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SOME INTEGRAL TRANSFORMS OF CHARACTERISTIC FUNCTIONS[1]

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1. Introduction. We shall deal with the basic convergence problem of sequences of probability distribution functions in relation to the celebrated Lévy continuity theorem. One of the simplest ways of proving this theorem was adopted by M. Loève [9]; it employs the integral characteristic function

$$f^I(x) = \int_0^x f(t)\,dt = \int_{-\infty}^{+\infty} (e^{ixu} - 1)/iu\,dF(u), \tag{1.1}$$

where $f(t)$ is the characteristic function of the distribution function $F(u)$. This method was also adopted in [7].

A standard form of the Lévy continuity theorem requires the continuity at the origin of the limiting function, $f(u)$, of a sequence of characteristic functions. L. Schmetterer [10] has shown that the continuity of $f(u)$ can be replaced by $(C, 1)$ summability of the Fourier series of $f(u)$ to one at the origin. So he considered the Fejér integral of a characteristic function $f(u)$:

$$L(\alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha t(\alpha t^2)^{-1} f(t)\,dt, \qquad (\alpha > 0). \tag{1.2}$$

Further, it was noticed by one of the authors [5], that the transform

$$L_0(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha(t - x)(\alpha(t - x)^2)^{-1} f(t)\,dt, \tag{1.3}$$

$\alpha > 0, -\infty < x < +\infty$, plays a role similar to that of $f^I(x)$ in the convergence problem of sequences of distribution functions.

Here, in place of (1.3), we will take the Fourier transform of the c.f. after multiplication by the Fejér kernel:

$$L(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha t(\alpha t^2)^{-1} f(t) e^{-itx}\,dt, \tag{1.4}$$

$\alpha > 0, -\infty < x < +\infty$. The use of this functional allows for simpler arguments, concerning the convergence problem, than (1.2) and, in some sense, even $f^I(x)$. This is based on the well-known and easily shown fact that

$$L(x, \alpha) = (2\alpha)^{-1} \int_0^{2\alpha} (F(x + u) - F(x - u))\,du, \tag{1.5}$$

where $F(x)$ is the df corresponding to the cf $f(t)$ in (1.4). In Section 2 we will illustrate the different roles played by (1.4) and (1.5).

As pointed out by L. Schmetterer [10], the Fejér kernel may be replaced by other summability kernels. Even so, it is worthwhile to approach this problem also with the Poisson integral of a characteristic function

$$f(x, y) = \int_{-\infty}^{+\infty} P(t - x, y) f(t)\,dt, \tag{1.6}$$

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$y > 0, -\infty < x < +\infty$, where

$$P(u, v) = v/\pi(v^2 + u^2), \qquad v > 0, \qquad -\infty < u < +\infty. \tag{1.7}$$

This is accomplished in Section 3.

Equation (1.6) obviously defines a function which is harmonic in the upper half plane, $y > 0$, and so leads in a natural way to the problem of characterizing certain harmonic functions to be the Poisson integrals of positive definite functions. We shall consider this question in Section 4.

2. The Fejér integral of a characteristic function. For convenience we suppose throughout that all non-decreasing functions are standardized:

$$F(u) = \tfrac{1}{2}\{F(u + 0) + F(u - 0)\}, \qquad -\infty < u < +\infty. \tag{2.1}$$

Suppose that a sequence $\{F_n(u)\}$ of distribution functions is given. The corresponding characteristic functions are denoted by $f_n(t)$, $n = 1, 2, \cdots$, and we set

$$L_n(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} (\sin^2 \alpha t/\alpha t^2) f_n(t) e^{-itx}\, dt, \quad \alpha > 0, \quad -\infty < x < +\infty. \tag{2.2}$$

THEOREM 1. *$F_n(x)$ converges to a non-decreasing function $F(x)$ up to additive constants if, and only if, $L_n(x, \alpha)$ converges for every $\alpha > 0$ and x, $-\infty < x < +\infty$. When $L_n(x, \alpha)$ converges, the limit function, $L(x, \alpha)$, can be written in the form* (1.4) *with $f(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$.*

This theorem is an analogue of Loève's weakcon vergence theorem ([9], p. 190).

