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THE OUTPUT PROCESS OF A STATIONARY $M/M/s$ QUEUEING SYSTEM

BY P. J. BURKE

Bell Telephone Laboratories, Holmdel, New Jersey

0. Introduction. We consider a stationary $M/M/s$ queueing system (Poisson input at rate λ, exponential service time with rate μ, s servers, with $\lambda < s\mu$) with service in order of arrival. In the sequel the terms "queueing system" or "system," if unqualified, always mean such a stationary queueing system, while the term "queue" will designate the calls (customers) waiting for service in such a system.

For a typical call c in such a queueing system, we discuss the *lifetime* L_c, the sum of the delay waiting for service, if any, plus the service time, i.e., the total time spent in the system by c. We are concerned particularly with the relationship of L_c to the *state* of the system $N(t)$, the number of calls in the system either waiting or being served at an epoch t, and to the *departure sequence*, T, the set of interdeparture intervals prior to the departure of c. The essence of our main results may be summarized as follows:

Let c have arrival time t_c' and departure time t_c; then $L_c = t_c - t_c'$. Let $F_k^{\text{arr}}(x) = \Pr\{L_c \leqq x \mid N(t_c' - 0) = k\}$ and

$$F_k^{\text{dep}}(x) = \Pr\{L_c \leqq x \mid N(t_c + 0) = k\}.$$

Then Theorem 1 states that $F_k^{\text{arr}}(x) = F_k^{\text{dep}}(x)$. If we now let $F^{\text{dep}}(x, T) = \Pr\{L_c \leqq x \mid T\}$ and $F(x) = \Pr\{L_c \leqq x\}$, Theorem 2 states that $F^{\text{dep}}(x, T) = F(x)$, i.e., the lifetime of a call is independent of the departure sequence prior to its departure epoch. The application of Theorem 2 is to tandem queues. Let Q_1 and Q_2 be a pair of queueing systems in tandem (such that the output of Q_1 is the input of Q_2) with Q_2 generalized to have an arbitrary service time distribution. Then the conclusion of Theorem 3 (Corollary 1) is that the lifetimes of an individual call in Q_1 and Q_2 are mutually independent.

The present results were found in the course of an attempt to generalize a result found by Edgar Reich for queueing systems in tandem. Reich discovered the intuitively unexpected and important fact that in a sequence of tandem single-server systems (which have a joint stationary state distribution by a result of R. R. P. Jackson [3]) the lifetimes of a particular call are mutually independent [4], [5]. This result is all the more remarkable in view of the fact that the independence of the lifetimes is not part of a general pattern of independence in the separate systems. In fact for these single-server systems in tandem the delays for service, as distinct from the lifetimes, have been shown to be dependent by an explicit calculation [2].

The question as to whether Reich's result for single-server systems is true also for many-server systems has not been given an immediate answer, however,

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apparently because Reich's method cannot be applied to queueing systems which are not strictly "first in, first out." In a many-server system it is clear that calls may pass each other even though initiation of service is in order of arrival. It was conjectured by the present writer ([7], p. 456) nevertheless, that Reich's theorem can be generalized to the many-server case.

1. Preliminaries. We collect here some facts concerning a queueing system and its state process. If t is an arrival epoch, we take $N(t)$ to be $N(t - 0)$, while if t is a departure epoch, $N(t) = N(t + 0)$. Since a sample function of $N(t)$ (a.s.) has only a finite number of discontinuities in any finite interval, we are thus requiring the sample functions to be (a.s.) lower semicontinuous.

The stationary state probabilities, $p_k = \Pr\{N(t) = k\}$, are given by

$$(1) \qquad \begin{aligned} p_k &= p_0(\lambda/\mu)^k/k!, & k &= 0, \cdots, s, \\ &= p_0(\lambda/\mu)^k/(k!\, s^{k-s}), & k &= s+1, \cdots, \end{aligned}$$

p_0 being found by the normalizing condition $\sum p_k = 1$. (See [6] p. 101 for these probabilities with some obvious changes in notation.) It is a routine exercise to show that these same state probabilities apply to the discrete parameter chains associated with arrival or departure epochs.

Since $N(t)$ is a stationary birth-and-death process, it is reversible. A stochastic process $X(t)$ is said to be reversible if the finite-dimensional distributions of $X(-t)$ are the same as those of $X(t)$. For a proof that a stationary birth-and-death process is reversible, see Reich [4].

