The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1654–1660

APPROACH TO DEGENERACY AND THE EFFICIENCY OF SOME MULTIVARIATE TESTS[1]

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1. Introduction and summary. When testing p-variate distributions for a shift in location, two important nonparametric competitors of Hotelling's T^2 are the multivariate extensions W of the Wilcoxon test and M of the normal score test. Bounds on their asymptotic relative efficiency (ARE) have been investigated by Hodges-Lehmann [6] and Chernoff-Savage [4] in the univariate case and by Bickel [3] and Bhattacharyya [1] in the multivariate case.

The univariate normal score test has the commendable property that for all continuous distributions, its ARE with respect to the t-test exceeds 1 and with respect to the Wilcoxon test it exceeds $\pi/6$. This naturally raises the question of whether or not the multivariate extension M inherits this property and if not, what the lower bounds on its ARE with respect to W and T^2 are. In this paper, we answer this question by providing an example where the ARE of M with respect to both W and T^2 is arbitrarily close to zero for some direction. The example consists of a gross error distribution which places most of its mass on a hyperplane and has marginals with high sixth moments.

Bickel [3] mentioned a similar property of the ARE of W with respect to T^2. His proof for the case $p = 2$ is, however, incorrect. We show that for the type of gross error model considered by Bickel, the above ARE is bounded strictly away from zero. We correct his proof by constructing a distribution which also places high mass on a line but is not of the gross error type.

2. Results for the M-test. For the sequence of local shift alternatives $\boldsymbol{\delta} N^{-\frac{1}{2}}$, the ARE of a test A relative to a test B depends, in general, on the parent distribution $\mathbf{F}$ as well as the direction $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_p)$. Denote this ARE by $e_{A:B}(\boldsymbol{\delta}, \mathbf{F})$. Let $\mathfrak{F}_p$ be the class of all nonsingular continuous p-variate distributions. Our main results for the M test are summarized in the following theorem.

Theorem 2.1. *For* $p \geqq 2, \mathbf{F} \varepsilon \mathfrak{F}_p$ *and* $\boldsymbol{\delta} \neq \mathbf{0}$

$$\inf_{\mathbf{F}} \inf_{\boldsymbol{\delta}} e_{M:T}(\boldsymbol{\delta}, \mathbf{F}) = 0, \tag{1}$$

$$\inf_{\mathbf{F}} \inf_{\boldsymbol{\delta}} e_{M:W}(\boldsymbol{\delta}, \mathbf{F}) = 0. \tag{2}$$

Before proceeding with the proof of the theorem, we make some preliminary remarks and calculations.

Let $\mathfrak{F}_p^0$ be the subclass of $\mathfrak{F}_p$ consisting of those p-variate distributions whose univariate marginals are all identical and whose bivariate joint marginals are all identical. Let $F(x, y)$ and $G(x)$ denote the common bivariate and univariate

Received 12 February 1968.

[1] This research was sponsored in part by the Wisconsin Alumni Research Foundation.

marginal cdf's of $\mathbf{F} \varepsilon \mathfrak{F}_p^0$ and assume that G has density g satisfying the conditions of Lemma 3 of Hodges-Lehmann [5]. Denote the standard normal cdf by Φ and its density by ϕ. Equations (2.7) and (2.8) of [1] give the following expressions for the infimum of the ARE over $\boldsymbol{\delta}$.

$$\text{(3a)} \quad \inf_{\boldsymbol{\delta}} e_{M:T}(\boldsymbol{\delta}, \mathbf{F}) = \sigma^2\theta^2 \inf_{\boldsymbol{\delta}} (\boldsymbol{\delta}\mathbf{R}^{-1}\boldsymbol{\delta}')/(\boldsymbol{\delta}\mathbf{R}^{*-1}\boldsymbol{\delta}') = \sigma^2\theta^2 \min [(1-\rho)/(1-\rho^*), \{1+(p-1)\rho\}/\{1+(p-1)\rho^*\}],$$

$$\text{(3b)} \quad \inf_{\boldsymbol{\delta}} e_{M:W}(\boldsymbol{\delta}, \mathbf{F}) = \gamma^2\theta^2 \inf_{\boldsymbol{\delta}} (\boldsymbol{\delta}\mathbf{R}^{0-1}\boldsymbol{\delta}')/(\boldsymbol{\delta}\mathbf{R}^{*-1}\boldsymbol{\delta}') = \gamma^2\theta^2 \min [(1-\rho')/(1-\rho^*), \{1+(p-1)\rho'\}/\{1+(p-1)\rho^*\}]$$