PROOF. The "only if" part of the theorem and the last statement are obvious from (1.5) in the form

$$L_n(x, \alpha) = (2\alpha)^{-1} \int_0^{2\alpha} \{F_n(x + u) - F_n(x - u)\}\, du, \qquad \alpha > 0, \qquad -\infty < x < +\infty. \tag{2.3}$$

The converse is a consequence of Helly's selection theorem and the convergence of $L_n(x, \alpha)$ as given in (2.3). The convergence of $L_n(x, \alpha)(\alpha > 0, -\infty < x < +\infty)$ implies that $\int_0^\xi \{F_n(x + u) - F_n(x - u)\}\, du$ converges for all ξ, $-\infty < \xi < +\infty$. Because of Helly's theorem, $\{F_n(y)\}$ contains a subsequence $\{F_{n_k}(y)\}$ convergent to a non-decreasing function $F(y)$. If there exist two convergent sub-sequences $\{F_{n_k}(y)\}$ and $\{F_{m_k}(y)\}$ with limiting functions $F(y)$ and $G(y)$ respectively, then

$$\int_0^\xi \{F(x + u) - F(x - u)\}\, du = \int_0^\xi \{G(x + u) - G(x - u)\}\, du \quad (-\infty < \xi < +\infty).$$

This implies that $F(x + \xi) - F(x - \xi) = G(x + \xi) - G(x - \xi)$ for every x and almost every ξ (where the exceptional set E_x may depend on x).

It follows easily that $F(y) = G(y) +$ constant for all y, $-\infty < y < +\infty$. (e.g., for any real x, choose a sequence $\xi_n \notin E_x$ such that $\xi_n \downarrow 0$ $(n \to +\infty)$ and pass to the limit in $F(x + \xi_n) - F(x - \xi_n) = G(x + \xi_n) - G(x - \xi_n)$).

The last statement of the theorem follows by letting $f(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$;

the limit of (2.3) is $L(x, \alpha) = (2\alpha)^{-1} \int_0^{2\alpha} \{F(x+u) - F(x-u)\}\, du$ which is equal to (1.4). This completes the proof.

It is of some interest to note that we may avoid direct use of Helly's selection theorem and use instead some basic properties of convex functions. This is because $\int_0^{\xi} \{F_n(x+u) - F_n(x-u)\}\, du$, being a continuous convex function, has a limiting function $J(x, \xi)$ which is also convex and continuous in ξ for each real x. But a continuous convex function has right and left derivatives at every point and they are equal a.e., the exceptional set being at most *countable*. The convergence of $F_n(u) - F_n(v)$ follows for all u, v except those in this countable set.

THEOREM 2. *$F_n(x)$ converges to a distribution function $F(x)$ if, and only if,*

(i) *$L_n(x, \alpha)$ converges for every $\alpha > 0$ and x, $-\infty < x < +\infty$, and*

(ii) *$L_n(0, \alpha) = L_n(\alpha)$ converges uniformly for all $\alpha \geqq A$, for some $A > 0$.*

PROOF. Because of Theorem 1, it is sufficient to show that if $\{F_n(x)\}$ converges to a bounded non-decreasing function $F(x)$, up to additive constants, then (ii) is a n.a.s.c. for $F(x)$ to be a distribution function.

By Theorem 1, $L_n(x, \alpha)$ converges and the limit may be written as

$$L(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} (\sin^2 \alpha t/\alpha t^2) f(t) e^{-itx}\, dt, \tag{2.4}$$

where $f(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$.

Suppose that (ii) holds and set $L(0, \alpha) = L(\alpha)$. Clearly, $\lim_{\alpha \to +\infty} L_n(\alpha) = f_n(0) = 1$ for each n. The uniform convergence then implies that $\lim_{\alpha \to +\infty} L(\alpha) = 1$. But $\lim_{\alpha \to +\infty} L(\alpha) = f(0) = F(+\infty) - F(-\infty)$. Hence $F(x)$ is a distribution function.

Conversely, suppose that $F(x)$ is a distribution function. Then $L(+\infty) = 1$ from (2.4) and for each $\alpha > 0$, $\lim_{n \to +\infty} L_n(\alpha) = L(\alpha)$. Moreover, for each n, $L_n(\alpha)$ is from (2.3) a non-decreasing function on $\alpha \geqq A$, for any $A > 0$, $\lim_{n \to \infty} L_n(+\infty) = L(+\infty)$ and $\lim_{n \to +\infty} L_n(A) = L(A)$. Hence, since $L(\alpha)$ is a continuous function, it follows that the convergence of $\{L_n(\alpha)\}$ to $L(\alpha)$ is uniform on $[A, +\infty)$. This completes the proof of the theorem.