Let the instants of change of state be denoted $\cdots \tau_{-1}, \tau_0, \tau_1, \cdots, \tau_i, \cdots$. Let $j_s = \min[j, s]$. If $N(\tau_i + 0) = j$, then the occupation time $\tau_{i+1} - \tau_i$ has the conditional distribution function (df)

$$(2) \qquad G_j(t) = 1 - e^{-(j_s\mu+\lambda)t}.$$

It is clear that $\tau_{i+1} - \tau_i$ has the same conditional df given $N(\tau_{i+1} - 0) = j$.

Let $P_{j,k} = \Pr\{N(\tau_{i-1} - 0) = k \mid N(\tau_i - 0) = j\}$. Then by the reversibility of $N(t)$

$$(3) \qquad P_{j,k} = \Pr\{N(\tau_i + 0) = k \mid N(\tau_{i-1} + 0) = j\},$$

and hence

$$P_{j,j+1} = \lambda/(j_s\mu + \lambda), \qquad P_{j,j-1} = j_s\mu/(j_s\mu + \lambda),$$

while otherwise $P_{j,k} = 0$.

The conditional df of the occupation time $\tau_i - \tau_{i-1}$ given that $N(\tau_{i-1} + 0) = j$, and given a sequence of interarrival intervals $T_\alpha' = \{t_1', \cdots, t_\alpha'\}$ following τ_{i-1}, with t_1' the time interval from τ_{i-1} to the first subsequent call arrival, is

$$(4) \qquad \begin{aligned} R_j(t, t_1') &= 1, & t &\geqq t_1', \\ &= 1 - e^{-j_s\mu t}, & 0 &\leqq t < t_1'. \end{aligned}$$

It is clear that T_i', except for t_1', is irrelevant for R_j. By the reversibility of

$N(t)$, the df of $\tau_i - \tau_{i-1}$, given that $N(\tau_i - 0) = j$ and given a sequence of interdeparture intervals $T_\alpha = \{t_1, \cdots t_\alpha\}$ preceding τ_i, with t_1 the interval from the last such departure to τ_i, is given by $R_j(t, t_1)$.

2. An independence relation between delays and departure sequences. We first consider the relationship of the delay of a delayed call to the number of calls behind it in line at a departure instant. Let c_j denote a generic delayed call which finds at least $j - 1$ calls, $j = 1, \cdots$ in the waiting line at its arrival instant t'_{c_j} and let $t_{c,j}$ be the instant that c_j advances from jth to $(j - 1)$th in line (where $t_{c,1}$ is the instant that c_j enters the service mechanism). Let K denote the number of calls in line behind c_j at $t_{c,j}$. We denote the conditional df of $t_{c,j} - t'_{c_j}$, given $K = k$, by $H_{j,k}(t)$. Let $[\cdot]^{*k}$ denote the k-fold convolution with itself of the df in the square brackets, with the convention (used in the sequel) that $[\cdot]^{*k}$ is the df of the distribution concentrated at zero for $k \leqq 0$.

Our first result is that $H_{j,k}(t)$ is independent of j and is the same as the df of the delay of a delayed call given there are k delayed calls in line at its arrival instant.

LEMMA 1. $H_{j,k}(t) = H_k(t)$ *where*

$$H_k(t) = [1 - e^{-s\mu t}]^{*(k+1)}, \qquad k = 0, 1, \cdots. \tag{5}$$

PROOF. Let $p(k \mid t)$ be the probability that $K = k$, given that $t_{c,j} - t'_{c_j} = t$. Then $p(k \mid t) = (\lambda t)^k e^{-\lambda t}/k!$ since there will be k calls behind c_j at $t_{c,j}$ if and only if there were k arrivals in the interval $(t'_{c_j}, t_{c,j})$. The frequency function for the event that c_j will advance from jth to $(j - 1)$th in line at $t_{c,j} = t'_{c_j} + t$ is found as

$$\textstyle\sum_{i=0}^{\infty} p_{s+j-1+i} s\mu (s\mu t)^i e^{-s\mu t}/i! = p_{s+j-1} s\mu e^{-(s\mu-\lambda)t}, \tag{6}$$

with the help of (1). The probability of k calls in line behind c_j at $t_{c,j}$ is simply $p_{s+j-1}(\lambda/(s\mu))^k$, by (1). By Bayes's Theorem and a modicum of algebra we find the density corresponding to $H_{j,k}(t)$ to be $s\mu(s\mu t)^k e^{-s\mu t}/k!$, which implies (5).