where ρ and σ^2 are the common correlation and variance in F. Also

$$\text{(4)} \quad \theta = \int_{-\infty}^{\infty} (g^2(x)/\phi\{\Phi^{-1}[G(x)]\})\, dx, \qquad \gamma = [(12)^{\frac{1}{2}} \int_{-\infty}^{\infty} g^2(x)\, dx]^{-1},$$

$$\text{(5)} \quad \rho' = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x)G(y)\, dF(x, y) - 3,$$

$$\rho^* = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^{-1}[G(x)]\Phi^{-1}[G(y)]\, dF(x, y)$$

and $\mathbf{R}$, $\mathbf{R}^*$ and $\mathbf{R}^0$ are the correlation matrices with off diagonal elements ρ, ρ^* and ρ' respectively. Here ρ' is the grade correlation and ρ^* is the normal score correlation. Since $\mathfrak{F}_p^0 \subset \mathfrak{F}_p$, we have

$$\inf_{\mathbf{F}\varepsilon\mathfrak{F}_p} \inf_{\boldsymbol{\delta}} e_{M:T}(\boldsymbol{\delta}, \mathbf{F}) \leqq \inf_{\mathbf{F}\varepsilon\mathfrak{F}_p^0} \inf_{\boldsymbol{\delta}} e_{M:T}(\boldsymbol{\delta}, \mathbf{F})$$

$$\text{(6)} \quad = \min \{\inf_{\mathbf{F}\varepsilon\mathfrak{F}_p^0} \theta^2\sigma^2(1-\rho)(1-\rho^*)^{-1}, \inf_{\mathbf{F}\varepsilon\mathfrak{F}_p^0} \theta^2\sigma^2 \cdot [1+(p-1)\rho][1+(p-1)\rho^*]^{-1}\}.$$

To prove (1) it is thus sufficient to produce an example of $\mathbf{F} \varepsilon \mathfrak{F}_p^0$ such that that $\theta^2\sigma^2(1-\rho)(1-\rho^*)^{-1}$ is less than any preassigned positive number. By the same argument, the proof of (2) will be accomplished by showing the existence of an $\mathbf{F} \varepsilon \mathfrak{F}_p^0$ for which $\gamma^2\theta^2(1-\rho')(1-\rho^*)^{-1}$ is arbitrarily close to zero.

Let $\{\Lambda_c : c > 0\}$ be the class of cdf's defined by

$$\text{(7)} \qquad \Lambda_c(x) = \Phi(cx[1+x^2]), \qquad -\infty < x < \infty.$$

For every c, the density $\lambda_c(x) = c\phi(cx[1+x^2])(1+3x^2)$ is symmetric ('symmetry' shall always mean symmetry about 0). If X has cdf Λ_c then $Z = cX(1+X^2)$ has cdf Φ. Letting $v(c) = \operatorname{Var}(X)$, we have $v(1) < 1$ and $v(c) < \infty$ for all $c > 0$. Making the transformation $y = -c^2x^2(1+x^2)^2/2$, we obtain $v(c) > 2\int_1^{\infty} x^2\lambda_c(x)\, dx > (4\pi^3c^4)^{-1/6} \int_{2c^2}^{\infty} y^{-1/6} \exp(-y)\, dy \to \infty$ as $c \to 0$. Since $v(c)$ is continuous, there exists a c_0 $(0 < c_0 < 1)$ satisfying $v(c_0) = 1$. For notational simplicity, we shall denote Λ_{c_0} by Λ and λ_{c_0} by λ.

