The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2123–2127

ON THE LOCAL BEHAVIOR OF MARKOV TRANSITION PROBABILITIES[1]

BY DAVID BLACKWELL AND DAVID FREEDMAN

University of California, Berkeley

1. Introduction. Let $P(t) = P(t, i, j)$ be a semigroup of stochastic matrices on the countable set $I = \{i, j, \cdots\}$. Suppose

(1) $$\lim_{t\to 0} P(t, i, i) = 1 \quad \text{for each} \quad i \varepsilon I.$$

Fix one state $a \varepsilon I$ and abbreviate

$$f(t) = P(t, a, a).$$

(2) THEOREM. *Suppose* $0 < \epsilon < 1$ *and* $f(1) \leqq 1 - \epsilon$. *Then*

$$\int_0^1 f(t)\, dt < 1 - \tfrac{1}{2}\epsilon.$$

(3) THEOREM. *Suppose* $0 < \epsilon < \frac{1}{4}$ *and* $f(1) \geqq 1 - \epsilon$. *Then for all* t *in* $[0, 1]$,

$$f(t) \geqq [1 + (1 - 4\epsilon)^{\frac{1}{2}}]/2 = 1 - \epsilon - \epsilon^2 - O(\epsilon^3).$$

COROLLARY. *Suppose* $0 < \epsilon < \frac{1}{4}$ *and* $\int_0^1 f(t)\, dt \geqq 1 - \epsilon$. *Then* $f(t) > 1 - 2\epsilon$ *for* $0 \leqq t \leqq 1$.

NOTE. (3) can be restated (using algebra) in this more attractive form: if $0 < \delta < \frac{1}{2}$ and $f(1) \geqq 1 - \delta + \delta^2$, then $f(t) \geqq 1 - \delta$ for $0 \leqq t \leqq 1$.

Perhaps it is worth noting explicitly that the bounds in (2) and (3) hold for all stochastic semigroups satisfying (1). The bounds in (3) are not supposed to be sharp, but they cannot be improved much, as (4) and (5) show.

(4) EXAMPLE. For any $\delta > 0$, there is an f with

$$f(t) \leqq \delta \quad \text{for} \quad \delta \leqq t \leqq 1 - \delta \quad \text{and} \quad f(1) > e^{-1}.$$

The right coefficient for ϵ^2 in (3) is not known to us, but

(5) EXAMPLE. For $K < \frac{3}{8}$ and small $\epsilon > 0$, there is an f with

$$f(\tfrac{2}{3}) < 1 - \epsilon - K\epsilon^2 \quad \text{and} \quad f(1) > 1 - \epsilon.$$

The constant $\frac{1}{2}$ in (2) is sharp, as shown by

(6) EXAMPLE. For $K < 2$, there is an f with

$$1 - f(1) > K[1 - \int_0^1 f(t)\, dt].$$

2. Proof of Theorem (2). Sippose a is not absorbing, so $0 < f(t) < 1$ for all $t > 0$. Suppose

(7) $$f(1) = 1 - \delta.$$

Received 19 February 1968.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 1312-67.

The problem is to show

$$\int_0^1 f(t)\,dt < 1 - \tfrac{1}{2}\delta. \tag{8}$$

But $f(t)f(1-t) \leqq f(1)$, for $0 < t < 1$. Moreover,

$$a + b - 1 < ab \quad \text{for} \quad 0 < a, b < 1, \tag{9}$$

because $0 < (1-a)(1-b) = 1 - a - b + ab$. Thus

$$f(t) + f(1-t) < 2 - \delta. \tag{10}$$

Integrate (10) from 0 to 1 and divide by 2, to get (8).□

We are grateful to Professor G. E. H. Reuter for removing the unnecessary half of a previous proof.