The main result of this section is

LEMMA 2. *The partial delay* $t_{c,j} - t'_{c_j}$, *given* $K = k$, *is independent of the sequence of departure intervals preceding* $t_{c,j}$.

PROOF. We show that the df of $t_{c,j} - t'_{c_j}$, given $K = k$ and given the departure sequence previous to $t_{c,j}$, is $H_k(t)$. Let $T_{i,j}$ represent the i interdeparture intervals immediately preceding $t_{c,j}$, and let $H_{j,k}(t, T_{i,j})$ be the df of $t_{c,j} - t'_{c_j}$ given $T_{i,j}$ and $K = k$. Let t_1 be the length of the last interdeparture interval preceding $t_{c,j}$. Then for $t \leqq t_1$,

$$H_{j,0}(t, T_{i,j}) = 1 - e^{-s\mu t} \tag{7}$$

by (4). For $t > t_1$,

$$H_{j,0}(t, T_{i,j}) = 1 - e^{-s\mu t_1} + e^{-s\mu t_1} H_{j+1,0}(t - t_1, T_{i-1,j+1}). \tag{8}$$

Since $H_{j,k}(t, T_{0,j}) = H_k(t)$, by Lemma 1, it follows by induction on i that (7) holds for all t. For $k > 0$ and $t \leqq t_1$,

$$H_{j,k}(t, T_{i,j}) = \int_0^t s\mu e^{-s\mu x} H_{j,k-1}(t - x, T_{i,j})\, dx \tag{9}$$

and for $t > t_1$,

$$H_{j,k}(t, T_{i,j}) = \int_0^{t_1} s\mu e^{-s\mu x} H_{j,k-1}(t - x, T_{i,j})\, dx + e^{-s\mu t_1} H_{j+1,k}(t - t_1, T_{i-1,j+1}). \tag{10}$$

It is obvious that

$$H_{j,k}(t, T_{i,j}) = [1 - e^{-s\mu t}]^{*(k+1)} = H_k(t), \tag{11}$$

satisfies (9) since it is in the form of a convolution; and the calculations necessary to show that (11) satisfies (10) are fairly trivial. Since by Lemma 1, for $i = 0$, (11) is true for all t and all j, k; it follows by induction on i and k together that (11) is true in general.

3. Distribution of the lifetime given the departure state.

THEOREM 1. *The conditional* df *of* L_c *given* $N(t_c) = k$ *is the same as that given* $N(t_c') = k$, i.e.,

$$F_k^{\text{dep}}(t) = F_k^{\text{arr}}(t) = F_k(t) = [1 - e^{-\mu t}] * [1 - e^{-s\mu t}]^{*[k-s+1]}, \tag{12}$$

where the symbol $[\cdot] * [\cdot]$ *indicates ordinary convolution.*

PROOF. We view the process in reversed time, proceeding backward from t_c. The first state (last in forward time) preceding t_c is $k + 1$, which has occupation-time df G_{k+1} given by (2). The remainder of the history of c depends both on the value of k and on whether the first state is preceded by state k or state $k + 2$.

Consider first the case $k \leqq s - 2$. If the first transition is to state k, which occurs with probability $P_{k+1,k}$, c either had just arrived at the system, with probability $1/(k + 1)$; or had arrived previously, with probability $k/(k + 1)$, and its previous lifetime has the same distribution as that of a call whose departure state is $k - 1$. These statements follow readily from the renewal property ([8], p. 9) of the exponential distribution, which implies that the last call to begin service has the same probability of being the first call to complete service as any other call. If, on the other hand, the first transition is to state $k + 2$, which occurs with probability $P_{k+1,k+2}$, c's previous lifetime will be distributed as that of one whose departure state is $k + 1$, since there is no queue of delayed calls at this instant of transition. These facts can be summarized by the recurrence,

$$F_k = P_{k+1,k}(k + 1)^{-1}(G_{k+1} + kG_{k+1} * F_{k-1}) + P_{k+1,k+2}G_{k+1} * F_{k+1}, \quad k = 0, \cdots, s - 2. \tag{13}$$

For $k = s - 1$, by a similar argument we have

$$F_{s-1} = P_{s,s-1}s^{-1}[G_s + (s - 1)G_s * F_{s-2}] + P_{s,s+1}s^{-1}[(s - 1)G_s * F_s + G_s * H_0]. \tag{14}$$