Proof of Theorem 2.1. For $p \geqq 2$ and $0 < \epsilon < 1$, consider the model

$$\text{(8)} \quad \mathbf{F}_\epsilon(x_1, \cdots, x_p) = (1-\epsilon)\mathbf{Q}^*(x_1, x_2, \cdots, x_p) + \epsilon \prod_{i=1}^{p} \Psi(x_i)$$

where $\mathbf{Q}^*$ is a p-variate distribution which concentrates its entire mass on the line $x_1 = x_2 = \cdots = x_p$ and has univariate marginal $Q(x)$. $\Psi(x)$ is a univariate continuous cdf having density ψ. According to the argument following (6), we need only consider the bivariate marginal of (8) given by

$$F_\epsilon(x, y) = (1 - \epsilon)Q^*(x, y) + \epsilon\Psi(x)\Psi(y). \tag{9}$$

Now we take $Q(x) = \Lambda(x)$ and assume that Ψ has unit variance and a bounded symmetric density ψ which satisfies $\psi(x) = \lambda(x)$ on $[|x| \geqq M]$ for some positive M. Specification of ψ on $[|x| < M]$ will be postponed until the need arises.

Letting $\mu_{(r)\psi} = \int_{-\infty}^{\infty} x^r \, d\Psi(x)$ it is clear that $\mu_{(r)\psi} < \infty$ for all $r > 0$ and $\mu_{(r)\psi} = 0$ for r odd. Let $e_1(\epsilon)$ denote the quantity $\theta^2\sigma^2(1 - \rho)(1 - \rho^*)^{-1}$ for the cdf F_ϵ. The relations (3a) and (5) give

$$e_1(\epsilon) = \theta_\epsilon^2\epsilon[1 - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^{-1}[G_\epsilon(x)]\Phi^{-1}[G_\epsilon(y)] \, dF_\epsilon(x, y)]^{-1} \tag{10}$$

where $G_\epsilon = (1 - \epsilon)\Lambda + \epsilon\Psi$ and θ_ϵ is as defined in (4) for the cdf G_ϵ. By the symmetry of G_ϵ and Ψ, we have $\int_{-\infty}^{\infty} \Phi^{-1}(G_\epsilon) \, d\Psi = 0$. Also $\int_{-\infty}^{\infty} [\Phi^{-1}(\Lambda)]^2 \, d\Lambda = 1$. Substituting (9) into (10) gives

$$e_1(\epsilon) = \theta_\epsilon^2[\epsilon^{-1}\int_{-\infty}^{\infty}\{[\Phi^{-1}(\Lambda)]^2 - [\Phi^{-1}(G_\epsilon)]^2\} \, d\Lambda + \int_{-\infty}^{\infty} [\Phi^{-1}(G_\epsilon)]^2 \, d\Lambda]^{-1}. \tag{11}$$

The fact that Λ and Ψ have the same tails allows us to apply the dominated convergence theorem and obtain $\theta_\epsilon \to 4c_0$ and $\int_{-\infty}^{\infty} [\Phi^{-1}(G_\epsilon)]^2 \, d\Lambda \to 1$ as $\epsilon \to 0$. An expansion of the integrand in the first term of (11) followed by integration by parts and an application of the dominated convergence theorem gives

$$\lim_{\epsilon\to0} e_1(\epsilon) = 16[\mu_{(2)\psi} + 2\mu_{(4)\psi} + \mu_{(6)\psi}]^{-1}. \tag{12}$$

Due to (12), the proof of (1) would be complete if we can produce, for every $\eta > 0$, a cdf Ψ which satisfies all the foregoing properties and makes the right hand side of (12) smaller than η. Equivalently, for any positive arbitrarily large N, we need to exhibit a continuous cdf Ψ having the following properties: (i) Ψ has unit variance and bounded symmetric density ψ, (ii) $\Psi(x) = \Lambda(x)$ on $[|x| \geqq M]$ for some positive M and (iii) $\mu_{(6)\psi} > N$.

Let $d_M = \int_M^{\infty} d\Lambda(x)$, $d_M' = \int_M^{\infty} x^2 \, d\Lambda(x)$ and define L_M and K_M by

$$L_M = M^2(1 - 3\,d_M' + d_M)/(M^2 - 1), K_M = \tfrac{1}{2} - d_M - L_M(M - 1)M^{-3}. \tag{13}$$

Clearly, $d_M \to 0$, $d_M' \to 0$, $L_M \to 1$ and $K_M \to \frac{1}{2}$ as $M \to \infty$. Now, we completely specify Ψ by the following density

$$\begin{aligned} \psi(x) &= K_M && \text{if } 0 \leqq x < 1 \\ &= M^{-3}L_M && \text{if } 1 \leqq x < M \\ &= \lambda(x) && \text{if } M \leqq x \end{aligned} \tag{14}$$

and $\psi(x) = \psi(-x)$ for all x. By straightforward calculation the conditions (i) and (ii) can be shown to hold. Also, $\mu_{(6)\psi} > \{2K_M/7 + 2L_M(M^4 - M^{-3})/7\} \to \infty$ as $M \to \infty$, so that (iii) holds for a suitable choice of M. We conclude that

$e_1(\epsilon)$ can be made arbitrarily close to zero for the distribution (9) by choosing ϵ sufficiently small and M sufficiently large.

