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OPTIMAL STOPPING FOR FUNCTIONS OF MARKOV CHAINS[1]

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1. The Introduction. The purpose of this paper is to prove the existence of finite optimal stopping rules for certain problems (Theorem 1 and Theorem 2), that are generalizations of a problem introduced by Y. S. Chow and H. Robbins [1] and subsequently generalized by A. Dvoretzky [3].

The problem of Y. S. Chow and H. Robbins is stated as follows: let S_n be the excess of the number of heads over the number of tails in the first n tosses of a fair coin. Does there exist a finite stopping rule for which the expected average gain is maximal? They proved the existence of such a stopping rule; subsequently A. Dvoretzky considered a sequence $X_1, X_2, \cdots$, of independent identically distributed random variables with finite variance, and proved the existence of a finite stopping rule which maximizes $E(S_t/t)$ where $S_n = X_1 + X_2 + \cdots + X_n$. Our method of proof consists of looking at the rate at which the expected tail-income $\sup_{t\varepsilon T_\infty} E(S_t/(a + t))$ goes to zero as $a \to \infty$ (where T_∞ is the class of all stopping rules). Then we use this information to show that there is an improvement for any stopping rule which continues indefinitely with positive probability.

2. Definitions and preliminaries. Let $\{X_n, F_n, n = 1, 2, \cdots\}$ be a stochastic sequence defined on a probability space (Ω, F, P), (i.e., (F_n) is an increasing sequence of sub-sigma-algebras of F, and for each $n \geqq 1$, X_n is a random variable measurable F_n), with $E|X_n| < \infty$ for $n = 1, 2, \cdots$, and $E(\sup_n X_n^+) < \infty$. Let T_∞ = class of all stopping rules with respect to (F_n), i.e., class of all $t:\Omega \to \{1, 2, \cdots, \infty\}$ such that $[t = k] \varepsilon F_k$ for $k = 1, 2, \cdots$. $T = \{t \varepsilon T_\infty : t < \infty \text{ a.s.}\}$. Given a $\tau \varepsilon T$ let,

$$T_\infty^{(\tau)} = \text{class of all random variables } t:\Omega \to \{0, 1, \cdots, \infty\}$$

such that $[t = k] \varepsilon F_{\tau+k}$.

If $t \varepsilon T_\infty$, following D. O. Siegmund we adopt the convention that $X_t = \limsup_{n\to\infty} X_n$ if $t = \infty$.

For this class of stochastic sequences D. O. Siegmund has shown (Theorem 4 of [5]), that if:

$$s = \text{first } n \geqq 1 \text{ such that } X_n = f_n$$
$$= \infty \quad \text{if no such } n \text{ exists,}$$

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where f_n is "the best we can expect to do as a function of the first n observations if we do not stop before time n, and we may continue indefinitely with positive probability." Then s is optimal stopping rule, i.e., $E(X_s) \geqq E(X_t)$ for any other stopping rule t. We shall refer to this stopping rule as the "functional equation rule." The sequence of rewards that we are considering here satisfy $X_n \to 0$ as $n \to \infty$, and $E(\sup_n |X_n|) < \infty$. So we will be interested in proving the finiteness of the "functional equation rule."

We shall use the following lemma that appears in [4].

LEMMA 1. *Let $Y_1, Y_2, \cdots, F_0, F_1, F_2, \cdots$ be a stochastic sequence with:*

$$E(Y_{n+1} \mid F_n) = 0 \quad \text{and} \quad E(Y_{n+1}^2 \mid F_n) = U_n < \infty \qquad \text{for} \quad n = 0, 1, 2, \cdots$$

and

$$(Y_n + \cdots + Y_{n+k})/(U_n + \cdots + U_{n+k}) \to_{\text{a.s.}} 0 \quad \text{as} \quad k \to \infty \qquad \text{for } n = 1, 2, \cdots .$$

For a fixed n let u be an F_{n-1}-measurable random variable with $u > 0$ a.s. Then

$$E\{|(Y_n + \cdots + Y_{n+t})/(u + U_n + \cdots + U_{n+t})| \mid F_{n-1}\} < u^{-\frac{1}{2}}$$

uniformly in $t \varepsilon T_\infty^{(n)}$.