Finally,

$$(15)\quad F_k = P_{k+1,k}G_{k+1} * F_{k-1} + P_{k+1,k+2}s^{-1}[(s-1)G_{k+1} * F_{k+1} + G_{k+1} * H_{k+1-s}], \quad k \geqq s.$$

The Laplace-Stieltjes transform of F_k, $\int_0^\infty e^{-\theta t}\, dF_k(t)$, will be denoted $\varphi_k(\theta)$. Substituting the values of G_k, $P_{j,k}$ and H_k given by (2), (3), and (4) respectively into the above recurrences, taking transforms and simplifying somewhat, one obtains the linear recurrences

$$(13')\quad \varphi_k(\theta) = [(k+1)\mu + \lambda + s]^{-1}[\mu + k\mu\varphi_{k-1}(\theta) + \lambda\varphi_{k+1}(\theta)], \quad 0 \leqq k \leqq s-2,$$

$$(14')\quad \varphi_{s-1}(\theta) = (s\mu + \lambda + s)^{-1}\{\mu + (s-1)\mu\varphi_{s-2}(\theta) + (\lambda/s)[(s-1)\varphi_s(\theta) + s\mu/(s\mu+\theta)]\},$$

$$(15')\quad \varphi_k(\theta) = (s\mu + \lambda + s)^{-1}\{s\mu\varphi_{k-1}(\theta) + (\lambda/s)[(s-1)\varphi_{k+1}(\theta) + (s\mu/(s\mu+\theta))^{k+2-s}]\}, \quad k \geqq s.$$

Letting $P_j = \sum_{k=0}^{j} p_k$, where p_k is given by (1), we may write the unconditional lifetime df as

$$(16)\quad F(t) = P_{s-1}[1 - e^{-\mu t}] + (1 - P_{s-1})[1 - e^{-\mu t}] * [1 - e^{-(s\mu-\lambda)t}].$$

Thus there is the additional condition

$$(16')\quad \sum_{k=0}^{\infty} p_k\varphi_k(\theta) = P_{s-1}(\mu/(\mu+\theta)) + (1 - P_{s-1})(\mu/(\mu+\theta))\cdot((s\mu-\lambda)/(s\mu-\lambda+\theta)).$$

The values of φ_k, $k > 0$, can be found in terms of φ_0 by successive elimination in (13')–(15'). Then φ_0 is determined by (16'). Hence if a solution exists it is unique. It is a routine matter to verify that

$$(17)\quad \varphi_k(\theta) = \mu(\mu+\theta)^{-1}[s\mu/(s\mu+\theta)]^{[k-s+1]^+},$$

where $[x]^+$ means max $(0, x)$, is a solution to these equations, and by inversion of φ_k we have the statement of the theorem.

The fact that in the stationary queueing system studied the conditional lifetime distribution given the departure state is identical with that given the same arrival state justifies calling the lifetime arrival-departure symmetric. It is fairly obvious that the delay proper is not arrival-departure symmetric, since the delay is zero with probability one if the the call arrives when at least one server is idle and otherwise is positive with probability one. On the other hand, the conditional probability of zero delay is positive given any departure state, say k. This is so since the following event has positive probability: the call arrives when the state of the system is zero and exactly k other calls arrive during its service interval, all of which have service times exceeding that of the call in question. For a single-server system the probability of zero delay given that the departure state is k is readily calculated to be $[\mu/(\mu + \lambda))]^{k+1}$.

4. Independence of the lifetime and previous departure process.

THEOREM 2. *The lifetime of a call is independent of the sequence of departures previous to its departure instant.*

PROOF. Let $T_i = \{t_1, \cdots, t_i\}$ be the sequence of the lengths of the last i interdeparture intervals preceding t_c, with t_1 the length of the last such interval and let $F_k(t, T_i)$ be the conditional df of L_c given $N(t_c) = k$ and given T_i.

