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SOME RESULTS ON PÓLYA TYPE 2 DISTRIBUTIONS

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A family of distributions of a real random variable x with density $p(x, \theta)$ depending on a real parameter θ is said to belong to P_2 (Pólya Type 2) if

$$p(x_1, \theta_1)p(x_2, \theta_2) - p(x_1, \theta_2)p(x_2, \theta_1) \geqq 0 \tag{1}$$

for all $x_1 < x_2$ and $\theta_1 < \theta_2$. Let $f(x, \mu)$ and $g(x, \mu, \nu)$ be two density functions with respect to some σ-finite measure $d \wedge (x)$ and let

$$h(\mu, \nu) = \int g(x, \mu, \nu)f(x, \mu)\, d \wedge (x).$$

Then the following theorem is true.

THEOREM 1. *If relation* (1) *is satisfied for* $g(x, \mu, \nu)$ *between* (i) x *and* μ *when* ν *is fixed,* (ii) μ *and* ν *when* x *is fixed,* (iii) ν *and* x *when* μ *is fixed and for* $f(x, \mu)$ *between* (iv) x *and* μ *then relation* (1) *is satisfied for* $h(\mu, \nu)$ *between* μ *and* ν.

PROOF. Let $\mu < \mu'$, $\nu < \nu'$ and

$$\lambda(x, \mu, \nu) = g(x, \mu, \nu)f(x, \mu)/h(\mu, \nu).$$

Then (1) is satisfied for $\lambda(x, \mu, \nu)$ between x and μ when ν is fixed and

$$\int \lambda(x, \mu, \nu)\, d \wedge (x) = 1.$$

Therefore, there exists a value x_0 (say) of x for a given value of ν, such that

$$\lambda(x, \mu', \nu) - \lambda(x, \mu, \nu) \leqq (\geqq) 0 \tag{2}$$

according as $x \leqq (\geqq) x_0$.

Now,

$$\begin{aligned} h(\mu, \nu') &= \int g(x, \mu, \nu')f(x, \mu)\, d \wedge (x) \\ &= h(\mu, \nu) \int [g(x, \mu, \nu')/g(x, \mu, \nu)]\lambda(x, \mu, \nu)\, d \wedge (x) \\ &\leqq h(\mu, \nu) \int [g(x, \mu', \nu')/g(x, \mu', \nu)]\lambda(x, \mu, \nu)\, d \wedge (x) \end{aligned} \tag{3}$$

by (ii) and

$$h(\mu', \nu') = h(\mu', \nu) \int [g(x, \mu', \nu')/g(x, \mu', \nu)]\lambda(x, \mu', \nu)\, d \wedge (x). \tag{4}$$

From (3) and (4) we have

$$\begin{aligned} & h(\mu', \nu')h(\mu, \nu) - h(\mu, \nu')h(\mu', \nu) \\ & \geqq h(\mu', \nu)h(\mu, \nu) \int [g(x, \mu', \nu')/g(x, \mu', \nu)](\lambda(x, \mu', \nu) - \lambda(x, \mu, \nu))\, d \wedge (x) \\ & \geqq h(\mu', \nu)h(\mu, \nu)[g(x_0, \mu', \nu')/g(x_0, \mu', \nu)] \int (\lambda(x, \mu', \nu) - \lambda(x, \mu, \nu))\, d \wedge (x) \\ & = 0 \end{aligned}$$

by (iii) and (2). This proves the theorem.

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Lehmann [4] has stated the above theorem without condition (i) and the proof given by him (Theorem 3, page 410) is incorrect. That conditions (ii), (iii) and (iv) together are inadequate can be seen by the following example. Let

$$f(x, \mu) = x^2 \exp(-\tfrac{1}{2}(x-\mu)^2)/(2\pi)^{\frac{1}{2}}(1+\mu^2),$$
$$g(x, \mu, \nu) = (2/\pi)^{\frac{1}{2}} \exp(-2(x+\mu-\nu)^2),$$

$-\infty < x, \mu, \nu < \infty$. It is easy to see that condition (i) of the theorem does not hold, but the other conditions are satisfied. Straight forward computation gives

$$h(\mu, \nu) = \int_{-\infty}^{\infty} g(x, \mu, \nu) f(x, \mu)\, dx$$
$$= [2/25(10\pi)^{\frac{1}{2}}](5 + (3\mu - 4\nu)^2) \exp((3\mu-4\nu)^2/10 - 2(\mu-\nu)^2 - \mu^2/2).$$

Taking partial derivatives with respect to μ and ν we have

$$\partial^2 \log(h(\mu, \nu))/\partial\mu\partial\nu$$
$$= \tfrac{8}{5} - 24(5 - (3\mu - 4\nu)^2)/(5 + (3\mu - 4\nu)^2)^2 < 0 \quad \text{for} \quad 3\mu = 4\nu$$

which shows that $h(\mu, \nu)$ does not satisfy (1).