3. Proof of Theorem (3). Suppose $0 < \epsilon < \frac{1}{4}$ and $f(1) \geqq 1 - \epsilon$. Begin by establishing

$$f(t) \geqq 1 - \epsilon^{\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq 1. \tag{11}$$

Indeed, for $0 \leqq t \leqq 1$,

$$\begin{aligned} 1 - \epsilon \leqq f(1) &= f(t)f(1-t) + \textstyle\sum_{j \neq a} P(t, a, j)P(1-t, j, a) \\ &\leqq f(t)f(1-t) + \textstyle\sum_{j \neq a} P(t, a, j) \\ &= f(t)f(1-t) + 1 - f(t), \end{aligned}$$

so that

$$f(t)[1 - f(1-t)] \leqq \epsilon. \tag{12}$$

Plainly,

$$\text{If} \quad x, y \geqq 0 \quad \text{and} \quad xy \leqq \epsilon \quad \text{then} \quad x \leqq \epsilon^{\frac{1}{2}} \quad \text{or} \quad y \leqq \epsilon^{\frac{1}{2}}. \tag{13}$$

Consequently,

$$\text{for each} \quad t \in [0, 1], \quad \text{either} \quad f(t) \leqq \epsilon^{\frac{1}{2}} \quad \text{or} \quad f(1-t) \geqq 1 - \epsilon^{\frac{1}{2}}. \tag{14}$$

Similarly, or by putting $1 - t$ for t,

$$\text{for each} \quad t \in [0, 1], \quad \text{either} \quad f(t) \geqq 1 - \epsilon^{\frac{1}{2}} \quad \text{or} \quad f(1-t) \leqq \epsilon^{\frac{1}{2}}. \tag{15}$$

Now suppose $f(t) > \epsilon^{\frac{1}{2}}$ Relation (14) implies $f(1-t) \geqq 1 - \epsilon^{\frac{1}{2}} > \epsilon^{\frac{1}{2}}$, because $\epsilon < \frac{1}{4}$. Then relation (15) implies $f(t) \geqq 1 - \epsilon^{\frac{1}{2}}$. That is,

$$\text{for each } t \text{ in } [0, 1], \text{ either } f(t) \leqq \epsilon^{\frac{1}{2}} \text{ or } f(t) \geqq 1 - \epsilon^{\frac{1}{2}}. \tag{16}$$

An easy continuity argument now establishes (11).

Introduce $\theta(x) = 1 - (\epsilon/x)$. Let $b_0 = 1 - \epsilon^{\frac{1}{2}}$ and $b_{n+1} = \theta(b_n)$. If $f(t) \geqq b_n$ for all $t \in [0, 1]$, use (12) with $s = 1 - t$ to check that $f(s) \geqq b_{n+1}$ for all s in $[0, 1]$. By algebra, θ is convex and has fixed points

$$b_{\pm} = [1 \pm (1 - 4\epsilon)^{\frac{1}{2}}]/2.$$

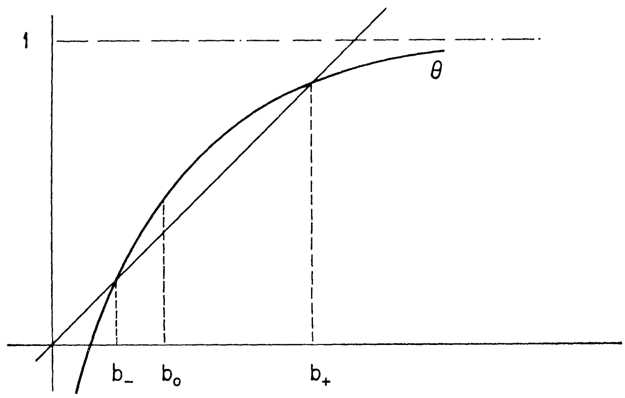

FIG. 1

Moreover, $b_- < b_0 < b_+$. Hence $b_n \uparrow b_+$.□

4. Proof of Corollary to (3) and remarks. By way of contradiction, suppose $0 < \epsilon < \frac{1}{4}$ and $\int_0^1 f(t)\,dt \geqq 1 - \epsilon$ and $f(t) \leqq 1 - 2\epsilon$ for some t in (0, 1). By (2),

