

NOTES

A NOTE ON THE ADMISSIBILITY OF POOLING IN THE ANALYSIS OF VARIANCE¹

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1. Summary. Consider, for the moment, a balanced fixed two way layout of the analysis of variance assuming interactions and replications. Suppose a hypothesis of interest is that the row effects are all equal. A test statistic for such a hypothesis is the ratio of the mean square for rows (MSR) divided by the mean square for error (MSE). Another test procedure frequently used in practice is as follows: Test whether the interaction effects are zero. If it is decided that these effects are not zero, then test for the equality of row effects by MSR/MSE. If it is decided that the interaction effects are zero, then test for the equality of row effects by MSR divided by the pooled mean square error. That is, the pooled mean square error consists of the sum of squares for error plus the sum of squares for interaction divided by the sum of degrees of freedom for error and interaction. This latter type of procedure is called a "sometimes pooling" procedure. For a more general description and discussion of such procedures see Bozivich, Bancroft, and Hartley (1956).

In this note we consider the general linear hypothesis model. We prove, in this general framework, that the "sometimes pooling" procedure is an admissible test procedure. The proof follows from a well known invariance result and a theorem of Matthes and Truax (1967). The "sometimes pooling" procedure can be viewed as a test procedure which depends on the outcome of a preliminary test. It is interesting to note that estimation procedures which depend on a preliminary test were found to be inadmissible for the squared error loss function. (See Cohen (1965).) In the next section we state the model and prove the admissibility result.

2. Main result. Consider the canonical form of the general linear hypothesis model. That is, let $z = (z_1, z_2, \dots, z_n)'$ be a random vector distributed according to the multivariate normal distribution with mean vector ζ and covariance matrix $\sigma^2 I$. Here ζ is such that $\zeta_i = 0, i = r + 1, \dots, n$. We wish to test the hypothesis $H_0 : \zeta_i = 0, i = 1, 2, \dots, k, k \leq r$, against the alternative that not all $\zeta_i = 0, i = 1, 2, \dots, k$. Write $z' = (z^{(1)'}, z^{(2)'}, z^{(3)'})$ where $z^{(1)}$ is $k \times 1, z^{(2)}$ is $(r - k) \times 1$ and $z^{(3)}$ is $(n - r) \times 1$. Let $\varphi(z)$, the probability of rejecting

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H_0 , be the procedure defined as follows:

$$\begin{aligned}
 \varphi(z) &= 1, && \text{if } (z^{(2)'z^{(2)}/z^{(3)'z^{(3)}} > c_1 \text{ and } (z^{(1)'z^{(1)}/z^{(3)'z^{(3)}} > c_2, \\
 (2.1) \quad &= 1, && \text{if } (z^{(2)'z^{(2)}/z^{(3)'z^{(3)}} \leq c_1 \\
 &&& \text{and } (z^{(1)'z^{(1)}/z^{(2)'z^{(2)}} + z^{(3)'z^{(3)}} > c_3, \\
 &= 0, && \text{otherwise.}
 \end{aligned}$$

Here c_1, c_2, c_3 are positive constants determined by the size α of the test. (Recall the size of the test is defined as the supremum of the probabilities of rejecting when the null hypothesis is true.) The main result of this note is that the test procedure $\varphi(z)$, given in (2.1), is admissible. In order to prove this we will show that $\varphi(z)$ is admissible among all procedures which are invariant with respect to the compact group G , of transformations, which are orthogonal transformations of $z^{(1)}$. It will then follow from a version of the well known Hunt-Stein theorem, that $\varphi(z)$ is admissible among all procedures. Note that the set of statistics $(z^{(1)}, z^{(2)}, z'z)$ is sufficient and has a multivariate exponential distribution. Furthermore it is clear that $\varphi(z)$ is a function of the sufficient statistics and may be written $\varphi(z^{(1)}, z^{(2)}, z'z)$. Now we state the