The relation (1), more commonly called the monotone likelihood ratio property, is of importance in distribution theory and has been applied by Karlin [2], [3], Hall [1], Savage [5] and others to various problems. A familiar result of (1) is that if $\Psi(x)$ is monotone non-increasing (non-decreasing) function of x then $E_\theta\Psi(x)$ is monotone non-increasing (non-decreasing) function of θ, where E_θ represents expectation.

For an application of Theorem 1 we consider the distribution of a random variable which is the product of two independent random variables having gamma distributions with different degrees of freedom. Let u_n denote a gamma random variable with n degrees of freedom whose density function is given by

$$u_n(y) = [y^{n-1}/\Gamma(n)]e^{-y}; \qquad y > 0, \qquad n > 0,$$

and for $0 < m < n$, $(n-m)/(n+m) < \nu < 1$ let

$$v_\nu = u_{m+\nu m} u_{n-\nu n},$$

where $u_{m+\nu m}$ and $u_{n-\nu n}$ are independently distributed. The density function of v_ν is given by

$$v_\nu(y) = y^{m+\nu m-1} \int_0^\infty x^{n-m-\nu n-\nu m-1} e^{-x-y/x}\, dx/\Gamma(m+\nu m)\Gamma(n-\nu n)$$
$$(5) \qquad = (y^{\frac{1}{2}})^{n+m-\nu t-2} \int_0^\infty x^{t-\nu n-\nu m-1} e^{-y^{\frac{1}{2}}(x+1/x)}\, dx/\Gamma(m+\nu m)\Gamma(n-\nu n)$$
$$= (y^{\frac{1}{2}})^{n+m-\nu t-2} \int_0^1 x^{-1}(x^{t-\nu n-\nu m} + x^{-t+\nu n+\nu m}) e^{-y^{\frac{1}{2}}(x+1/x)}\, dx$$
$$\div\ \Gamma(m+\nu m)\Gamma(n-\nu n)$$

where $t = n - m$. For $0 < x \leqq 1$,

$$f(x, \mu) = e^{-\frac{1}{2}(x+1/x)}$$

and (writing ν for $-\nu$)

$$g(x, \mu, \nu) = (\mu^{\frac{1}{2}})^{n+m+\nu t-2}x^{-1}(x^{t+\nu n+\nu m} + x^{-t-\nu n-\nu m})$$

the conditions of Theorem 1 are satisfied. It follows from (5) that the distribution of v_ν has monotone likelihood ratio property in ν.

Simple applications of Theorem 1 arise where $g(x, \mu, \nu)$ does not depend upon μ. For example, consider the density function of a non-central chi-square variable with p degrees of freedom and the non-centrality parameter θ^2. It is given by

$$f_p(x, \theta^2) = e^{-\theta^2} \sum_{r=0}^{\infty} (\theta^{2r}/r!) f_{p+2r}(x),$$

$x > 0$, where $f_{p+2r}(x)$ represents the density function of a central chi-square variable with $p + 2r$ degrees of freedom. As $f_{p+2r}(x)$ satisfies (1) between x and r and $(\theta^{2r}/r!)e^{-\theta 2}$ satisfies (1) between θ^2 and r it follows from Theorem 1 that the density $f_p(x, \theta^2)$ satisfies (1) between x and θ^2. For another example, in Bayesian analysis $g(x, \mu, \nu)\, d \wedge (x) = g(x, \nu)\, d \wedge (x)$ may represent the density of the *a priori* distribution of x and $f(x, \mu)$ may represent the conditional density of μ given x. Then $h(\mu, \nu)$ represents the density of the marginal distribution of μ which satisfies (1) between μ and ν if $f(x, \mu)$ and $g(x, \nu)$ satisfy (1).

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