REMARK 1. Let $\{f_n(t)\}$ be a sequence of cf's and $\{F_n(x)\}$ the corresponding sequence of df's. If $f_n(t) \to f^*(t)$ for every t, $-\infty < t < +\infty$, then $L_n(x, \alpha)$ converges on $\alpha > 0$, $-\infty < x < +\infty$, and so by Theorem 1, $F_n(x) \to F(x)$ up to additive constants and $L_n(\alpha) \to L(\alpha)$, where

$$L(\alpha) = \pi^{-1} \int_{-\infty}^{+\infty} (\sin^2 \alpha t/\alpha t^2) f(t)\, dt, \qquad (\alpha > 0), \tag{2.5}$$

with $f(t) = \int_{-\infty}^{+w} \exp(itx)\, dF(x)$. On the other hand, from the form of the Fejér integral, the condition that $f_n(t) \to f^*(t)$ implies that $L(\alpha)$ equals

$$\pi^{-1} \int_{-\infty}^{+\infty} (\sin^2 \alpha t/\alpha t^2) f^*(t)\, dt. \tag{2.6}$$

Hence, (2.5) and (2.6) are equal. (The equality of (2.5) and (2.6) for all $\alpha > 0$, implies only that $\operatorname{Re} f(t) = \operatorname{Re} f^*(t)$, $-\infty < t < +\infty$.)

Thus, the $(C, 1)$-summability of the Fourier series of $f^*(t)$ at $t = 0$ to one, implies that $L(\alpha) \to 1$ as $\alpha \to +\infty$. But this implies that $F(x)$ is a df as noted in the proof of Theorem 2. This gives L. Schmetterer's result [10].

REMARK 2. If $f_n(t)$ converges to $f^*(t)\,(-\infty < t < +\infty)$, then (i) of Theorem 2 holds and if the convergence is uniform on $-\delta \leqq t \leqq \delta$ for some $\delta > 0$, then condition (ii) of Theorem 2 also holds and so $\{F_n(x)\}$ converges to a df. This is Lévy's continuity theorem.

3. Sequences of Poisson integrals. Let $\{F_n(x)\}$ be a sequence of df's $\{f_n^*(t)\}$ the corresponding sequence of cf's and $f_n(x, y)$ the Poisson integral of $f_n^*(t)\,(n = 1, 2, \cdots)$. We shall now prove the Poisson integral analogues of Theorems 1 and 2.

THEOREM 3. *$\{F_n(x)\}$ converges to a non-decreasing function $F(x)$ up to additive constants if, and only if, $\{f_n(x, y)\}$ converges for every $y > 0$ and x, $-\infty < x < +\infty$. In this case, the limit $f(x, y)$, $-\infty < x < +\infty$, $y > 0$, of $\{f_n(x, y)\}$ is the Poisson integral of the Fourier-Stieltjes transform of $F(u)$.*

PROOF. It is easily seen that

$$f_n(x, y) = \int_{-\infty}^{+\infty} e^{ixu-y|u|}\, dF_n(u) \tag{3.1}$$

$y > 0$, $-\infty < x < +\infty$, $n = 1, 2, \cdots$.

Hence, if $\{F_n(x)\}$ converges to $F(x)$ up to additive constants, then by the extended Helly-Bray theorem ([9], p. 181) $\{f_n(x, y)\}$ converges to $f(x, y)$, where

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{+\infty} e^{ixu-y|u|}\, dF(u) \\ &= \int_{-\infty}^{+\infty} P(t - x, y) f^*(t)\, dt, \qquad y > 0, \qquad -\infty < x < +\infty, \end{aligned} \tag{3.2}$$