$$\int_0^t f(u)\,du < t(1 - \epsilon).$$

Moreover, $f(u) \leqq 1 - 2\epsilon + 4\epsilon^2$ for $t \leqq u \leqq 1$ by the second form of (3), since $2\epsilon < \frac{1}{2}$. Consequently

$$\begin{aligned}\int_t^1 f(u)\,du &\leqq (1 - t)(1 - 2\epsilon + 4\epsilon^2)\\ &= (1 - t)(1 - \epsilon) - (1 - t)(\epsilon - 4\epsilon^2).\end{aligned}$$

Adding,

$$\int_0^1 f(u)\,du < (1 - \epsilon) - (1 - t)(\epsilon - 4\epsilon^2).$$

But $\epsilon - 4\epsilon^2 > 0$.□

REMARK. In (2), if $f(1) = 1 - \delta$, then $f(\frac{1}{3}) \leqq 1 - \frac{1}{3}\delta$, so $f(t) \leqq 1 - \frac{1}{3}\delta$ for $\frac{1}{3} \leqq t \leqq \frac{2}{3}$ (mod δ^2). Now

$$\begin{aligned}f(t) + f(1 - t) &= f(t)\cdot f(1 - t) + 1 - [1 - f(t)][1 - f(1 - t)]\\ &\leqq 2 - \delta \text{ everywhere}\\ &\leqq 2 - \delta - \tfrac{1}{9}\delta^2 \quad \text{on} \quad (\tfrac{1}{3}, \tfrac{2}{3}) \ (\text{mod } \delta^3).\end{aligned}$$

So $\int_0^1 f(t)\,dt \leqq 1 - \frac{1}{2}\delta - \frac{1}{54}\delta^2 + O(\delta^3)$. The best value for $\frac{1}{54}$ is unknown to us.

REMARK. If is easy to check that the inequalities $0 \leqq f(t) \leqq 1$ and $f(t)f(1 - t) \leqq 1$ were fully exploited in (2). For example, consider this function:

$$\begin{aligned}f(t) &= 1 \quad \text{for} \quad 0 \leqq t < \tfrac{1}{2}\\ &= 1 - \delta \quad \text{for} \quad \tfrac{1}{2} \leqq t \leqq 1.\end{aligned}$$

It is almost as easy to check that the continuity of f and the inequalities $0 \leqq$

$f(t) \leqq 1$ and $f(t)f(1-t) \leqq f(1) \leqq f(t)f(1-t) + 1 - f(t)$ were fully exploited in (3). For example, consider a function of this form:

$$f(0) = 1, \quad f(\tfrac{1}{2}) = [1 + (1-4\epsilon)^{\frac{1}{2}}]/2, \quad f(1) = 1 - \epsilon;$$

f is continuous and strictly decreasing on $[0, \frac{1}{2}]$;

f is continuous and strictly increasing on $[\frac{1}{2}, 1]$;

$f(1-t) = \theta[f(t)]$ for $0 \leqq t \leqq \frac{1}{2}$;

where

$$\theta(x) = 1 - (\epsilon/x)$$

5. A construction. Let $0 < q, c < \infty$. Let $\tau_1, \tau_2, \cdots$ be independent, expo-

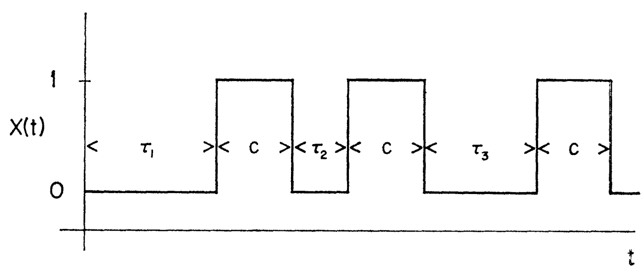

FIG. 2

nential random variables on $(\Omega, \mathfrak{F}, \mathcal{P})$, with common parameter q, so $\mathcal{P}\{\tau_n \geqq t\} = e^{-qt}$. Define a stochastic process $X(t)$ as follows:

