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ON A CHARACTERIZATION OF SYMMETRIC STABLE PROCESSES WITH FINITE MEAN[1]

By B. L. S. Prakasa Rao
University of Illinois

1. Introduction. Laha [1] studied the characterization of symmetric stable laws through regression properties and he has proved the following theorem.

Theorem 1.1. *Let X and Y be two independent nondegenerate random variables whose expectations exist and are zero. Suppose that a structure is given by $U = aX + bY$, $V = cX + dY$ with a, b, c, d different from zero and $ad \neq bc$. Then both X and Y have symmetric stable distributions with the same exponent $\lambda > 1$, if and only if,*

(i) *there exists a constant $\delta > 0$ such that the relation $E[V \mid U] = \beta U$ a.e. holds for all a such that $0 < |a| < \delta$, and*

(ii) $$\beta = (ca^{-1}\alpha_1 |a|^\lambda + db^{-1}\alpha_2 |b|^\lambda)(\alpha_1 |a|^\lambda + \alpha_2 |b|^\lambda)^{-1}$$

where α_1 and α_2 are the scale parameters of the distributions of X and Y respectively.

Our aim in this paper is to derive a similar result to characterize symmetric stable processes with finite mean function. While this paper was in preparation, we noticed that Lucaks [2] has given a different characterization of symmetric stable processes.

2. Definitions. We shall now present some definitions and some results concerning stochastic processes and stochastic integrals. Let T be any bounded interval. We shall take $T = [0, 1]$ unless otherwise stated.

A stochastic process $\{X(t), t \varepsilon T\}$ is said to be a homogeneous process with independent increments if the distribution of the increments $X(t + h) - X(t)$ depends only on h but is independent of t and if the increments over non-overlapping intervals are independent.

Let $\{X(t), t \varepsilon T\}$ be a homogeneous process with independent increments. Let $\varphi(u; h)$ denote the characteristic function of $X(t + h) - X(t)$. It is well known that $\varphi(u; h)$ is infinitely divisible and $\varphi(u; h) = [\varphi(u; 1)]^h$. It can be shown that the stochastic integral,

$$\int_0^1 a(t)\, dX(t), \tag{2.1}$$

can be defined in the sense of convergence in probability for a large class of functions $a(\cdot)$ on [0, 1] which includes the class of infinitely differentiable functions on [0, 1]. This can be done by defining (2.1) for simple functions $a(\cdot)$ in the obvious manner and then extending the definition to functions which can be approximated by simple functions uniformly.

Let $\{X(t), t \varepsilon T\}$ be a homogeneous process with independent increments and

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let $X(0) = 0$. The process is said to be a symmetric stable process with exponent α if the increments of the process have symmetric stable laws with exponent α.

3. Characterization.

THEOREM 3.1. *Let* $\{X(t),\ t \varepsilon T\}$ *be a homogeneous process with independent increments. Further suppose that* (i) $X(0) = 0$, (ii) $E[X(t)] = 0$, *for all* t, *and* (iii) *the increments of the process have non-degenerate symmetric distributions. Let*

$$Y_\lambda = \int_0^1 t^\lambda \, dX(t) \tag{3.1}$$

for any $\lambda > 0$. *Then the process is a symmetric stable process with exponent* $\alpha > 1$, *if and only if, for some positive numbers* λ *and* μ, $\lambda \neq \mu$,

$$E[Y_\lambda \mid Y_\mu] = \beta Y_\mu \qquad \text{a.e.} \tag{3.2}$$

for some constant β *depending on* λ, μ. *Furthermore* α, λ, μ *and* β *are connected by the relation*

$$(\mu\alpha + 1) = \beta(\lambda - \mu + \mu\alpha + 1). \tag{3.3}$$

We shall state two lemmas which will be used in the sequel before we give a proof of the theorem. Proofs for these lemmas can be found in Lucaks and Laha [3].

LEMMA 3.2. *Let* U *and* V *be two random variables with* $E(U) = E(V) = 0$. *Let* $h(u, v)$ *denote the characteristic function of* (U, V). *Then* $E(V \mid U) = \beta U$ a.e., *if and only if,*

$$\partial h(u, v)/\partial v \,|_{v=0} = \beta \, dh(u, 0)/du$$

for all real u.

LEMMA 3.3. *Let* $\{X(t),\ t \varepsilon T\}$ *be a homogeneous process with independent increments. Let*

$$Y = \int_0^1 a(t) \, dX(t); \qquad Z = \int_0^1 b(t) \, dX(t)$$

for any two infinitely differentiable functions $a(t)$ *and* $b(t)$ *on* $[0, 1]$. *Let* $\varphi(u; h)$ *and* $\theta(u, v)$ *denote the characteristic functions of* $X(t + h) - X(t)$ *and* (Y, Z) *respectively.*

Then $\theta(u, v)$ *is different from zero for all real* u *and* v, *and*

$$\log \theta(u, v) = \int_0^1 \psi[ua(t) + vb(t)] \, dt$$

where $\psi(u) = \log \varphi(u; 1)$.