with

$$f^*(t) = \int_{-\infty}^{+\infty} e^{itu}\, dF(u). \tag{3.3}$$

Conversely, suppose that $\{f_n(x, y)\}$ converges to $f(x, y)$ on $y > 0$, $-\infty < x < +\infty$. By Helly's selection theorem every subsequence $\{F_{n_k}(u)\}$ contains a subsequence $\{F_{n_k'}(u)\}$, $-\infty < x < +\infty$, converging to a bounded non-decreasing function $F(u)$. The extended Helly-Bray theorem then shows that $f_{n_k'}(x, y)$ converges to $\tilde{f}(x, y) = \int_{-\infty}^{+\infty} e^{ixu-y|u|}\, dF(x)$, $y > 0$, $-\infty < x < +\infty$. Hence, $f(x, y) = \tilde{f}(x, y)$ in the upper half plane and so $F(u)$ is uniquely determined (up to additive constants) by $f(x, y)$, $-\infty < x < +\infty$, $y > 0$. Therefore, every convergent subsequence has the same limit up to additive constants. The proof is complete once it is noted that the stated representation as a Poisson integral of the Fourier-Stieltjes transform of F follows from the integral representation of $\tilde{f}(x, y)$.

REMARK 3. It is clear from the proof of this theorem that the results continue to hold if the sequence of df's, $\{F_n(x)\}$, is replaced by any uniformly bounded sequence of non-decreasing functions on $(-\infty, +\infty)$.

THEOREM 4. *$\{F_n(u)\}$ converges to a* df *$F(u)$ if, and only if,*

(i) *$\{f_n(x, y)\}$ converges to $f(x, y)$ for every x, $-\infty < x < +\infty$ and $y > 0$.*

(ii) *$\{f_n(0, y)\}$ converges uniformly for $y > 0$.*

PROOF. Suppose that (i) and (ii) hold. We have that $\lim_{y\to 0+} f_n(0, y) = f_n^*(0) = 1$ for each n, $n = 1, 2, \cdots$. Hence, (ii) implies that $f(0, y) \to 1$ as

$y \to 0+$. Since from (3.2), $f(0, y) \to f^*(0)$, as $y \to 0+$, where $f^*(t)$ is given by (3.3), the sufficiency of (i) and (ii) follows from Theorem 3 and $f^*(0) = 1 = F(+\infty) - F(-\infty)$.

Conversely suppose that $F_n(u) \to F(u)$, where $F(u)$ is a df. Then by Theorem 3, (i) holds. That (ii) also holds follows from the uniform convergence of the cf's $f_n^*(t)$ in $[-\delta, \delta]$, $\delta > 0$, and the inequality

$$|f_n(0, y) - 1| \leqq \int_{|t| \leqq \delta} P(t, y)|f_n^*(t) - 1| \, dt + 2 \int_{|t| > \delta} P(t, y) \, dt,$$

for any $\delta > 0$.

REMARK 4. In Theorem 3, $f(x, y)$ as the Poisson integral of the Fourier-Stieltjes transform $f^*(t)$ of $F(u)$ has the property that

$$\lim_{y \to 0+} f(0, y) = \alpha \tag{3.4}$$

if $F(+\infty) - F(-\infty) = \alpha$.

Equation (3.4) is the same as saying that the Fourier series of $f^*(t)$ is Abel summable to α at $t = 0$.

4. Some classes of harmonic function. Here we consider functions which are harmonic in the upper half plane and have representation as Poisson integrals of characteristic functions, or more generally, of Fourier-Stieltjes transforms of functions of bounded variation.

We will require a characterizing property for positive definite functions. The following one is convenient and does not seem to be explicitly mentioned in the literature. Its proof is standard (e.g. H. Cramér [2], T. Kawata [6], González Dominguez [3]).

THEOREM 5. *Let $f(x)$ be a bounded continuous function on $(-\infty, +\infty)$. Then $f(x)$ is positive definite if, and only if,*

$$0 \leqq \int_{-\infty}^{+\infty} \varphi(x) f(x) \, dx \tag{4.1}$$

for all $\varphi(x) \varepsilon L_1$, for which

$$\hat{\varphi}(u) = \int_{-\infty}^{+\infty} e^{itu} \varphi(t) \, dt \geqq 0, \qquad -\infty < u < +\infty. \tag{4.2}$$

PROOF. The necessity of (4.1) follows from Parseval's relation and (4.2).

The sufficiency of (4.1) may be observed by first noting that

$$\int_{-\alpha}^{\alpha} e^{-itx}(1 - |t|/\alpha) f(t) \, dt \geqq 0, \qquad -\infty < x < +\infty,$$

since the function $\varphi(t) = e^{-itx}(1 - |t|/\alpha)$ for $|t| < \alpha$ and $\varphi(t) = 0$, $|t| \geqq \alpha$ has a non-negative Fourier transform, and then proceeding as in the proof of S. Bochner's theorem on positive definite functions (e.g. Loéve [9], p. 208 or Yu. Linnik [8], p. 42).