3. A lemma.

LEMMA 2. *Let $X_1, X_2, \cdots$, be a sequence of random variables defined on a probability space (Ω, F, P) and let G be a class of random variables also defined on (Ω, F, P) and with values in the set $\{1, 2, \cdots, \infty\}$. Suppose that there exists a constant $\sigma > 0$ with the following property:*

$$E(S_t/(a + t)) < \sigma a^{-\frac{1}{2}} \quad \text{for all} \quad a > 0 \quad \text{and all} \quad t \varepsilon G.$$

If for some positive numbers K, a, b and for a $t \varepsilon G$ we have $b \geqq K a^{\frac{1}{2}}$ and $P(t > a) > 2\sigma/K$ then, $E((b + S_t)/(a + t)) < b/a$.

PROOF.

$$\begin{aligned} E((b + S_t)/(a + t)) &< bE((a + t)^{-1}) + \sigma a^{-\frac{1}{2}} \\ &\leqq b((2a)^{-1} + P(t \leqq a)/(2a)) + \sigma a^{-\frac{1}{2}} \\ &= b(a^{-1} - P(t > a)/(2a)) + \sigma a^{-\frac{1}{2}} < b/a. \end{aligned}$$

4. Markov chain case. Let $X_1, X_2, \cdots$, be a stationary Markov chain with countable state space J forming a positive recurrent class and stationary initial distribution $\{\pi(j)\}_{j\varepsilon J}$. Let $\tau_1(j) < \tau_2(j) < \cdots$, be the times at which $X_n = j (j \varepsilon J)$. f and g real valued functions defined on J and satisfying: $\sum_{i\varepsilon J} \pi(i)|f(i)| < \infty$, $\sum_{i\varepsilon J} \pi(i)f(i) = 0$, $E(\sum_{n=\tau_1(j)}^{\tau_2(j)-1} |f(X_n)|)^2 < \infty$, $0 < c < g$, $\sum_{i\varepsilon J} \pi(i)g(i) < \infty$ (where c is a constant).

THEOREM 1. *For the conditions just stated and for any initial distribution, the* *"functional equation rule" for the stochastic sequence*

$$\{(f(X_1) + \cdots + f(X_n))/(g(X_1) + \cdots + g(X_n)), F_n\}_{n=1,2,\cdots}$$

where $F_n = \sigma(X_1, X_2, \cdots, X_n)$ *is finite.*

PROOF. Without loss of generality we may suppose $c = 1$. Fix a state $j \varepsilon J$ and in what follows τ_n refers to $\tau_n(j)$.

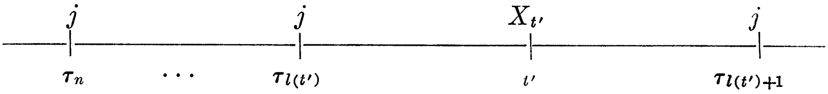
CLAIM 1. *There exists a positive constant* σ *such that,*

$$E((f(X_{\tau_n+1}) + \cdots + f(X_{\tau_n+t}))/(g(X_1) + \cdots + g(X_{\tau_n}) + t) \mid F_{\tau_n}) < \sigma(g(X_1) + \cdots + g(X_{\tau_n}))^{-\frac{1}{2}}$$

for all $n = 1, 2, \cdots$ *and all* $t \varepsilon T_\infty^{(\tau_n)}$ *with* $t \geqq 1$ a.s.

PROOF OF CLAIM 1. We divide the proof of Claim 1 in two cases.