To establish (2), we again consider the gross error model (8) with $Q = \Lambda$ and the same Ψ as defined above. Letting $e_2(\epsilon)$ denote the quantity $\gamma^2\theta^2(1 - \rho') \cdot (1 - \rho^*)^{-1}$ for the distribution F_ϵ and using (5) and (10), we have

$$e_2(\epsilon) = e_1(\epsilon)\gamma_\epsilon^2 h(\epsilon) \tag{15}$$

where γ_ϵ denotes the γ defined by (4) for the distribution G_ϵ. Here

$$h(\epsilon) = \epsilon^{-1}(4 - 12 \textstyle\int G_\epsilon^2\, d\Lambda) + 12 \int G_\epsilon^2\, d\Lambda - 12(\int G_\epsilon\, d\Psi)^2 \tag{16}$$

and

$$\lim_{\epsilon\to 0} h(\epsilon) = 12 - 12(\textstyle\int \Lambda\, d\Psi)^2 - 24 \int \Lambda\Psi\, d\Lambda$$
$$= 12[\textstyle\int \Lambda^2\, d\Psi - (\int \Lambda\, d\Psi)^2] = 12 \operatorname{Var}_\Psi [\Lambda(Z)]$$

where $\operatorname{Var}_\Psi [\Lambda(Z)]$ denotes the variance of $\Lambda(Z)$ when Z has distribution Ψ.

Due to symmetry of Λ,

$$\operatorname{Var}_\Psi [\Lambda(Z)] \leqq \textstyle\int_{-\infty}^{0} (0 - \frac{1}{2})^2\, d\Psi(z) + \int_0^\infty (1 - \frac{1}{2})^2\, d\Psi(z) = \frac{1}{4}.$$

Also

$$\lim_{\epsilon\to 0} \gamma_\epsilon^{-1} = (12)^{\frac{1}{2}} \textstyle\int_{-\infty}^{\infty} \lambda^2(x)\, dx = (3/\pi)^{\frac{1}{2}} c_0 \int_{-\infty}^{\infty} (1 + 3x^2)\lambda_{c_0 2^{\frac{1}{2}}}(x)\, dx$$
$$= (3/\pi)^{\frac{1}{2}} c_0[1 + 3v(c_0 2^{\frac{1}{2}})].$$

Hence

$$\lim_{\epsilon\to 0} e_2(\epsilon) \leqq \{16\pi c_0^{-2}[1 + 3v(c_0 2^{\frac{1}{2}})]^{-2}\}\{\mu_{(2)\psi} + 2\mu_{(4)\psi} + \mu_{(6)\psi}\}^{-1}. \tag{17}$$

The first factor on the right hand side is a finite constant and hence a choice of sufficiently small ϵ and sufficiently large M would make $e_2(\epsilon)$ less than any preassigned positive number. This completes the proof of the theorem.

3. Results for the W-test. We first examine the performance of the W test relative to T^2 for a general p-variate gross error model (8) where the marginals Q and Ψ are now unspecified. It is still assumed, however, that Q and Ψ have zero mean, unit variance and square integrable symmetric densities.