The following corollary is an analogue of a result due to S. Bochner [1] and I. J. Schoenberg [11] concerning Fourier-Stieltjes transforms of functions of bounded variation over $(-\infty, +\infty)$.

COROLLARY 1. *Let $f(x)$ be a bounded continuous function on $(-\infty, +\infty)$. If*

$$0 \leqq \int_{-\infty}^{+\infty} \varphi(x)f(x)\,dx \leqq k \max_{-\infty<u<+\infty} \hat{\varphi}(u) \tag{4.3}$$

for every $\varphi(x)$ in L_1 for which $\hat{\varphi}(x) \geqq 0$, where k is a constant independent of $\varphi(x)$, then there exists a bounded non-decreasing function $F(x)$ such that

$$f(t) = \int_{-\infty}^{+\infty} e^{itx}\,dF(x) \tag{4.4}$$

and

$$F(+\infty) - F(-\infty) \leqq k. \tag{4.5}$$

The converse holds.

PROOF. Suppose that (4.3) holds, then from the lefthand side of this inequality we have the existence of a bounded non-decreasing function $F(x)$ satisfying (4.4) by Theorem 5. Hence, we need only show that (4.5) holds. For this purpose we introduce the function

$$\varphi(u) = (\pi\epsilon u^2)^{-1}\{\cos(u(A-\epsilon)) - \cos uA\},$$

where A and ϵ are fixed positive constants with $0 < \epsilon < A$. Then $\varphi(x)$ belongs to L_1 and has Fourier transform $\hat{\varphi}(t)$ given by

$$\begin{aligned} \hat{\varphi}(t) &= 1, && |t| \leqq A-\epsilon, \\ &= (A-|t|)/\epsilon, && A-\epsilon \leqq |t| \leqq A, \\ &= 0, && |t| > A. \end{aligned} \tag{4.6}$$

Since $\hat{\varphi}(t) \geqq 0$ for all t $(-\infty < t < +\infty)$ and $\max \hat{\varphi}(t) = 1$, we have from (4.3) that

$$0 \leqq \int_{-\infty}^{+\infty} \hat{\varphi}(t)\,dF(t) \leqq k.$$

It then follows from (4.6) that

$$0 \leqq \int_{-A+\epsilon}^{A-\epsilon} dF(x) \leqq k;$$

since k is independent of A, (4.5) is immediate.

The converse is obvious and so the proof is complete.

We now introduce the family of functions H_c by requiring that $f(x, y) \in H_c$ if, and only if,

(i) $f(x, y)$ is defined and harmonic on $-\infty < x < +\infty$, $y > 0$;

(ii) the mapping $x \to f(x, y)$ is bounded on $(-\infty, +\infty)$ for each $y > 0$;

(iii) for each $\varphi(x) \in L_1$ for which $\hat{\varphi}(t) \geqq 0$

$$0 \leqq \int_{-\infty}^{+\infty} \varphi(x)f(x, y)\,dx \leqq M \max_{-\infty<u<+\infty} \hat{\varphi}(u), \tag{4.7}$$

where M is a constant independent of $\varphi(x)$ and y $(y > 0)$.

Theorem 6 will show that the members of H_c are Poisson integrals of positive definite functions. For the proof of this theorem we require a uniqueness theorem for functions which are bounded and harmonic in the upper half plane.

LEMMA. *If* $H(x, y)$ *is a bounded harmonic function on* $y > 0$, $-\infty < x < +\infty$ *and* $\lim_{y\to 0+} H(x, y) = 0$ *for almost all* x, *then* $H(x, y)$ *vanishes on* $y > 0$, $-\infty < x < +\infty$.

This lemma follows from the corresponding result concerning bounded harmonic functions on the unit disc [4] applying a linear transformation taking the upper half plane onto the (open) unit disc.