THEOREM. *The test procedure $\varphi(z)$ given in (2.1) is admissible.*

PROOF. Since the group G of transformations mentioned above leaves the problem invariant, from the above remarks we need to prove that $\varphi(z)$ is admissible among all invariant procedures. Suppose then that $\varphi(z)$ is not admissible among all invariant procedures. Then there exists an invariant procedure $\psi(z^{(1)}, z^{(2)}, z'z)$ which is better than $\varphi(z)$. Furthermore such a ψ can be found which is admissible. This follows since the space of decision procedures for this problem is compact with respect to regular (weak) convergence; the weak limit of a sequence of invariant procedures is invariant, thus making the invariant procedures compact; and the fact that the admissible invariant procedures form a minimal complete class of procedures among invariant procedures. Now by the Hunt-Stein theorem again, any admissible invariant procedure is admissible. Also from Matthes and Truax (1967), Theorem 3.1, a complete class of procedures for this problem consists of procedures whose acceptance regions have convex sections in $z^{(1)}$ for fixed $(z^{(2)}, z'z)$. Hence, if $\varphi(z)$ is inadmissible there exists an invariant procedure $\psi(z^{(1)}, z^{(2)}, z'z)$, which is better than $\varphi(z)$, with an acceptance region whose $(z^{(2)}, z'z)$ sections are convex. The invariance property of ψ can be described by

$$(2.2) \quad \psi(z^{(1)}, z^{(2)}, z^{(3)}) = \psi(gz^{(1)}, z^{(2)}, z^{(3)}),$$

where g represents an orthogonal transformation on $z^{(1)}$.

Now for any fixed $(z^{(2)}, z'z)$, let $S(z^{(2)}, z'z)$ be the set of points $z^{(1)}$ for which $\psi(z^{(1)}, z^{(2)}, z'z) = 0$. Then S is either the whole space of possible values of $z^{(1)}$, or, from (2.2) and the fact that S must be convex, S is a sphere centered at the origin.

Note that if ψ is at least as good as φ , then

$$(2.3) \quad \int (\psi - \varphi) f(z^{(1)}, z^{(2)}, z'z) dz^{(1)} dz^{(2)} d(z'z) \geq 0,$$

for all (ζ, σ) with equality whenever $\zeta^{(1)} = 0$. Here $f(z^{(1)}, z^{(2)}, z'z)$ is the distribution of $(z^{(1)}, z^{(2)}, z'z)$. The equality whenever $\zeta^{(1)} = 0$ follows from the continuity of the power function of φ and ψ . Hence, by completeness of the family of densities of $z^{(1)}, z^{(2)}, z'z$ it follows from (2.3) that

$$(2.4) \quad \int (\psi(z) - \varphi(z)) \mu(dz^{(1)}; z^{(2)}, z'z) = 0,$$

for every $(z^{(2)}, z'z)$, where $\mu(dz^{(1)}; z^{(2)}, z'z)$ is the conditional distribution of $z^{(1)}$ given $z^{(2)}, z'z$, when $\zeta^{(1)} = 0$. But (2.4) implies that for every given $(z^{(2)}, z'z)$, ψ and φ must have the size with respect to each conditional distribution of $z^{(1)}$, given $(z^{(2)}, z'z)$. Since the acceptance regions, for any given $(z^{(2)}, z'z)$, are spheres with centers at the origin for both ψ and φ , (see 2.1), (2.4) implies that these spheres must coincide. Thus $\psi \equiv \varphi$, and φ cannot be dominated by any invariant ψ . This completes the proof of the theorem.

REMARK. The referee has pointed out that the result can be obtained without using the theorem of Matthes and Truax. For if φ is invariant and inadmissible there exists an invariant test ψ which beats it and ψ has convex $(z^{(2)}, z'z)$ sections. This is so since $(z^{(1)'}, z^{(1)}, z^{(2)}, z'z)$ is a maximal invariant and the conditional distribution of $z^{(1)'}, z^{(1)}$ given $(z^{(2)}, z'z)$ has a monotone likelihood ratio in $\zeta^{(1)'}, \zeta^{(1)}/\sigma^2$. Hence if ψ did not have convex sections it could be dominated conditionally by a test ψ_1 whose acceptance region is $z^{(1)'}, z^{(1)} \leq k(z^{(2)}, z'z)$. The function k is determined uniquely by a relation like (2.4) and is easily seen to be measurable since the conditional size of ψ is measurable.

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