$$\begin{aligned} X(t) = 0 \quad &\text{for} \quad 0 \leqq t < \tau_1 \\ &\text{and} \quad \tau_1 + c \leqq t < \tau_1 + c + \tau_2 \\ &\text{and} \quad \tau_1 + c + \tau_2 + c \leqq t < \tau_1 + c + \tau_2 + c + \tau_3 \\ &\qquad \vdots \end{aligned}$$

while

$$X(t) = 1 \text{ elsewhere.}$$

Let $g(t) = \mathcal{P}\{X(t) = 0\}$.

The process X is not Markov. However, there are stochastic semigroups P_n satisfying (1) such that $P_n(t, 0, 0) \to g(t)$ uniformly on bounded t-sets. This will be argued later.

Clearly,

$$g(t) = e^{-qt} \quad \text{for} \quad 0 \leqq t \leqq c, \tag{17}$$

and

$$g(t) = e^{-qt} + \int_0^{t-c} qe^{-q(t-c-s)}\, g(s)\, ds \quad \text{for} \quad c \leqq t. \tag{18}$$

In particular, g is continuous. Combine (17) and (18):

$$g(t) = e^{-qt} + q(t-c)e^{-q(t-c)} \quad \text{for} \quad c \leqq t \leqq 2c. \tag{19}$$

Discussion of Example (4). Let $c < 1$ increase to 1, and let $q = 1/(1-c)$. Then g tends to 0 uniformly on $[\delta, 1-\delta]$, while $g(1)$ decreases to e^{-1}.□

Discussion of Example (5). Fix $c = \frac{2}{3}$ and let q decrease to 0. Then

$$g(\tfrac{2}{3}) = 1 - \tfrac{2}{3}q + \tfrac{2}{9}q^2 + O(q^3)$$

while

$$g(1) = 1 - \tfrac{2}{3}q + \tfrac{7}{18}q^2 + O(q^3).$$

Thus

$$\begin{aligned} g(1) - g(\tfrac{2}{3}) &= \tfrac{1}{6}q^2 + O(q^3) \\ &= \tfrac{3}{8}[1 - g(1)]^2 + O(q^3) \end{aligned}$$

□

The Approximation Argument. If $\sigma_1, \sigma_2, \cdots, \sigma_n$ are independent and exponential with parameter n/c, then $\sigma_1 + \cdots + \sigma_n$ has mean c and variance c^2/n, so $\sigma_1 + \cdots + \sigma_n$ is practically c. Define a continuous time Markov process Y_n with state space $\{0, 1, \cdots, n\}$ as follows: from i the process jumps to $i+1$, except that from n the process jumps to 0. The holding time in 0 is exponential with parameter g. The holding time in $i = 1, \cdots, n$ is exponential with parameter n/c. It is not hard to verify that $P_n(t, 0, 0) \to g(t)$ uniformly on bounded t-sets, where P_n is the transition matrix of Y_n and satisfies (1).□

Discussion of Example (6). Let $q > 0$. Consider a Markov process with state space $\{0, 1\}$, such that the holding time parameter in 0 is q while 1 is absorbing. Then $f(t) = P(t, 0, 0) = e^{-qt}$, so for q near 0,

$$f(1) = e^{-q} = 1 - q + O(q^2)$$

and

$$\int_0^1 f(t)\, dt = (1 - e^{-q})/q = 1 - \tfrac{1}{2}q + O(q^2).$$

□

Obviously, the same reasoning applies in a much more general class of examples. For instance, let P be one stochastic semigroup for which (1) holds and $0 > P'(0, a, a) > -\infty$. Then study the semigroups P_λ, where $P_\lambda(t) = P(\lambda t)$, as $\lambda \to 0$.