THEOREM 6. $f(x, y) \varepsilon H_c$ *if and only if* $f(x, y)$ *is the Poisson integral of a continuous positive definite function.*

PROOF. Suppose that $f(x, y) \varepsilon H_c$, then for fixed $y > 0$, $x \to f(x, y)$ is a bounded continuous function on $(-\infty, +\infty)$. Hence, (4.7) and Corollary 1 imply the existence of a bounded non-decreasing function $F_y(u)$ on $(-\infty, +\infty)$ and a constant $M > 0$, such that

$$f(x, y) = \int_{-\infty}^{+\infty} e^{ixu} \, dF_y(u) \tag{4.8}$$

and

$$F_y(+\infty) - F_y(-\infty) \leqq M, \tag{4.9}$$

where M is independent of y. It follows that $f(x, y)$ is bounded on $-\infty < x < +\infty, y > 0$.

Now let $\delta > 0$ be fixed and construct the Poisson integral

$$f_1(x, y; \delta) = \int_{-\infty}^{+\infty} P(x - t, y - \delta) f(t, \delta) \, dt, \qquad -\infty < x < +\infty, \qquad y > \delta.$$

Since $f(x, \delta)$ is continuous for $\delta > 0$, from a well-known result concerning Poisson integrals [4], p. 123–124, we obtain

$$\lim_{y \to \delta+} f_1(x, y; \delta) = f(x, \delta) \tag{4.10}$$

for every x.

Then, since $f_1(x, y; \delta)$ and $f(x, y)$ have the same boundary values on $y = \delta$, $-\infty < x < +\infty$ and $f(x, y)$ is bounded on the upper half plane, the lemma implies that

$$f(x, y) = \int_{-\infty}^{+\infty} P(t - x, y - \delta) f(t, \delta) \, dt \tag{4.11}$$

for all $y \geqq \delta$, $-\infty < x < +\infty$.

Using (4.8) and (4.11), a simple calculation yields

$$f(x, y) = \int_{-\infty}^{+\infty} e^{ixu - (y-\delta)|u|} \, dF_\delta(u) \tag{4.12}$$

for all $y \geqq \delta$, $-\infty < x < +\infty$.

Let $y > 0$ be arbitrary, but fixed and $\{\delta_n\}$ be a numerical sequence, $\delta_n \downarrow 0$ $(n \to +\infty)$, with $y > \delta_1$. Set $F_{\delta_n}(x) = F_n(x)$. Then (4.12) can be written in the form

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{+\infty} e^{ixu - y|u|} \, dF_n + \int_{-\infty}^{+\infty} e^{ixu - y|u|}(e^{\delta_n|u|} - 1) \, dF_n(u) \\ &= f_n(x, y) + I_n(x, y), \end{aligned}$$

where $f_n(x, y)$ is the Poisson integral of the Fourier-Stieltjes transform of $F_n(u)$, and

$$I_n(x, y) = \int_{-\infty}^{+\infty} e^{ixu-y|u|}(e^{\delta_n|u|} - 1)\, dF_n(u),$$

$(n = 1, 2, \cdots)$. It is easily seen that $I_n(x, y) \to 0$ for any (x, y) $-\infty < x < +\infty$, $y > 0$ as $n \to +\infty$.

Hence, since $0 \leqq F_n(+\infty) - F_n(-\infty) \leqq M$, $(n = 1, 2, \cdots)$, and $f_n(x, y) \to f(x, y)$ as $n \to +\infty$, $-\infty < x < +\infty$, $y > 0$, Theorem 3 and Remark 3 apply and $f(x, y)$ is the Poisson integral of a continuous positive definite function. Since the converse is obvious the proof is complete.

REMARK 5. Let $f(x, y) \varepsilon H_c$ and $f^*(t)$ be the positive definite function referred to in Theorem 6. From a familiar property of Poisson integrals and the continuity of $f^*(t)$, $\lim_{y\to 0+} f(x, y) = f^*(x)$ for all x, $-\infty < x < +\infty$. For a given α, $0 < \alpha < +\infty$, we will denote by $H_c(\alpha)$ that sub-family of H_c consisting of those $f(x, y) \varepsilon H_c$ for which $f^*(t)$ satisfies $f^*(0) = \alpha$. We then have the following:

COROLLARY 2. *The class $H_c(1)$ coincides with the class of Poisson integrals of* cf's.