Proceeding in the same manner as in the derivation of (3a) and (3b), we now have

$$\inf_{\boldsymbol{\delta}} e_{W:T}(\boldsymbol{\delta}, \mathbf{F}_\epsilon) = 12[\textstyle\int_{-\infty}^{\infty} f_\epsilon^2(x)\, dx]^2 \min [(1 - \rho_\epsilon)(1 - \rho_\epsilon')^{-1}, \{1 + (p - 1)\rho_\epsilon\}\{1 + (p - 1)\rho_\epsilon'\}^{-1}] \tag{18}$$

where f_ϵ is the marginal density, ρ_ϵ is the correlation and ρ_ϵ' is the grade correlation of the bivariate marginal $F_\epsilon(x, y)$ given by (9). Note that $\rho_\epsilon = (1 - \epsilon) \to 1$ and $\rho_\epsilon' \to 1$ as $\epsilon \to 0$. Further, replacing Λ by Q in (16), we have $\epsilon^{-1}(1 - \rho_\epsilon') \to 12 \operatorname{Var}_\Psi [Q(Z)] \leqq 3$. Also $\int_{-\infty}^{\infty} f_\epsilon^2(x)\, dx \to \int_{-\infty}^{\infty} q^2(x)\, dx$ where q is the density of

Q. Hence as $\epsilon \to 0$ in (18), the right hand side tends to

$$(19) \qquad 12\left(\int_{-\infty}^{\infty} q^2(x)\,dx\right)^2 \min\left[\{12 \operatorname{Var}_\Psi [Q(Z)]\}^{-1}, 1\right] \geqq .288.$$

The inequality above follows from the bound $12(\int_{-\infty}^{\infty} q^2(x)\,dx)^2 \geqq .864$, (cf. Hodges-Lehmann [6]) and the fact that $\operatorname{Var}_\Psi [Q(Z)] \leqq \frac{1}{4}$.

Thus with approach to degeneracy ($\epsilon \to 0$) in the model (8), inf $e_{W:T}(\boldsymbol{\delta}, \mathbf{F}_\epsilon)$ does not fall below .288 for any Q and Ψ. Bickel [2], [3] investigated the behavior of this infimum as $\epsilon \to 0$ and $b \to 1$ in the closely related model ($p = 2$)

$$(20) \quad F_{\epsilon,b}(x, y) = (1 - \epsilon)Q^*(x, y) + \epsilon b\Psi^*(x, y) + \epsilon(1 - b)\Psi(x)\Psi(y)$$

where Ψ^* degenerates on the line $x = y$ and has marginal Ψ. Using essentially the argument given above, it is easy to see that this limit is identical to the expression (19) and consequently the additional complication in the model is unnecessary. Inequality (19) shows that the proof of Theorem 6.2 of Bickel [2] is incorrect. In particular, the expression (6.5) of [2] is identically zero contrary to the claim that it can be made nonzero.

We state the theorem here for the sake of completeness and present a correct proof.

THEOREM 3.1. *For* $p = 2$, $F(x, y) \,\varepsilon\, \mathfrak{F}_p$ *and* $\boldsymbol{\delta} \neq \mathbf{0}$

$$\inf_{F \varepsilon \mathfrak{F}_2} \inf_{\boldsymbol{\delta}} e_{W:T}(\boldsymbol{\delta}, F) = 0.$$

PROOF. For $0 < a < 1$, consider the following sequence of continuous bivariate cdfs $F_n(x, y)$ on the unit square

$$\begin{aligned} F_n(x, y) &= an^2xy && \text{if } 0 \leqq t, t' \leqq n^{-1}, \\ (21) \qquad &= \text{ant} && \text{if } 0 \leqq t \leqq n^{-1}, \\ & && \quad n^{-1} < t' \leqq 1, \\ &= a + (1 - a)(n - 1)^{-1}(nt - 1) && \text{if } n^{-1} < t, t' \leqq 1, \end{aligned}$$

where $t = \min(x, y)$ and $t' = \max(x, y)$. F_n distributes mass a uniformly on the square $[0, n^{-1}] \times [0, n^{-1}]$ and the remaining mass $(1 - a)$ uniformly on the line connecting the points $x = n^{-1}, y = n^{-1}$ and $x = 1, y = 1$. The common marginal cdf $G_n(x)$ is given by

$$\begin{aligned} (22) \qquad G_n(x) &= anx && \text{if } 0 \leqq x \leqq n^{-1} \\ &= a + (1 - a)(n - 1)^{-1}(nx - 1) && \text{if } n^{-1} < x \leqq 1. \end{aligned}$$

Consider the random variables $U_n = G_n(X_n)$ and $V_n = G_n(Y_n)$ where (X_n, Y_n) has cdf $F_n(x, y)$ given by (2.1). Denoting the joint cdf of (U_n, V_n) by $H(u, v)$, we have

$$\begin{aligned} (23) \qquad H(u, v) &= P[X_n \leqq G_n^{-1}(u), Y_n \leqq G_n^{-1}(v)] \\ &= \begin{cases} uv/a & \text{on } [0 \leqq u \leqq a] \cap [0 \leqq v \leqq a] \\ \min(u, v) & \text{on } [a < u \leqq 1] \cup [a < v \leqq 1] \end{cases} \end{aligned}$$

and this distribution is independent of n.