Case 1. Suppose that $[f < 0]$ is a finite subset of J. For a $t \varepsilon T_\infty^{(\tau_n)}$, let $t' = \tau_n + t$ and let $l(t') = \sum_{n=1}^{t'} I_{[X_n=j]}$



let

$$Y_k = \sum_{n=\tau_k}^{\tau_{k+1}-1} f(X_n), \qquad k = 1, 2, \cdots,$$

and

$$S^{t'} = \sum_{n=t'+1}^{\tau_{l(t')+1}-1} f(X_n) \quad \text{if} \quad t' + 2 \leqq \tau_{l(t')+1} \quad \text{and} \quad t' < \infty$$
$$= 0 \quad \text{otherwise.}$$

By Theorem I.14.3 of [2] the random variables $Y_1, Y_2, \cdots$ are independent and identically distributed with $E(Y_1) = 0$ and $E(Y_1^2) = u < \infty$ we have:

$$\begin{aligned} & E(|(Y_n + \cdots Y_{n+l(t')-n})(1 + g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1}| \mid F_{\tau_n}) \\ & \leqq E(|(Y_n + \cdots + Y_{n+l(t')-n}) \\ (1) \qquad & \qquad (1 + g(X_1) + \cdots + g(X_{\tau_n}) + l(t') - n)^{-1}| \mid F_{\tau_n}) \\ & = u \cdot E(|(Y_n + \cdots + Y_{n+l(t')-n}) \\ & \qquad (u \cdot (g(X_1) + \cdots + g(X_{\tau_n})) + u_n + \cdots + u_{n+l(t')-n})^{-1}| \mid F_{\tau_n}) \end{aligned}$$

where $u_1 = u_2 = \cdots = u = EY_1^2$ and the stochastic sequence $Y_1, Y_2, \cdots, F_0', F_1', F_2', \cdots$ with $F_k' = \sigma(X_1, \cdots, X_{\tau_{k+1}})$ satisfies the conditions of Lemma 1, that is: $E(Y_{k+1} \mid F_k') = 0$ and $E(Y_{k+1}^2 \mid F_k') = u$ for $k = 1, 2, \cdots$. Moreover $l(t') - n$ is in $T_\infty'^{(n)}$. Hence:

$$(1) < u^{\frac{1}{2}}(g(X_1) + \cdots + g(X_{\tau_n}))^{-\frac{1}{2}}.$$

Now let

$$t_1 = \text{first } n \geqq t' \text{ such that } X_n \varepsilon [f < 0]$$
$$= \infty \text{ if no such } n \text{ exists}$$

and

$$\bar{t}_1 = \min(t_1, \tau_{l(t')+1})$$

then,

$$E(S^{t'}(1 + g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1} | F_{\tau_n})$$
$$\geqq E(I_{[t_1<\tau_{l(t')+1}]}(f(X_{t_1}) + \cdots + f(X_{\tau_{l(t')+1}-1}))$$
$$(1 + g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1} | F_{\tau_n})$$
$$\geqq -ME(I_{[t_1<\tau_{l(t')+1}]}(\tau_{l(t')+1} - t_1)$$
$$\cdot(1 + g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1} | F_{\tau_n})$$
where $-M = \min_{i\varepsilon[f<0]} f(i)$
$$\geqq -ME((\tau_{l(t')+1} - \bar{t}_1)(g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1} | F_{\tau_n})$$
$$\geqq -M(g(X_1) + \cdots + g(X_{\tau_n}))^{-1}$$
$$\cdot \sum_{i\varepsilon[f<0]} \int_{X_{\bar{t}_1}=i} E(\tau_{l(t')+1} - \bar{t}_1 | X_{\bar{t}_1} = i, F_{\bar{t}_1})\, dP(\cdot | F_{\tau_n})$$
$$\geqq -M(g(X_1) + \cdots + g(X_{\tau_n}))^{-1} \sum_{i\varepsilon[f<0]} m_{ij} P(X_{\bar{t}_1} = i | F_{\tau_n})$$
$$\geqq -Mm(g(X_1) + \cdots + g(X_{\tau_n}))^{-1}$$

where m_{ij} is the expected length of the block from $X_1 = i$ to the first n such that $X_n = j$ and $m = \max_{i\varepsilon[f<0]} m_{ij}$. Hence

$$E((f(X_{\tau_n}) + \cdots + f(X_{\tau_{n+t}}))(1 + g(X_1) + \cdots + g(X_{\tau_n}) + t)^{-1} | F_{\tau_n})$$
$$- Mm(g(X_1) + \cdots + g(X_{\tau_n}))^{-1} < u^{\frac{1}{2}}(g(X_1) + \cdots + g(X_{\tau_n}))^{-\frac{1}{2}}$$

and the proof of Case 1 of Claim 1 follows easily.

