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A DELICATE LAW OF THE ITERATED LOGARITHM FOR NON-DECREASING STABLE PROCESSES

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1. Introduction and results. Until quite recently, the only analogues for stable processes of the law of the iterated logarithm were somewhat crude. The difficulty is basically this: from Khinchin's paper [1] it is easy to deduce that for $X(t)$ any stable process of exponent $\alpha < 2$, and $\varphi(t)$ any monotonic function

$$\limsup_{t\to\infty} X(t)/t^{1/\alpha}\varphi(t)$$

is either zero, almost surely, or infinity, almost surely. This was the way the matter rested until Fristed's work in 1964 [2] where he proved, that for $X(t)$ a non-decreasing stable process with $\alpha < 1$,

$$\liminf_{t\to\infty} X(t)/t^{1/\alpha}(\log\log t)^{-\alpha/1-\alpha} = c \quad \text{a.s.}$$

where c is a finite positive constant. This sort of a result we call a delicate law of the iterated logarithm.

Actually, Fristed proved more than the above. For functions $\varphi(t) \downarrow 0$ he almost proved the analogue of the general law of the iterated logarithm by giving conditions on $\varphi(t)$ under which

$$P(X(t) \leqq t^{1/\alpha}\varphi(t) \text{ i.o.} \quad \text{a.s.} \quad t\to\infty)$$

equals zero or one. The two sets of conditions are close together but not the same.

The reason we cannot get a delicate law of the iterated logarithm for lim sup is that the process has upward jumps which are too large. The reason the delicate result holds for lim inf is that whenever $X(t)/t^{1/\alpha}\varphi(t)$ moves downward, it does so continuously. Following Mootoo [3] we can give a simple and elegant proof of the general law of the iterated logarithm which illuminates the above remarks. We make use of a simple time transformation to change $X(t)$ into a recurrent Markov process which has the property that it moves upward only in jumps and downward continuously. Then Mootoo's proof for Brownian motion can be followed, virtually word for word, to give a proof of the following theorem, which is our main result.

THEOREM 1. *Let $X(t)$, $t \geqq 0$, be the non-decreasing stable process of exponent α, $0 < \alpha < 1$. Take $\varphi(t) \downarrow 0$. Then*

$$P(\liminf_{t\to\infty} (X(t) - t^{1/\alpha}\varphi(t)) \leqq 0) = 1$$

if and only if

$$\int_1^\infty [\varphi(t)]^{-\lambda/2} e^{-\mu[\varphi(t)]^{-\lambda}}\, dt/t = \infty$$

where $\lambda = \alpha/(1-\alpha)$ and μ is related to the intensity m of the process, which is de-

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fined through the relation

$$\log (Ee^{-sX(t)}) = mt \int_0^\infty (e^{-sx} - 1)x^{-(1+\alpha)}\, dx$$

by

$$\mu = (\Gamma(1 - \alpha)m)^{1/(1-\alpha)}(1 - \alpha)/\alpha.$$

From this theorem we get the immediate corollary

COROLLARY 1.

$$\liminf_{t\to\infty} X(t)/(t^{1/\alpha}\mu^{1/\lambda}(\log\log t)^{-1/\lambda}) = 1.$$

By exactly the same methods used to prove Theorem 1, we can get a similar theorem for the behavior of $X(t)$ near $t = 0$.

THEOREM 2. *Take* $\varphi(t) \downarrow 0$ *as* $t \downarrow 0$, *then*

$$P(\liminf_{t\downarrow 0} (X(t) - t^{1/\alpha}\varphi(t)) \leqq 0) = 1$$

if and only if

$$\int_0^1 [\varphi(t)]^{-\lambda/2} e^{-\mu[\varphi(t)]^{-\lambda}} t^{-1}\, dt = \alpha$$

where λ *and* μ *are as defined in Theorem* 1.

