), we have, for any choice of (j, b),

$$E|F_{ia}(Y_j, Y_b) - F(Y_j)F(Y_b)| \leqq \alpha(|i - a|). \tag{3.4}$$

Now, following Volkonskii and Rozanov [7], it can be proved easily that for random variables η measurable with respect to $\mathfrak{M}_{-\infty}^{a}$, $|\eta| \leqq 1$, and ξ measurable with respect to $\mathfrak{M}_{a+k}^{\infty}$, $|\xi| \leqq 1$, we have

$$|E\eta\xi - E\eta E\xi| \leqq 4\alpha(k). \tag{3.5}$$

Hence, G being a distribution function,

$$|\text{Cov}\,[G(X_i), G(X_a)]| \leqq 4\alpha(|i - a|). \tag{3.6}$$

In view of (3.4) and (3.6), (2.12) holds with $h(k) = 5\alpha(k)$. Similarly, (2.13) holds for this choice of the function h. Since $\sum \alpha(k) < \infty$, the conditions of Theorem 2.1 are fulfilled and the result follows.

We now state two central limit theorems apropos to regularity conditions (I) and (II). The first result is Theorem 1.6 of [4]. The second result follows from Theorem 7.2 and Lemma 5.1 of [6] in conjunction with (3.5) above. (See [4] or [6] for definitions of stationarity.)

LEMMA 3.2 (Ibragimov). *Let* $\{\xi_i\}_{-\infty}^{\infty}$ *be a bounded strictly stationary sequence satisfying regularity condition* (II) *with*

$$\sum \alpha(k) < \infty \quad \text{and} \quad \alpha(k) < M/k \log k.$$

Then $n^{-\frac{1}{2}} \sum_1^n (\xi_i - E\xi_1)$ *is asymptotically normal with mean* 0 *and variance* σ^2 *given by*

$$\lim_{k\to\infty} k^{-1} \operatorname{Var} \left(\sum_1^k \xi_i\right) = \operatorname{Var}(\xi_1) + 2 \sum_1^{\infty} \operatorname{Cov}[\xi_1, \xi_{1+i}]. \tag{3.7}$$

LEMMA 3.3. *Let* $\{\xi_i\}_{-\infty}^{\infty}$ *be a bounded weakly stationary sequence satisfying regularity condition* (I) *with* $\phi(k) = O(k^{-\lambda})$, $\lambda > 1$. *Then* $n^{-\frac{1}{2}} \sum_1^n (\xi_i - E\xi_1)$ *is asymptotically normal with mean* 0 *and variance* σ^2 *given by* (3.7).

We see that the stationarity assumption is more stringent in Lemma 3.2 than in Lemma 3.3, whereas the latter requires a slightly stronger regularity assumption.

Turning attention to the statistic W, we see that if the sequences $\{G(X_i)\}$ and $\{F(Y_j)\}$ each satisfy the conditions of either of Lemmas 3.2 and 3.3, then W is asymptotically normal as m and $n \to \infty$ such that $m/n \to c \neq 0$. We remark that if a sequence $\{\xi_i\}$ satisfies regularity condition (I) or (II), then so does the sequence $\{f(\xi_i)\}$, and that if a sequence $\{\xi_i\}$ is strictly stationary of order 2, then so is the sequence $\{f(\xi_i)\}$. Therefore, by Lemmas 3.1, 3.2 and 3.3, the following results hold.

THEOREM 3.1. *Let* $\{X_i\}_{-\infty}^{\infty}$ *and* $\{Y_i\}_{-\infty}^{\infty}$ *be independent sequences of random variables, each sequence being strictly stationary of order* 2 *and satisfying regularity condition* (I) *with* $\phi(k) = O(k^{-\lambda})$, $\lambda > 1$. *Let* F *and* G *denote the respective continuous marginal distributions. Then, as* m *and* $n \to \infty$ *such that* m/n *has a limit* $c \neq 0$, *the statistic* $Z = m^{\frac{1}{2}}(U - \gamma)$ *is asymptotically normal with mean zero and variance*

$$A^2 = 4 \lim_{k\to\infty} k^{-1} \operatorname{Var} \left(\sum_1^k G(X_i)\right) + 4c \lim_{k\to\infty} k^{-1} \operatorname{Var} \left(\sum_1^k F(Y_i)\right). \tag{3.8}$$

THEOREM 3.2. *Let* $\{X_i\}_{-\infty}^{\infty}$ *and* $\{Y_i\}_{-\infty}^{\infty}$ *be independent sequences of random variables, each sequence being strictly stationary and strongly mixing with* $\sum \alpha(k) < \infty$ *and* $\alpha(k) < M/(k \log k)$. *Let* F *and* G *denote the respective continuous marginal distributions. Then, as* m *and* $n \to \infty$ *such that* m/n *has a limit* $c \neq 0$, *the statistic* $Z = m^{\frac{1}{2}}(U - \gamma)$ *is asymptotically normal with mean zero and variance* A^2 *given by* (3.8).

4. Applications.

4.1. *Comparison of rates of occurrence.* Two stationary series of events may be compared by forming for each series the sequence of time intervals between successive events and then comparing these sequences. The mean interval is the reciprocal of the rate of occurrence. Various approaches are described in Cox and Lewis [1].