Case 2. $[f < 0]$ is countable. In this case choose $k \in J$ such that $f(k) < 0$ and let $B = [f < 0] - \{k\}$ then,

$$-\pi(k)f(k)/\sum_{i\varepsilon[f\geqq 0]} \pi(i)f(i) = K^{-1}$$

where $K = 1 + \delta$, $\delta > 0$,

and,

$$-\sum_{i\varepsilon B} \pi(i)f(i) = \pi(k)f(k) + \sum_{i\varepsilon[f\geqq 0]} \pi(i)f(i) = -\delta\pi(k)f(k).$$

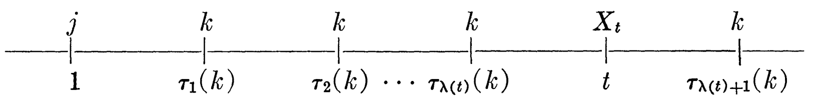
Now we define the functions f' and f'' by:

$$f'(i) = 0 \quad \text{if } i \varepsilon B$$
$$= f(k) \quad \text{if } i = k$$
$$= K^{-1}f(i) \quad \text{if } i \varepsilon [f \geqq 0]; \qquad \text{and}$$
$$f''(i) = 0 \quad \text{if } i \varepsilon [f \geqq 0]$$
$$= \delta f(k) \quad \text{if } i = k$$
$$= -f(i) \quad \text{if } i \varepsilon B$$

and clearly follows that $\sum_{i \varepsilon J} \pi(i) f'(i) = 0 = \sum_{i \varepsilon J} \pi(i) f''(i)$ and we have the identity, $f(\cdot) = Kf'(\cdot) - f''(\cdot)$. Case 1 takes care of Kf'. To finish with Case 2 and in order not to use much notation we shall only prove that for a function f satisfying the same properties as the function f at the beginning of paragraph 4 and with $f(k) > 0$ and $f(J - \{k\}) \leqq 0$:

$$E((f(X_1) + \cdots + f(X_t))(a + t)^{-1} \mid X_1 = j) < \sigma_0 a^{-\frac{1}{2}} \quad \text{for all} \quad a > 0$$

and all $t \varepsilon T_\infty$, where σ_0 is a positive constant. Let $\lambda(t) = \sum_{n=1}^{t} I_{[X_n = k]}$,



For this purpose it is clearly enough to consider stopping rules t such that $t \geqq \tau_1(k)$ and $f(X_1) + \cdots + f(X_t) \geqq 0$. Let

$$Z_\nu = \sum_{n=\tau_\nu(k)}^{\tau_{\nu+1}(k)-1} f(X_n), \qquad \nu = 1, 2, \cdots.$$

For these stopping rules we have;

$$E((f(X_1) + \cdots + f(X_t))(a + t)^{-1} \mid X_1 = j)$$
$$\leqq E((f(k) + Z_1 + \cdots + Z_{\lambda(t)-1})(a + t)^{-1} \mid X_1 = j).$$

$$E\{((Z_1 + \cdots + Z_{\lambda(t)-1})(a + t)^{-1})^2 \mid X_1 = j\} \leqq \sum_{n=1}^{\infty} W_n/(a + n)^2$$

where $W_n = \int_{t=n} (Z_1 + \cdots + Z_{\lambda(n)-1})^2 \, dP(\cdot \mid X_1 = j)$. Let

$$\beta_N = \lambda(t) \quad \text{if} \quad \lambda(t) \leqq N$$
$$= N \quad \text{if} \quad \lambda(t) > N.$$

Then,

$$\sum_{n=1}^{N} W_n \leqq 2E\{(Z_1 + \cdots + Z_{\beta_N})^2 \mid X_1 = j\} + 2E\{Z_{\beta_N}^2 \mid X_1 = j\} \leqq 4E(Z_1^2) \cdot N.$$

Now we argue as A. Dvoretzky did in Lemma 2 of [3].