Then by considerations similar to those discussed in the proof of Theorem 1, we have for $t \leqq t_1$,

$$F_0(t, T_i) = 1 - e^{-\mu t}, \tag{18}$$

while for $t > t_1$

$$F_0(t, T_i) = 1 - e^{-\mu t_1} + e^{-\mu t_1} F_1(t - t_1, T_{i-1}), \tag{19}$$

and induction on i gives

$$F_0(t, T_i) = 1 - e^{-\mu t} = F_0(t). \tag{20}$$

For $1 \leqq k \leqq s - 2$, and $t \leqq t_1$,

$$\begin{aligned} &F_k(t, T_i) \\ &\quad = \int_0^t (k + 1)\mu e^{-(k+1)\mu x}[(k + 1)^{-1} + (k/(k + 1))F_{k-1}(t - x, T_i)]\, dx. \end{aligned} \tag{21}$$

Similarly, for $t > t_1$

$$\begin{aligned} &F_k(t, T_i) \\ &\quad = \int_0^{t_1} (k + 1)\mu e^{-(k+1)\mu x}[(k + 1)^{-1} + (k/(k + 1))F_{k-1}(t - x), T_i)]\, dx \\ &\qquad + e^{-(k+1)\mu t_1} F_{k+1}(t - t_1, T_{i-1}). \end{aligned} \tag{22}$$

For $k = s - 1$, $t \leqq t_1$,

$$F_{s-1}(t, T_i) = \int_0^t s\mu e^{-s\mu x}[s^{-1} + ((s - 1)/s)F_{s-2}(t - x, T_i)]\, dx, \tag{23}$$

and for $t > t_1$

$$\begin{aligned} &F_{s-1}(t, T_i) \\ &\quad = \int_0^{t_1} s\mu e^{-s\mu x}[s^{-1} + ((s - 1)/s)F_{s-2}(t - x, T_i)]\, dx \\ &\qquad + e^{-s\mu t_1}[((s - 1)/s)F_s(t - t_1, T_{i-1}) + s^{-1}H_0(t - t_1, T_{i-1})]. \end{aligned} \tag{24}$$

For $k \geqq s$ and $t \leqq t_1$,

$$F_k(t, T_i) = \int_0^t s\mu e^{-s\mu x} F_{k-1}(t - x, T_i)\, dx \tag{25}$$

and for $t > t_1$,

$$F_k(t, T_i)$$
$$= \int_0^{t_1} s\mu e^{-s\mu x} F_{k-1}(t - x, T_i)\, dx \quad (26)$$
$$+ e^{-s\mu t_1}[((s - 1)/s)F_{k+1}(t - t_1, T_{i-1}) + s^{-1}H_{k-s+1}(t - t_1, T_{i-1})].$$

After substituting the value of $H_k(t, T_i)$ given by Lemmas 1 and 2, one may verify by routine calculation that the above equations are satisfied by

$$F_k(t, T_i) = F_k(t), \quad (27)$$

and hence by induction on i and k together (27) is indeed true. Therefore the lifetime, given the departure state, is conditionally independent of the previous departure sequence; and since the latter is independent of the departure state, it is independent of the lifetime and departure state jointly.

5. Independence of the lifetimes in tandem queues. We now consider a pair of systems Q_1 and Q_2 in tandem in that order. It is not necessary that Q_2 have exponential service times (or, trivially, have a stationary state distribution) but it must have order-of-arrival service and satisfy the remaining usual postulates of an $M/M/s$ system. (The Poisson input to Q_2 is guaranteed by the stationarity of Q_1.) Then we have

THEOREM 3. *Let t_j be the departure instant of the* jth *departing call in* Q_1, $j = \cdots, -1, 0, 1, \cdots$; *let w_{jk} be the corresponding lifetime in Q_k, $k = 1, 2$; and let $N_k(t)$ be the state of Q_k at time t. Then the pair $C_{j1} = [N_1(t_j), w_{j1}]$ is independent of $C_{j'2} = [N_2(t_{j'}), w_{j'2}]$, where $j' \leqq j$.*

PROOF. Let T_j denote the sequence of departure instants in Q_1 previous to t_j. By Theorem 2, C_{j1} is independent of T_j. Since $C_{j'2}$ is completely determined by T_j and the sequence of service times in Q_2, which are postulated to be jointly independent of C_{j1} and T_j, it follows that C_{j1} is independent of $C_{j'2}$.

COROLLARY 1. *The lifetimes of a particular call in Q_1 and Q_2 are independent.*

COROLLARY 2. *The lifetime of a call in Q_1 is independent of its delay in Q_2.*

If now we consider a sequence of queueing systems $\{Q_k\}$, $k = 1, \cdots, K$ with $K \geqq 3$, in tandem, and assume there is but a single server in each Q_j, $2 \leqq j \leqq K - 1$, then we can assert by arguments similar to the above that the lifetimes of an individual call in the $\{Q_k\}$ are mutually independent.