REMARK 6. It is clear from the proof of Theorem 6 that *if* (i) *$f(x, y)$ is a bounded harmonic function on $-\infty < x < +\infty$, $y > 0$ and* (ii) $0 \leqq \int_{-\infty}^{+\infty} \varphi(x) f(x, y)\, dx$ *for all $\varphi(x) \varepsilon L_1$ for which $\hat{\varphi}(t) \geqq 0$, then $f(x, y) \varepsilon H_c$ and conversely.*

Hence, Theorems 5 and 6 combine to yield the statement that *every function $f(x, y)$ which is bounded and harmonic on the upper half plane $y > 0$ and for which the mapping $x \to f(x, y)$ is positive definite for each $y > 0$ is the Poisson integral of a continuous positive definite function.*

In this form Theorem 6 is similar to a result of D. V. Widder [12]. He has shown that a *function $u(x, y)$ defined on the upper half plane $y > 0$ is harmonic, non-negative and absolutely integrable on $(-\infty, +\infty)$ for each $y > 0$ if, and only if, it is the Poisson integral of a (finite) positive measure.*

Where Widder's theorem is analogous to the Herglotz theorem on the unit disc [4], p. 34, Theorem 6 in the above form is linked in some sense to Fatou's theorem [4], p. 33, stating that *every bounded harmonic function in the unit disc is the Poisson integral of a bounded function on the circle.*

Finally, we introduce the family of functions H_v by requiring that $f(x, y) \varepsilon H_v$ if, and only if,

(i)′ $f(x, y)$ is defined and harmonic on $-\infty < x < +\infty$, $y > 0$;

(ii)′ the mapping $x \to f(x, y)$ is bounded on $(-\infty, +\infty)$ for each $y > 0$;

(iii)′ for each $\varphi(x) \varepsilon L_1(-\infty, +\infty)$

$$\left|\int_{-\infty}^{+\infty} \varphi(x) f(x, y)\, dx\right| \leqq M \max_{-\infty<u<+\infty} |\hat{\varphi}(u)|, \tag{4.13}$$

where M is a constant independent of $\varphi(x)$ and y $(y > 0)$.

Then we have the following

THEOREM 7. *$f(x, y) \varepsilon H_v$ if, and only if, $f(x, y)$ is the Poisson integral of the Fourier-Stieltjes transform of a function of bounded variation on $(-\infty, +\infty)$.*

This theorem is easily proved in the same way as Theorem 6 with the applications of Theorem 3 and Corollary 1 replaced by Helly's selection theorem and the following theorem of I. J. Schoenberg [11].

THEOREM A. *A bounded continuous function $f(x)$ is the Fourier-Stieltjes transform of a function, $G(u)$, of bounded variation on $(-\infty, +\infty)$ if, and only if, condition* (iii)′ *holds with $f(x)$ in place of $f(x,y)$. When* (4.13) *holds, then*

$$\int_{-\infty}^{+\infty} |dG(u)| \leqq M. \tag{4.14}$$

(The original proof of Theorem 7 was considerably simplified by a suggestion of R. P. Boas concerning the application of Theorem A.)

REFERENCES

[1] BOCHNER, S. (1934). A theorem on Fourier integrals. *Bull. Amer. Math. Soc.* **40** 271–276.

[2] CRAMÉR, H. (1939). On the representation of a function by certain Fourier integrals. *Trans. Amer. Math. Soc.* **46**.

[3] DOMINGUEZ, A. GONZÁLEZ (1940). The representation of functions by Fourier integrals. *Duke Math. J.* **6** 246–255.

[4] HOFFMAN, K. (1962). *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs.

[5] KAWATA, T. (1946). *Fourier Analysis and Probability Theory* (in Japanese). Chubunkan, Tokyo.

[6] KAWATA, T. (1950). Representation of a function by the Fourier-Stieltjes integral. *Ann. Inst. Statist. Math.* **1** 131–139.

[7] KAWATA, T. and UDAGAWA, M. (1949). On infinite convolutions. *Kōdai Math. Sem* **3**.

[8] LINNIK, YU. V. (1965). *Decomposition of Probability Distributions*. Oliver and Boyd, Edinburgh.

[9] LOÉVE, M. (1960). *Probability Theory* (2nd ed.). D. Van Nostrand, New York.

[10] SCHMETTERER, L. (1965). Some theorems on the Fourier analysis of positive definite functions. *Proc. Am. Math. Soc.* **16** 1141–1146.

[11] SCHOENBERG, I. J. (1934). A remark on the preceding note by Bochner. *Bull. Amer. Math. Soc.* **40** 277–278.

[12] WIDDER, D. V. (1963). Functions of three variables which satisfy both the heat equation and Laplace's equation in two variables. *J. Australian Math. Soc.* **3** 396–407.