Since $(a + n)^2 > 0$ and is strictly increasing with n,

$$(2) \qquad \sum_{n=1}^{\infty} W_n/(a + n)^2$$

is increased if some W_n is increased and a W_m, with $m > n$, is decreased by the same amount. Hence the maximum of (2) is obtained for, and only for, $W_1 = W_2 = \cdots = c^2 = 4E(Z_1^2)$. From where it follows that,

$$E((Z_1 + \cdots + Z_{\lambda(t)-1})(a + t)^{-1} \mid X_1 = j) \leqq c(\sum_{n=1}^{\infty} (a + n)^{-2})^{\frac{1}{2}} = ca^{-\frac{1}{2}}.$$

This ends the proof of Claim 1.

Next let

$$U_\nu = \sum_{n=\tau_\nu}^{\tau_{\nu+1}-1} |f(X_n)|, \qquad \nu = 1, 2, \cdots,$$

by Theorem I.14.3 of [2] the random variables $U_1, U_2, \cdots$, are independent identically distributed and with finite variance. Then

$$E(\sup_n |(f(X_1) + \cdots + f(X_n)(g(X_1) + \cdots + g(X_n))^{-1}| \mid X_1 = j)$$
$$\leqq E(\sup_m (U_1 + \cdots + U_m) m^{-1})$$

and by Lemma 9 of [3], $< \infty$. Next we prove that the "functional equation rule" s is finite.

CLAIM 2.

$$P(s = \infty \mid X_1 = j) = 0$$

(*in what follows we shall write* P_j *for* $P(\cdot \mid X_1 = j)$).

PROOF. If $P_j(s = \infty) > 0$ then $P_j(s = \infty) > 2\sigma K^{-1}$ for some positive number K. Let $\tau' =$ first τ_n such that

$$f(X_1) + \cdots + f(X_{\tau_n}) \geqq K(g(X_1) + \cdots + g(X_{\tau_n}))^{\frac{1}{2}}$$

and $= \infty$ if no such n exists. Then $\tau' \varepsilon T$. In order to see this, we note that the Law of the iterated logarithm holds for $\{Y_\nu\}_{\nu=1,2,\cdots}$, and

$$(g(X_1) + \cdots + g(X_{\tau_n}))n^{-1} \rightarrow_{\mathrm{a.s.}} c_2 > 0,$$

hence for large n,

$$K(g(X_1) + \cdots + g(X_{\tau_n}))^{\frac{1}{2}} \leqq K((11/10)\ c_2 n)^{\frac{1}{2}} \leqq \tfrac{1}{2}\sigma(Y)(2n \log\log n)^{\frac{1}{2}}.$$

Now let $\tau = \min(\tau', s)$ then $\tau \varepsilon T$ and for some n,

$$P_j\{\tau = \tau_n < s \quad \text{and} \quad P\ (s = \infty \mid F_{\tau_n}) > 2\sigma/K\} > 0$$

let us denote this event by Λ_{τ_n}.

In order to see this

$$\begin{aligned} 2\sigma K^{-1} < P_j(\tau < s \quad \text{and} \quad s = \infty) &= \textstyle\sum_{n=1}^{\infty} P_j(\tau = \tau_n < s \quad \text{and} \quad s = \infty) \\ &= \textstyle\sum_{n=1}^{\infty} \int_{\tau=\tau_n<s} P(s = \infty \mid F_{\tau_n})\, dP_j \end{aligned}$$

hence, for some n,

$$\int_{\tau=\tau_n<s} 2\sigma/K\ dP_j < \int_{\tau=\tau_n<s} P(s = \infty \mid F_{\tau_n})\, dP_j .$$

Now let us define

$$\begin{aligned} s^* &= s \quad \text{on} \quad \Lambda_{\tau_n}^c \\ &= \tau_n \quad \text{on} \quad \Lambda_{\tau_n} . \end{aligned}$$