2. Proof of Theorem 1. The essential observation in the proof of this theorem is:

PROPOSITION 1. *Let* $X(t)$ *be any stable process of exponent* α, $0 < \alpha < 2$, *then*

$$Z(t) = X(e^t)/e^{t/\alpha}$$

is a stationary Markov process with stationary transition probabilities given by

$$P(Z(t + s)\ \varepsilon\ dx \mid Z(s) = y) = P((X(e^t - 1) + y)/e^{t/\alpha}\ \varepsilon\ dx).$$

PROOF. That $Z(t)$ is Markovian is obvious. Now

$$\begin{aligned} P(Z(t + s)\ \varepsilon\ dx \mid Z(s) = y) &= P(X(e^{t+s})/e^{(t+s)/\alpha}\ \varepsilon\ dx \mid X(e^s) = e^{s/\alpha}y) \\ &= P((X(e^{t+s}) - X(e^s))/e^{(t+s)/\alpha}\ \varepsilon\ dx - e^{-t/\alpha}y). \end{aligned}$$

Since

$$(X(e^{t+s}) - X(e^s))/e^{(t+s)/\alpha}$$

has the same distribution as

$$X(e^t - 1)/e^{t/\alpha},$$

the process has stationary transition probabilities. Further, since $X(e^t)/e^{t/\alpha}$ has the same distribution for all t, the $Z(t)$ process is stationary.

Roughly, the $Z(t)$ process bears the same relationship to the $X(t)$ process as the stationary Ornstein-Uhlenbeck process does to Brownian motion. The essential reason that a delicate law of the iterated log holds for the limit inferior of

$$(X(t) - t^{1/\alpha}\varphi(t))$$

where $X(t)$ is a non-decreasing stable process, is that the $Z(t)$ process does not jump to the left. That is:

PROPOSITION 2. *If* $Z(t_1) = x$ *and* $Z(t_2) = y < x$, *then for any* z, $y < z < x$, *there is a* t, $t_1 < t < t_2$ *such that* $Z(t) = z$.

One easy way to see this is to use the well-known fact that $X(t)$ is the sum of its jumps up to time t.

If we start the $Z(t)$ process off at time zero at $x = 1$, then the times between successive returns to the point $x = 1$ are independent and identically distributed with finite expectation. Starting from $x = 1$, let t^* be the first exit time of the $Z(t)$ process from the interval $(z, 1]$, $0 < z < 1$, and

$$u(z) = P(Z(t^*) = z).$$

Then for $\psi(t)$ any non-increasing function, it follows from Mootoo's work [3] that

$$P(\liminf_{t\to\infty} (Z(t) - \psi(t)) \leqq 0) = 1$$

if and only if

$$\int_0^\infty u(\psi(t))\, dt = \infty.$$

Actually, Mootoo proves this result only for positive recurrent diffusions, but the proof goes through, word for word, for strong Markov processes whose sample paths satisfy Proposition 2 and which have finite expected recurrence times. The proof then reduces to finding $u(z)$. An expression in closed form seems difficult to come by (except in the case $\alpha = \frac{1}{2}$). But all we really need is:

THEOREM 3. *As* $z \downarrow 0$,

$$u(z) \sim z^{-\lambda/2} e^{-\mu z^{-\lambda}}.$$

With this result a simple change of variable in Mootoo's integral gives Theorem 1. Hence we complete the proof of the main result by now proving Theorem 3.

Take $f(x)$ to be any continuous bounded function on $[0, \infty)$ with a continuous bounded first derivative. The infinitesimal operator Sf is given by the limit as $t \downarrow 0$ of

$$t^{-1}(E_x f(Z(t)) - f(x)) = t^{-1}(f((X(e^t - 1) + x)/e^{t/\alpha}) - f(x)).$$

The infinitesimal operator for $X(t)$ is

$$m \int_0^\infty [f(x + y) - f(x)] y^{-(1+\alpha)}\, dy.$$

A Taylor expansion gives

$$(Sf)(x) = m \int_0^\infty [f(x + y) - f(x)] y^{-(1+\alpha)}\, dy - \alpha^{-1} x f'(x).$$