Now

$$(24)\quad \inf_{\boldsymbol{\delta}} e_{M:T}(\boldsymbol{\delta}, F_n) = 12\sigma_n^2\left(\int_0^1 f_n^2(x)\,dx\right)^2 \min\,[1 - \rho_n)(1 - \rho')^{-1}, (1 + \rho_n)(1 + \rho')^{-1}]$$
$$\leqq 12\sigma_n^2\left(\int_0^1 g_n^2(x)\,dx\right)^2(1 - \rho_n)(1 - \rho')^{-1}$$

where $\sigma_n^2 = \text{Var}(X_n)$, $g_n(x)$ is the density of G_n, $\rho_n = \text{corrl}(X_n, Y_n)$ and $\rho' = \text{corrl}(U_n, V_n)$. A straightforward computation leads to the expressions

$$\rho' = (1 - a^3), \qquad \int_0^1 g_n^2(x)\,dx = a^2 n + n(1 - a)^2(n - 1)^{-1},$$
$$\sigma_n^2(1 - \rho_n) = E(X_n^2) - E(X_n \cdot Y_n) = a[12n^2]^{-1}.$$

Substituting these expressions into the right hand side of (24) and taking the limit, we have $\lim_{n\to\infty} a^{-2}[a^2 + (1 - a)^2(n - 1)^{-1}]^2 = a^2$. Thus for the class of distributions F_n defined by (21), we can make $\inf_{\boldsymbol{\delta}} e_{W:T}(\boldsymbol{\delta}, F_n)$ arbitrarily close to zero by choosing a sufficiently small and n sufficiently large. This completes the proof.

4. Conclusions. The implication of Theorem 2.1 is that the behavior of the M test could be quite poor in comparison to both T^2 and W for some direction $\boldsymbol{\delta}$ of alternatives when a nonnormal multivariate distribution is almost degenerate on a line. Theorem 3.1 gives another degenerate situation where W behaves poorly with respect to T^2. Consideration of the infimum over directions of the ARE is however much too conservative a viewpoint in test comparison. On the other hand, a study of overall relative efficiency of multivariate tests is complicated by the fact that there is no satisfactory direction-free measure of ARE in the Pitman sense.

An extremely local measure has been proposed by Bickel [3]. It is defined as the limit of the inverse ratio of the sample sizes required by two tests of the same asymptotic size to have equal Gaussian curvature of the power surface at $\mathbf{0}$. For any two tests A and B and the parent cdf $\mathbf{F}$, let us denote this ARE by $E_{A:B}(\mathbf{F})$. It essentially follows from [3] that, for $p = 2$, $E_{M:T}(\mathbf{F}) = \theta^2\sigma^2(1 - \rho^2)^{\frac{1}{2}} \cdot (1 - \rho^{*2})^{-\frac{1}{2}}$. With $\mathbf{F} = F_\epsilon$ given by (8), we have $E_{M:T}(F_\epsilon) = \theta_\epsilon e_1^{\frac{1}{2}}(\epsilon)[(2 - \epsilon)\cdot(1 + \rho_\epsilon^*)^{-1}]^{\frac{1}{2}}$. The last factor tends to 1 as ϵ tends to 0. Hence the proof of Theorem 2.1 also yields $\inf_{\mathbf{F}} E_{M:T}(\mathbf{F}) = 0$ and similarly $\inf_{\mathbf{F}} E_{M:W}(\mathbf{F}) = 0$. In the same manner, the proof of Theorem 3.1 yields $\inf_{\mathbf{F}} E_{W:T}(\mathbf{F}) = 0$.

Acknowledgment. The authors wish to take this opportunity to thank Professor S. Stigler for extremely helpful discussion concerning the proof of Theorem 3.1.

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