It is easy to check that $s^* \varepsilon T_\infty$, and then:

$$\begin{aligned} &\int_{\Lambda_{\tau_n}} (f(X_1) + \cdots + f(X_s))(g(X_1) + \cdots + g(X_s))^{-1}\, dP_j \\ &\quad \leqq \int_{\Lambda_{\tau_n}} E\{(f(X_1) + \cdots + \cdots f(X'_{\tau_n}) + f(X_{\tau_n+1}) + \cdots + f(X_{\tau_n+s-\tau_n})) \\ &\qquad \cdot (g(X_1) + \cdots + g(X_{\tau_n}) + s - \tau_n)^{-1} \mid F_{\tau_n}\}\, dP_j \\ &\quad < \int_{\Lambda_{\tau_n}} (f(X_1) + \cdots + f(X_{\tau_n}))(g(X_1) + \cdots + g(X_{\tau_n}))^{-1}\, dP_j \\ &\quad = \int_{\Lambda_{\tau_n}} (f(X_1) + \cdots + f(X_{s^*}))(g(X_1) + \cdots + g(X_{s^*}))^{-1}\, dP_j \end{aligned}$$

(by Lemma 2)

which is in contradiction with the fact that s is optimal. Therefore $P_j(s = \infty) = 0$. This proves Claim 2.

So far we have shown that if the Markov chain starts with probability 1 at a state j, then the "functional equation rule" s is finite, and from here the proof of the theorem follows easily.

5. Independent case. Let $(X_1, Y_1), (X_2, Y_2), \cdots$, be a sequence of independent random vectors with $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2, 0 < c_0^2 \leqq \sigma_n^2 \leqq c_1^2 < \infty$ for $n = 1, 2, \cdots$, and

(1) $\mathcal{L}((X_1 + \cdots + X_n)(\sigma_1^2 + \cdots + \sigma_n^2)^{-\frac{1}{2}}) \rightarrow N(0, 1)$.

Also, $0 < d_1 \leqq Y_n$, and $\limsup_n (Y_1 + \cdots + Y_n)n^{-1} \leqq d_2 < \infty$ where c_0, c_1, d_1, d_2 are constants.

THEOREM 2. *For the conditions just stated the "functional equation rule" for the stochastic sequence* $\{(X_1 + \cdots + X_n)(Y_1 + \cdots + Y_n)^{-1}, F_n\}_{n=1,2,\cdots}$ *where* $F_n = \sigma(X_1, Y_1, \cdots, X_n, Y_n)$ *is finite,*

This proof is analogous to that given for the M.C. case, so we will omit it.

The following example shows that if the condition (1) fails to be satisfied, then the assertion of Theorem 2 is not necessarily true.

EXAMPLE. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with the following distributions

$$P(X_n = a_n) = P_n, \qquad P(X_n = -b_n) = q_n, \qquad n = 1, 2, \cdots,$$

where

$$\begin{aligned} a_n &= 1 \quad \text{if} \quad n = 1 & b_n &= 1 \quad \text{if} \quad n = 1 \\ &= 2^{-n} \quad \text{if} \quad n \geqq 2, & &= 2^n \quad \text{if} \quad n \geqq 2, \end{aligned}$$

$$\begin{aligned} P_n &= \tfrac{1}{2} \quad \text{if} \quad n = 1 & q_n &= \tfrac{1}{2} \quad \text{if} \quad n = 1 \\ &= (1 + \tfrac{1}{2}^{2n})^{-1} \quad \text{if} \quad n \geqq 2, & &= \tfrac{1}{2}^{2n}(1 + \tfrac{1}{2}^{2n})^{-1} \quad \text{if} \quad n \geqq 2, \end{aligned}$$

then, $E(X_n) = 0$ and $E(X_n^2) = 1$ for $n = 1, 2, \cdots$. On the other side we see that $\sup_{t \varepsilon T_\infty} E(S_t/t) = \frac{1}{2}$, and

$$\begin{aligned} s &= 1 \quad \text{if} \quad X_1 = 1 \\ &= \infty \quad \text{if} \quad X_1 \neq 1 \end{aligned}$$

if the optimal stopping rule.

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