The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1753–1754

A MIXTURE OF RECURRENT RANDOM WALKS NEED NOT BE RECURRENT

BY J. MINEKA

New York University

For our purposes a sequence $\{X_i\}_{i=1}^{\infty}$ of independent identically distributed random variables is said to generate a recurrent random walk if the partial sums $S_n = X_1 + X_2 + \cdots + X_n$ are recurrent in the sense that for any $\epsilon > 0$.

$$P\{|S_n| \leqq \epsilon \quad \text{for infinitely many} \quad n\} = 1.$$

The purpose of this note is to display two recurrent sequences of mutually independent identically distributed symmetric random variables $\{X_i\}$ and $\{Y_i\}$ such that

(A) if $Z_i = X_i + Y_i$, $\{Z_i\}$ is not recurrent, and

(B) if $\{W_i\}$ is a sequence of independent identically distributed random variables, independent of the first two sequences, where $P\{W_i = 1\} = P = 1 - P\{W_i = 0\}$, $P \neq 1, 0$, and $V_i = (1 - W_i)X_i + W_iY_i$, then $\{V_i\}$ is not recurrent.

Crucial to our counterexample is a result of Polya: If $\varphi(0) = 1$, $\varphi(t)$ is even, and $\varphi(t)$ is concave for $t > 0$, then $\varphi(t)$ is the Fourier transform of a probability distribution.

The classical Chung-Fuchs criterion [1] for recurrence, when applied to a sequence of i.i.d. symmetric random variables, reduces by the monotone convergence theorem to the following result:

If $\{X_i\}$ have Fourier transform $\varphi(t)$ then the partial sums of the sequence are recurrent iff for all $\delta > 0$,

$$\int_0^\delta dt/[1 - \varphi(t)] = \infty. \tag{1}$$

We construct two Fourier transforms $\varphi_1(t)$ and $\varphi_2(t)$, even and concave, satisfying condition (1), but such that

$$\int_0^\delta dt/[1 - \tfrac{1}{2}(\varphi_1(t) + \varphi_2(t))] < \infty \tag{2}$$

and also, since $\varphi_1(t)$ is monotonic for $t > 0$,

$$\int_0^\delta dt/[1 - \varphi_1(t)\varphi_2(t)] = \int_0^\delta dt/[(1 - \varphi_2(t))\varphi_1(t) + 1 - \varphi_1(t)] < \infty. \tag{3}$$

This shows that the sequences $\{Z_i\}$ and $\{V_i\}$ defined earlier are not recurrent, if $P = \frac{1}{2}$. The case $P \neq \frac{1}{2}$ is treated similarly.

Let $u(t)$ and $l(t)$ be two functions, such that

(a) $$u(0) = l(0) = 1;$$

(b) $$u(t) \quad \text{and} \quad l(t) \quad \text{are concave for} \quad t > 0;$$

Received 29 February 1968.

(c) $$\lim_{t\to 0^+} u'(t) = \lim_{t\to 0^+} l'(t) = \infty;$$

(d) $$u(t) > l(t) \quad \text{for} \quad t > 0;$$

(e) $$\int_0^\delta [1 - u(t)]^{-1}\, dt = \infty, \quad \text{while} \quad \int_0^\delta [1 - l(t)]^{-1}\, dt < \infty.$$

For instance, we might take, for $1 \geqq t > 0$,

$$1 - u(t) = t(\log 1/t)^\alpha, \quad 1 - l(t) = t(\log 1/t)^\beta$$

where $\alpha \leqq 1$ and $\beta > 1$.

We now define $\varphi_1(t)$ and $\varphi_2(t)$. Select an arbitrary t_0 and find $t_1 < t_0$ such that

$$\int_{t_1}^{t_0} dt/[1 - u(t)] = 1.$$

At the abscissa t_1, draw a tangent to $u(t)$, which will intersect the curve $y = l(t)$ at a point with abscissa $t_2 < t_1$. From this point draw a tangent to the curve $u(t)$, calling the abscissa of the point of tangency t_3, $t_3 < t_2$. Choose $t_4 < t_3$ such that

$$\int_{t_4}^{t_3} dt/[1 - u(t)] = 1.$$

Continue the process. Define:

$$\varphi_1(t) = u(t),$$
$$\varphi_2(t) = l(t), \quad \text{for} \quad t_{6n} \geqq t > t_{6n+1};$$
$$\varphi_1(t) = u(t_{6n+1}) + u'(t_{6n+1})(t - t_{6n+1}),$$
$$\varphi_2(t) = l(t), \quad \text{for} \quad t_{6n+1} \geqq t > t_{6n+2};$$
$$\varphi_1(t) = l(t),$$
$$\varphi_2(t) = u(t_{6n+3}) + u'(t_{6n+3})(t - t_{6n+3}), \quad \text{for} \quad t_{6n+2} \geqq t > t_{6n+3};$$
$$\varphi_1(t) = l(t),$$
$$\varphi_2(t) = u(t), \quad \text{for} \quad t_{6n+3} \geqq t > t_{6n+4};$$
$$\varphi_1(t) = l(t),$$
$$\varphi_2(t) = u(t_{6n+4}) + u'(t_{6n+4})(t - t_{6n+4}), \quad \text{for} \quad t_{6n+4} \geqq t > t_{6n+5};$$
$$\varphi_1(t) = u(t_{6n+6}) + u'(t_{6n+6})(t - t_{6n+6}),$$
$$\varphi_2(t) = l(t), \quad \text{for} \quad t_{6n+5} \geqq t > t_{6n+6};$$

$\varphi_1(t)$ and $\varphi_2(t)$ are obviously concave and satisfy (1).

Since $1 - \frac{1}{2}(\varphi_1(t) + \varphi_2(t)) > \frac{1}{2} - \frac{1}{2}l(t)$, (2) is true, as well as (3).

REFERENCE

[1] Chung, K. L., and Fuchs, W. H. J. (1951). On the distribution of values of the sums of random variables. *Memoirs Am. Math. Soc.* **6** 1–12.