The results given above thus constitute only a partial generalization of Reich's result to many-server systems, since in these results there can be only one such system—and this must be the last one—in tandem after the first system. It is a curious, but nevertheless plausible, conjecture that the lifetimes of a call in three or more many-server systems in tandem are *not* pairwise independent, although the lifetimes in any two consecutive such systems are independent.

6. Proof of Theorems 1 and 2 by reversibility. The referee was perspicacious enough to assert that Theorems 1 and 2 could be proved by an appeal to the reversibility of the $N(t)$ process. The following argument (in which the single-server system is an implicit special case) carries out this suggestion.

Let $K = N(t_c')$ denote the arrival state of a call c. All other random variables in what follows are, if not explicitly so, implicitly understood to be conditional on K. The queue size at $t_c' + 0$ is $[K + 1 - s]^+$, and thus the possible departure epochs of c are the downward transition epochs, $t_1, \cdots, t_i, \cdots$ in the $N(t)$ process where t_1 is the $([K + 1 - s]^+ + 1)$th such epoch after t_c'. Let $N_{i,s} = \min\{N(t_i) + 1, s\}$, $i = 1, 2, \cdots$, let $\mathfrak{N} = \{N_{i,s}\}$, and let $\mathfrak{n} = \{n_{i,s}\}$ be a generic set of values of the components of $\mathfrak{N}$. The conditional probability that c departs at t_i given $\mathfrak{N} = \mathfrak{n}$ is

$$Q_{i|\mathfrak{n}} = n_{i,s}^{-1} \prod_{j=1}^{i-1} (1 - n_{j,s}^{-1}),$$

in which the empty product is unity. Here, as in the sequel, we use the "lack of memory" of the exponential distribution. Let $T_{i|\mathfrak{n}} = t_i - t_c'$, conditionally that $\mathfrak{N} = \mathfrak{n}$. Then we may write the lifetime of c, conditional on K, as

$$L_K = \sum_{\mathfrak{n}} \sum_i Q_{i|\mathfrak{n}} T_{i|\mathfrak{n}} \Pr\{\mathfrak{N} = \mathfrak{n}\}. \tag{28}$$

We now show that the lifetime, given the departure state, has the identical representation as a random variable on the reverse state process that L_K has on the forward process. We use K now to denote the departure state, $N(t_c)$, and thus the possible arrival epochs are the downward transition epochs in the reverse process starting with the $([K + 1 - s]^+ + 1)$th such epoch before t_c. We write the conditional lifetime in question as

$$L_K' = \sum_{\mathfrak{n}} \sum_i Q_{i|\mathfrak{n}}' T_{i|\mathfrak{n}} \Pr\{\mathfrak{N} = \mathfrak{n}\}, \tag{29}$$

in which the symbols, primed or not, are defined analogously in terms of quantities related to the reverse process as these same symbols were defined previously in terms of those related to the forward process. Identical (unprimed) symbols in (28) and (29) denote, by the reversibility, obviously indistinguishable random variables. To show that $Q_{i|\mathfrak{n}}' = Q_{i|\mathfrak{n}}$, we note that if $N(t_i) \leqq s - 1$ (where t_i is now ith possible arrival instant in reverse order) the probability that c arrived at t_i, given that c was in the system at $t_i + 0$ is $1/[N(t_i) + 1]$. If $N(t_i) \geqq s$, then c's arriving at t_i is equivalent to c's entering the service mechanism from the queue at some upward transition epoch (in the reverse process) completely determined by $N(t_i)$. The probability of c's having entered the service mechanism at this epoch, given that it is one among s calls in service immediately afterward, is $1/s$. Thus,

$$\begin{aligned} Q_{i|\mathfrak{n}}' &= n_{i,s}^{-1} \Pr\{c \text{ did not arrive at } t_1, \cdots, t_{i-1}\} \\ &= n_{i,s}^{-1} \prod_{j=1}^{i-1} (1 - n_{j,s}^{-1}) = Q_{i|\mathfrak{n}}. \end{aligned}$$

Hence, by the reversibility of $N(t)$, not only does L_K' have the same distribution as L_K but also, since L_K is independent of the arrival process subsequent to t_c', L_K' is independent of the departure process previous to t_c. The statements of the theorems then follow easily.

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