A renewal process cannot always be assumed. In applications such as machine stoppages, for example, appreciable correlation may be present in the sequence of intervals. Theorems 3.1 and 3.2 show, however, that a Wilcoxon procedure is of use, if the dependence "falls off" sufficiently fast, for a nonparametric test that the sequences of intervals have a common continuous marginal distribution.

4.2. *Nonparametric test against shift.* We may test for "shift" in an otherwise stationary process $\{\xi_i\}_{-\infty}^{\infty}$ by taking two portions $\{X_i\}_1^m$ and $\{Y_j\}_1^m$ sufficiently far apart to be considered independent and treating these as samples from continuous distributions $F(x)$ and $G(y) = F(y - \Delta)$, respectively. See Lehmann [5]. The hypothesis $\Delta = \Delta_0$ may be tested by applying the Wilcoxon two-sample procedure to the observations $\{X_i\}_1^m$ and $\{Y_j'\}_1^n$, where $Y_j' = Y_j - \Delta_0$. The distribution theory of the corresponding statistic is given by Theorem 3.1 or 3.2 under appropriate restrictions on the dependence in the sequence $\{\xi_i\}$.

4.3. *Robustness of the Wilcoxon two-sample statistic.* The use of Theorems 3.1 and 3.2 can provide a clue to the robustness of the Wilcoxon two-sample procedure under departures from the standard assumption of *random* samples. By (3.7), we may write

$$A^2 = 4\{\operatorname{Var} G(X_1) + 2 \textstyle\sum_1^\infty \operatorname{Cov}[G(X_1), G(X_{1+k})]\} + 4c\{\operatorname{Var} F(Y_1) + 2 \textstyle\sum_1^\infty \operatorname{Cov}[F(Y_1), F(Y_{i+k})]\}. \tag{4.1}$$

Under the null hypothesis, that $F \equiv G$, this quantity reduces to

$$A_0^2 = 4(1 + c)\{1/12 + 2 \textstyle\sum_1^\infty \operatorname{Cov}[F(X_1), F(X_{1+k})]\}. \tag{4.2}$$

For example, let the covariances $r_k = \operatorname{Cov}[F(X_1), F(X_{1+k})]$ decrease geometrically: $r_k = \rho^k/12$ $(k = 0, 1, \cdots)$, i.e., assume $\{F(X_i)\}$ is a wide-sense Markov process (Doob [2], 233). Then

$$A_0^2 = \tfrac{1}{3}(1 + c)((1 + \rho)/(1 - \rho)). \tag{4.3}$$

In this case, the null distribution of the test statistic depends upon the *grade correlation* ρ of the variables X_i, X_{i+1}.

5. Techniques of application. Assume that one of Theorems 3.1 and 3.2 is applicable and let $h(k)$ denote the function $\alpha(k)$ or $\phi(k)$, as the case may be. Then, by (4.1) and (3.5), an upper bound for A^2 is

$$B^2 = 16(1 + c)[h(0) + 2 \textstyle\sum_1^\infty h(k)]. \tag{5.1}$$

If the bound B^2 can be determined, then conservative tests and estimates of power may be based on (5.1).

Another approach may be possible if we are dealing with a class of distributions $\mathfrak{F} = \{F\}$ for which the covariance structure of a sequence $\{F(X_i)\}$ does not depend greatly upon the particular distribution F. Then A_0^2 will be virtually a constant for $F \in \mathfrak{F}$, so that once the single parameter A_0^2 is determined, the null hypothesis distribution theory will be nonparametric for F, G taken from $\mathfrak{F}$.

Thirdly, let us consider estimation of A^2 from the data. It can be shown (in straight-forward but uninteresting fashion) that a consistent estimate of A^2, for the case $m = n$, is given by

$$(5.2)\qquad \hat{A}^2 = N^{-3}\textstyle\sum_{i=1}^{N^3} T_i s(Y_{1+(i-1)N} - X_{1+(i-1)N}) - 2(2N+1)U^2,$$

where $N = [n^{\frac{1}{4}}]$, the greatest integer $\leqq n^{\frac{1}{4}}$, and

$$(5.3)\qquad T_i = s(Y_{1+iN} - X_{1+(i-1)N}) + 2\textstyle\sum_{j=1}^{N} s(Y_{1+iN} - X_{1+(i-1)N+j}) + s(Y_{1+(i-1)N} - X_{1+iN}) + 2\textstyle\sum_{j=1}^{N} s(Y_{1+(i-1)N+j} - X_{1+iN}).$$

REFERENCES

[1] Cox, D. R. and Lewis, P. A. W. (1966). *The Statistical Analysis of Series of Events.* Methuen, London.

[2] Doob, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[3] Hoeffding, Wassily (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.* **19** 293–325.

[4] Ibragimov, I. A. (1962). Some limit theorems for stationary processes. *Theor. Prob. Appl.* **7** 349–382.

[5] Lehmann, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[6] Serfling, R. J. (1968). Contributions to central limit theory for dependent variables. *Ann. Math. Statist.* **39** 1158–1175.

[7] Volkonskii, V. A. and Rozanov, Yu. A. (1959). Some limit theorems for random functions. *Theor. Prob. Appl.* **4** 178–197.