Let t^* be the first exit time from $(z, 1]$ and define

$$h(x) = P_x(Z(t^*) > 1).$$

I assert that as $x \downarrow z$

$$\lim h(x) = 0.$$

An elementary proof is to show that as $x \downarrow z$

$$P_x(Z(1 - z/x) \leqq z) \to 1.$$

Let

$$\log(1 + \epsilon) = (1 - z)/x,$$

then the above follows from a quick computation based on the fact that $X(\epsilon)/\epsilon^{1/\alpha}$ has the same distribution as $X(1)$. Define $h(x) = 1$, $x \geqq 1$, and $h(x) = 0$, $0 \leqq x \leqq z$. It is well-known (see [4], pg. 143) that if U is the characteristic operator for the process, that

$$(Uh)(x) = 0, \qquad z < x \leqq 1.$$

Notice also that $h(x)$ is continuous on $(z, 1]$. By using the minimum principle ([4], pg. 141) we can furthermore show uniqueness: Any function f agreeing with h outside of $(z, 1]$, satisfying $(Uf)(x) = 0$ on $(z, 1]$, and continuous on $(z, 1]$, is equal to h. We look for a function f satisfying these conditions which is, in addition, differentiable on $(z, 1]$. Then $(Sf)(x) = (Uf)(x)$ on $(z, 1]$, and the equation we want to solve is

$$\begin{aligned} xf'(x) &= \alpha m \int_0^\infty (f(x+y) - f(x))y^{-(1+\alpha)}\,dy \\ &= \alpha m \int_x^\infty (f(y) - f(x))(y-x)^{-(1+\alpha)}\,dy \\ &= \alpha m\left(\int_x^1 (f(y) - f(x))(y-x)^{-(1+\alpha)}\,dy + (1 - f(x))/\alpha(1-x)^\alpha\right). \end{aligned}$$

Integrate by parts in the integral, getting

$$xf'(x) = m\int_x^1 f'(y)(y-x)^{-\alpha}\,dy + (1 - f(1))m(1-x)^{-\alpha}.$$

Let $\sigma = 1 - f(1)$. To show that the integral equation

$$x\theta(x) = m\int_x^1 \theta(y)(y-x)^{-\alpha}\,dy + \sigma m(1-x)^{-\alpha}$$

has a unique solution on $(0, 1)$, break $(0, 1)$ up into intervals

$$I_1 = (x_1, 1), \qquad I_2 = (x_2, x_1], \cdots$$

such that for each I_k,

$$m x_k^{-1}\int_{I_k}(y-x)^{-\alpha}\,dy \leqq \gamma < 1.$$

Use successive approximations

$$x\theta_{n+1}(x) = m\int_x^1 \theta_n(y)(y-x)^{-\alpha}\,dy + \sigma m(1-x)^{-\alpha}$$

and verify that

$$\sup_{x \geqq x_k} |\theta_{n+1}(x) - \theta_n(x)| \leqq C(n+1)^k\gamma^{n+1}.$$

If we solve this integral equation putting $\sigma = 1$, denote the solution by $\theta(x)$. Then, in general, $f'(x) = \sigma\theta(x)$, and

$$f(1) = \int_z^1 f'(x)\,dx = (1 - f(1))\int_z^1 \theta(x)\,dx.$$

By definition, $u(z) = 1 - f(1)$, so for $z \downarrow 0$,

$$u(z) \sim c/\int_z^1 \theta(x)\,dx.$$

Suppose we can find a function $g(x)$ on $(0, \infty)$, vanishing for $0 \leqq x < z$, and satisfying for $x \geqq z$, the equation

$$xg(x) = m \int_0^x g(y)(x - y)^{-\alpha}\,dy + 1.$$

Multiply the integral equation for $\theta(x)$ by $g(x)$, and integrate from zero to one, getting

$$\int_z^1 \theta(x)g(x)x\,dx$$
$$= \int_0^1 m\theta(x)(\int_0^x g(y)(x - y)^{-\alpha}\,dy)\,dx + m\int_0^1 g(y)(1 - y)^{-\alpha}\,dy.$$

Therefore

$$\int_z^1 \theta(x)\,dx = m\int_0^1 g(y)(1 - y)^{-\alpha}\,dy.$$

Let the Laplace transform of $g(x)$ be $\hat{g}(s)$. The right hand side above is the convolution of $g(x)$ and m/x^α evaluated at one. Hence equals

$$m\Gamma(1 - \alpha)(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^s s^{\alpha-1}\hat{g}(s)\,ds.$$