4. Proof of Theorem 3.1. *"Only if" part.* Let $\{X(t),\ t \varepsilon T\}$ be a symmetric stable process with $X(0) = 0$ and $E[X(t)] = 0$ for all t. Let $\theta(u, v)$ denote the log of the characteristic function of (Y_λ, Y_μ). $\theta(u, v)$ is well-defined since the process $\{X(t),\ t \varepsilon T\}$ is infinitely divisible. Let $\psi(u)$ denote the logarithm of the characteristic function of $X(t + 1) - X(t)$. Since $E(Y_\lambda) = E(Y_\mu) = 0$, it follows from Lemma 3.3, that

$$\theta(u, v) = \int_0^1 \psi[ut^\lambda + vt^\mu] \, dt. \tag{4.1}$$

Hence,

$$\partial\theta(u, v)/\partial u\,|_{u=0} = \int_0^1 t^\lambda \psi'[vt^\mu]\,dt, \tag{4.2}$$

and

$$d\theta(0, v)/dv = \int_0^1 t^\mu \psi'[vt^\mu]\,dt \tag{4.3}$$

where ψ' denotes the derivative of ψ. The differentiations under integral sign in (4.2) and (4.3) are valid since ψ' is continuous. Since the process is symmetric stable with mean zero, it is well known that $\psi(u) = -c\,|u|^\alpha$ for some real number c and $\alpha > 1$. It is easy to check from (4.2) and (4.3) that

$$\partial\theta(u, v)/\partial u\,|_{u=0} = \beta\, d\theta(0, v)/dv$$

where β satisfies (3.3). This relation in turn proves that $E[Y_\lambda \mid Y_\mu] = \beta Y_\mu$ a.e. in view of Lemma 3.2.

"If part". Let us define $\theta(u, v)$ and $\psi(u)$ as before. Since $E[Y_\lambda \mid Y_\mu] = \beta Y_\mu$ a.e., it follows from Lemma 3.2 that

$$\partial\theta(u, v)/\partial u\,|_{u=0} = \beta\, d\theta(0, v)/dv$$

for all v, and hence,

$$(\partial/\partial u)\Big[\int_0^1 \psi[ut^\lambda + vt^\mu]\,dt\Big]_{u=0} = \beta(d/dv)\Big[\int_0^1 \psi(vt^\mu)\,dt\Big] \tag{4.4}$$

by Lemma 3.3. Since the process is infinitely divisible with finite mean, it follows that ψ is differentiable and we have from (4.4),

$$\int_0^1 t^\lambda \psi'[vt^\mu]\,dt = \beta \int_0^1 t^\mu \psi'[vt^\mu]\,dt \tag{4.5}$$

for all v. Integrating both sides with respect to v, we get that

$$\int_0^1 t^{\lambda-\mu}\psi(vt^\mu)\,dt = \beta \int_0^1 \psi(vt^\mu)\,dt \tag{4.6}$$

for all v since $\psi(0) = 0$. It is easily seen from this relation that

$$\int_0^v s^{\frac{1}{2}(\lambda+1-2\mu)}\psi(s)\,ds = \beta v^{(\lambda-\mu)\mu^{-1}} \int_0^v s^{(1-\mu)\mu^{-1}}\psi(s)\,ds$$

for any $v > 0$. Differentiating with respect to v and simplifying we have

$$v^{\mu^{-1}}\psi(v)[1-\beta] = \beta(\lambda-\mu)\mu^{-1}\int_0^v s^{(1-\mu)\mu^{-1}}\psi(s)\,ds \tag{4.7}$$

for all $v > 0$. Differentiating again with respect to v, we have

$$\mu(1-\beta)v\psi'(v) = \psi(v)[\beta(\lambda-\mu) - (1-\beta)].$$

Since $X(1)$ has a non-degenerate distribution, it follows that $\beta \neq 1$. Hence

$$\psi'(v)[\psi(v)]^{-1} = [\beta(\lambda-\mu) - (1-\beta)][\mu(1-\beta)]^{-1} \tag{4.8}$$

for all $v > 0$. Let

$$\alpha = [\beta(\lambda-\mu) - (1-\beta)][\mu(1-\beta)]^{-1}. \tag{4.9}$$

Solving the differential equation in (4.8) under the condition that $\psi(\cdot)$ is continuous at the origin, we get that

$$\psi(v) = -cv^{\alpha} \tag{4.10}$$

where c is a constant different from zero. Since ψ is the logarithm of the characteristic function of a symmetric distribution with finite mean, it follows that $c > 0, \alpha > 1$ and $\psi(v) = -c\,|\,v\,|^{\alpha}$ for all v. It is well known that $\psi(\cdot)$ is the characteristic function of a symmetric stable law with exponent α. Hence the process $\{X(t), t \varepsilon T\}$ is a symmetric stable process with finite mean. It is easy to see that (3.3) follows from (4.9).

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