The transform of the equation satisfied by $g(x)$ is

$$-\hat{g}'(s) = \Gamma(1 - \alpha)\hat{g}(s)s^{\alpha-1} + s^{-1}e^{-zs}.$$

The appropriate solution is

$$\hat{g}(s) = e^{-\beta s^\alpha}\int_s^\infty \xi^{-1}e^{-z\xi}e^{\beta\xi^\alpha}\,d\xi$$

where $\beta = \Gamma(1 - \alpha)m/\alpha$. Write

$$R(z) = \int_1^\infty \xi^{-1}e^{-z\xi}e^{\beta\xi^\alpha}\,d\xi, \qquad F(x, z) = \int_1^s \xi^{-1}e^{-z\xi}e^{\beta\xi^\alpha}\,d\xi.$$

Then

$$\int_z^1 \theta(x)\,dx = R(z)m\int_0^1 (1 - x)^{-\alpha}\,dx$$
$$+ m\Gamma(1 - \alpha)(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^s s^{\alpha-1}e^{-\beta s^\alpha}F(s, z)\,ds.$$

The second term approaches a constant as $z \downarrow 0$. Hence the whole thing revolves around the asymptotic behavior of $R(z)$. Let $\delta = 1/(1 - \alpha)$ and make the substitution $\xi = z^{-\beta}x$ in $R(z)$ to get

$$R(z) = \int_{z^\beta}^\infty e^{z^{-\lambda}(-x+\beta x^\alpha)}x^{-1}\,dx.$$

The asymptotic expansion for this integral is easily gotten by Laplace's method (see, for example [5]), with the result

$$R(z) \sim cz^{\lambda/2}e^{\mu z^{-\lambda}}$$

where

$$\mu = \max(-x + \beta x^\alpha) = (\alpha\beta)^\delta(1/\alpha - 1).$$

3. The situation as $t \downarrow 0$. To look at $X(t)$ as $t \downarrow 0$ we need to examine the oscillations of $Z(t)$ as $t \to -\infty$. By a slight reworking of Mootoo's proof we get that

$$P(\liminf_{t\to-\infty} (Z(t) - \psi(t)) \leqq 0) = 1$$

if and only if

$$\int_{-\infty}^{0} u(\psi(t))\, dt = \infty.$$

Changing variable in this integral and using Theorem 3 yields the result of Theorem 2.

4. Remarks. Obviously, these results indicate that a similar theorem should hold for sums of independent identically distributed non-negative random variables in the domain of attraction of a stable law of exponent less than one. In the second moment case, by using Skorohod's idea of embedding the sums in a Brownian motion, Strassen [6] showed how to get the law of the iterated logarithm for random variables from the theorem for Brownian motion. But it is not at all clear that anything like that kind of embedding holds for stable processes and random variables in the domain of attraction of a stable law.

However, as an indication that this should hold, look at this argument: let S_n be a sum of n independent identically distributed variables with zero means and finite variances σ^2. Define ladder variables n_k^* by

$$n_{k+1}^* = \min \{n; S_n - S_{n_k^*} > 0, n > n_k^*\}, \qquad n_0^* = 0.$$

Then write, with

$$\varphi(n) = [2\sigma^2 n \log (\log n)]^{\frac{1}{2}},$$

$$\begin{aligned} 1 &= \limsup S_n/\varphi(n) \\ &= \limsup S_{n_k^*}/\varphi(n_k^*) = \limsup (S_{n_k^*}/k)\cdot(k/\varphi(n_k^*)) \quad \text{a.s.} \end{aligned}$$

Now $S_{n_k^*}$ is the sum $Y_1 + \cdots + Y_k$ of independent, identically distributed variables given by $Y_k = S_{n_k^*} - S_{n_{k-1}^*}$. Since $EY_1 < \infty$, then the law of large numbers gives

$$\varphi(n_k^*)/k = EY_1 \quad \text{a.s.}$$

The ladder variables n_k^* are sums $X_1 + \cdots + X_k$ of non-negative independent random variables in the domain of attraction of a stable law with exponent $\frac{1}{2}$. The above relationship leads to

$$\liminf (X_1 + \cdots + X_k)/(k^2/\log\log k) = c \quad \text{a.s.}$$

Regarding the proof of Theorem 1, Sidney Port has pointed out to me that by very similar reasoning a delicate law of the iterated logarithm should hold for the completely asymmetric processes with exponent $\alpha \geqq 1$. I am indebted to the referee for pointing out some key references, especially Fristed's work.

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